The Threshold Theorem for the energy critical Yang-Mills Flow

Daniel Tataru

University of California, Berkeley

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Three linear wave equations

1. The wave equation for functions

$$u : \mathbb{R}^{n+1} \to \mathbb{R}$$

Lagrangian:

$$L(\phi) = \int \partial^\alpha u \cdot \partial_\alpha u \; dxdt$$

Euler-Lagrange equation:

$$\Box u = 0$$

D'Allembertian:

$$\Box = \partial^\alpha \partial_\alpha$$

Three linear wave equations

2. The Maxwell equation for 1-forms:
Electromagnetic potential:

$$A_\alpha : \mathbb{R}^{n+1} \to \mathbb{R}$$

Covariant differentiation:

$$D_\alpha = \partial_\alpha + iA_\alpha$$

Curvature:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

Lagrangian:

$$\mathcal{L}(A) := \frac{1}{2} \int_{\mathbb{R}^{4+1}} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle \, dxdt.$$

Maxwell system:

$$D^\alpha F_{\alpha\beta} = 0$$

Gauge freedom

$$A \to A + db, \qquad b : \mathbb{R}^{n+1} \to \mathbb{R}$$

Three linear wave equations

3. The covariant wave equation for functions

$$\phi : \mathbb{R}^{n+1} \to \mathbb{C}$$

Lagrangian:

$$L(\phi) = \int D^\alpha \phi \cdot \overline{D_\alpha \phi}\, dxdt$$

Euler-Lagrange equation:

$$\Box_A u = 0$$

D'Allembertian:

$$\Box_A = D^\alpha D_\alpha$$

Three geometric wave equations

- Wave-Maps
- Maxwell-Klein Gordon
- Yang Mills

Global well-posedness and scattering in two settings:

- small data in the critical Sobolev space
- large data in the energy critical dimension

Wave maps

Maps into a Riemannian manifold:

$$\phi : \mathbb{R}^{n+1} \to (M, g)$$

Lagrangian

$$L(\phi) = \int \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_g dxdt$$

Euler-Lagrange equation in local coordinates:

$$\Box \phi + \Gamma(\phi) \partial^\alpha \phi \partial_\alpha \phi = 0$$

Covariant formulation:

$$D^\alpha \partial_\alpha \phi = 0$$

Energy

$$E(\phi) = \int |\partial_t \phi|_g^2 + |\nabla_x \phi|_g^2 dx$$

Critical Sobolev space: $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$, energy critical dimension $n = 2$.

Maxwell-Klein-Gordon

Maxwell field A, with curvature $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.
Complex field ϕ in $\mathbb{R}^{n+1}$, with covariant differentiation $D_\alpha = \partial_\alpha + iA_\alpha$.
Lagrangian:

$$\int_{\mathbb{R}^{n+1}} \frac{1}{2} D^\alpha \phi \overline{D_\alpha \phi} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} dxdt$$

Euler-Lagrange equations:

$$\partial^\beta F_{\alpha\beta} = \Im(\phi \overline{D_\alpha \phi})$$
$$D^\alpha D_\alpha \phi = 0$$

Gauge invariance: $(A, \phi) \to (A - db, \phi e^{ib})$.
Energy:

$$E(A, \phi) = \int_{\mathbb{R}^n} \frac{1}{4} |F|^2 + \frac{1}{2} |D_A \phi|^2 dx$$

Critical Sobolev space: $\dot{H}^{\frac{n}{2}-1} \times \dot{H}^{\frac{n}{2}-2}$, energy critical dimension $n = 4$.

Yang-Mills

Connection form $A_\alpha : \mathbb{R}^{4+1} \to \mathfrak{g}$, semisimple Lie algebra.

$$D_\alpha B := \partial_\alpha B + [A_\alpha, B] \qquad \text{(covariant differentiation)}$$

Curvature tensor

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta],$$

Lagrangian action functional

$$\mathcal{L}(A_\alpha, \phi) := \frac{1}{2} \int_{\mathbb{R}^{4+1}} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle \, dxdt.$$

Covariant form of Euler-Lagrange equations:

$$D^\alpha F_{\alpha\beta} = 0.$$

Gauge invariance: $A_\alpha \to OAO^{-1} - \partial_\alpha O O^{-1}$.

Conserved energy:

$$E(A) = \int_{\mathbb{R}^4} |F|^2 dx$$

Critical Sobolev space: $\dot{H}^{\frac{n}{2}-1} \times \dot{H}^{\frac{n}{2}-2}$, energy critical dimension $n = 4$.

