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Kakeya needle for rectifiable sets

Talk I ①

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$E \subseteq \mathbb{R}^2$ has

Any position → **Strong Kakeya condition**

if it can be moved to any other position covering only small area.

$\exists$ a position → Kakeya condition $\exists$ shift $\neq$ id s.t. moving to this shifted location takes 0 area.

Examples: ① line segment has strong K-prop.

② full line, circle → has K-property but NOT strong.

③ circular arcs have the strong K-prop.