Small data results

Theorem

(WM) is globally well-posed for small data in $\dot H^{\frac{n}{2}} \times \dot H^{\frac{n}{2}-1}$, $n \geq 2$.
(Tao '01 $\mathbf{S}^n$, *Krieger '03* $\mathcal{H}^2$, *T.'05* (M,g)*)*

Theorem

(MKG) is globally well-posed for small data in $\dot H^{\frac{n}{2}-1} \times \dot H^{\frac{n}{2}-2}$, $n \geq 4$.
[Coulomb gauge] (Rodnianski-Tao '05 ($n \geq 6$), Krieger-Sterbenz-T. '13)

Theorem

(YM) is globally well-posed for small data in $\dot H^{\frac{n}{2}-1} \times \dot H^{\frac{n}{2}-2}$, $n \geq 4$.
[Coulomb gauge] (Sterbenz-Krieger '06 ($n \geq 6$), Krieger-T. '15)

- quasilinear well-posedness (continuous dependence on data)
- modified scattering

The large data energy critical problem

Energy critical dimension: $n = 2$ (WM), $n = 4$ (MKG), (YM)

Potential obstructions to large data global well-posedness:

- Blow-up solutions (e.g. self-similar)
- Stationary solutions (solitons)

Ground state = lowest energy nontrivial steady state

Conjecture (Threshold Conjecture)

Global well-posedness and scattering holds in energy critical problems for data below the ground state energy (globally, if there is no steady state solution).

Three large data results

Theorem

(WM) is globally well-posed for data below the ground state energy, $n = 2$. (Sterbenz-T. '08 (M, g), Krieger-Schlag '08 $\mathcal{H}^2$, Tao '08 $\mathcal{H}^n$)

Theorem

(MKG) is globally well-posed for finite energy data, $n = 4$. [Coulomb gauge] (Oh-T. '15, Krieger-Luhrmann '15)

Theorem

(YM) is globally well-posed for data below the ground state energy. [caloric gauge] (Oh-T. '17, main goal of these lectures)

- no self-similar blow-up scenario

The gauge choice

Gauge freedom:

$$A \to A - db \qquad (MKG)$$

$$A_\alpha \to OAO^{-1} - \partial_\alpha O O^{-1} \qquad (YM).$$

Objectives of gauge fixing:

- preserve hyperbolic structure
- capture null structure of equations
- globally defined (large data)

Gauge choices:

- Lorenz gauge $\partial^\alpha A_\alpha = 0$.
- temporal gauge $A_0 = 0$.
- Coulomb gauge $\partial_j A_j = 0$. [(MKG), small data (YM)]
- Caloric gauge -defined via covariant heat flow. [Large data (YM)]

The heat flow and the local caloric gauge

Covariant Yang-Mills heat flow:

$$F_{j\alpha} = D^k F_{k\alpha}$$

Gauge choices:

- De Turck gauge $A_s = \partial^j A_j$.
  - strongly parabolic flow, but
  - not clear that solutions are global for large data
- local caloric gauge $A_s = 0$
  - degenerate parabolic
  - global solutions
  - solutions decay to flat conection A_∞, i.e. $F_\infty = 0$.

Theorem (Threshold theorem for Yang-Mills heat flow)

Data with energy below the ground state energy E_0 yield global solutions in the local caloric gauge.

The caloric gauge

A state A_x is called caloric if its parabolic flow satisfies

$$A(s = \infty) = 0$$

Caloric manifold $\mathcal{C}$ of class C^1 below E_0.
Wave Yang Mills data $(A_x, \partial_0 A_x) \in TC$.
Generalized Coulomb condition:

$$\partial^j A_j = Q(A, A) + R(A, A, A)$$

with Q quadratic and explicit, and R cubic and higher.
Yang-Mills equation in the caloric gauge:

$$\Box_A A_j = Q_j(A, \partial A) + R_j(A, A, A)$$

$$\Delta_A A_0 = Q_0(A, \partial A) + R_0(A, A, A)$$

Small data: semilinear vs quasilinear

Nonlinear wave equation:

$$\Box A = N(A)$$

Perturbative (semilinear) set-up:

$$\Box A = N(A) \qquad (perturbative)$$

Paradifferential (quasilinear) set-up:

$$\Box A_k + 2[A^{\alpha}_{<k}, \partial_\alpha A_k] = N_{pert}(A)_k$$

Two key difficulties:

A. Scale invariant function spaces for the perturbative part.

B. Parametrix construction for the paradifferential part.

Function spaces S, N

Here the goal is to have two properties:

$$\|A\|_S \lesssim \|A[0]\|_E + \|\Box A\|_N \qquad \text{(linear mapping)}$$

$$\|N_{pert}(A)\|_N \lesssim C(\|A\|_S) \qquad \text{(nonlinear mapping)}$$

1. Strichartz norms
   - scale invariant
   - do not capture null structure
2. $X^{s,b}$ spaces
   - capture null structure
   - no good scaling.
3. U^2 and V^2 spaces (T. '00, Koch-T. '04)
   - scale invariant refinements of $X^{\frac{1}{2}}$ spaces.
4. Null frame spaces (T, Tao '00)
   - combine Strichartz norms with multiscale frequency localizations adapted to the null cone.

The parametrix construction

Here the goal is to have the linear bound

$$\|A_k\|_S \lesssim \|A_k[0]\|_E + \|\Box A_k + \|_N$$

Renormalization: approximately conjugate the paradifferential flow to the flat wave flow.

$$R(A_k + 2[A^{\alpha}_{<k}, \partial_\alpha A_k]) - \Box R A_k = perturbative$$

with good mapping and invertibility properties

$$R : S \to S, \qquad R : N \to N$$

(i) (WM) (Tao '01, T. '04) Multiplicative renormalization

$$R = R(t,x) : \mathbb{R}^{n+1} \to SO(d)$$

(ii) (MKG) (Rodnianski-Tao '05) (YM) (Sterbenz-Krieger '06).

$$R = R(t,x,D),$$

pseudodifferential operator with rough symbol.

Large data: three related methods

Goal: To prove an estimate

$$\|A\|_S \lesssim F(E(A)), \qquad E < E_0 \quad \text{(ground state energy)}$$

1. Direct induction on energy (Bourgain), combines perturbative and nonperturbative elements in a single induction step.
2. Concentration compactness method (Kenig-Merle). Two step approach, by contradiction:
    - prove the existence of a minimal energy blow-up solution, with good compactness properties.
    - disprove the existence of a minimal energy blow-up solution, by Morawetz style (nonconcentration) estimates.
3. Energy dispersion method (Sterbenz-T.). Two step approach, direct method:
    - prove that energy dispersed solutions are global and scatter.
    - prove that all solutions are either energy dispersed or have pockets of energy convergent to a steady state.

Energy dispersed solutions

Energy dispersion norm:

$$\|A\|_{ED} = \sup_k 2^{-k} \|A_k\|_{L^\infty},$$

- scales like the energy
- measure of pointwise concentration
- measured in a time interval, not at fixed time

Theorem (Energy dispersed solutions)

For each $E < E_0$ there exist $\epsilon(E)$, $F(E)$ so that for each solution A of energy E in a time interval I we have:

$$\|A\|_{ED} \leq \epsilon(E) \implies \|A\|_S \leq F(E)$$

An induction on energy proof

The key steps are as follows:

1. Improved bilinear/multilinear estimates: ED yields gains for all *balanced* frequency interactions.
2. Improved paradifferential bound:

$$\|A_k\|_S \lesssim \|A_k[0]\|_E + \|\Box A_k + 2[A^\alpha_{<k}, \partial_\alpha A_k]\|_N$$

   with the *frequency gap* $m \gg_{\|\phi\|_S} 1$ as a proxy for smallness.
3. Divisibility estimate: For any solution ϕ of energy E and S size F we can split the time interval into $N \lesssim_F 1$ so that

$$\|\phi\|_{S[I_k]} \lesssim E$$

4. Induction on energy $E \to E + c$ with $c = c(E)$.

A concentration dichotomy

Theorem

For any finite energy solution in a cone we have the following dichotomy. Either $E(A) \to 0$ at the tip of the cone, or, on a subsequence,

$$A\left(\frac{x - x_n}{r_n}, \frac{t - t_n}{r_n}\right) \to L A_{steady}$$

with L Lorentz and A_{steady} steady state.

Proof ideas:

- Energy-flux relation: Flux converges to 0 at the tip of the cone.
- Morawetz identity with vector field $X_0 = \dfrac{t\partial_t + x\partial_x}{\sqrt{t^2 - x^2}}$ and translates.
- Eliminate null concentration scenario
- Show concentration persists away from cone.
- Extract concentration profile by multiple pidgeonhole arguments.
- Exclude self-similar solutions.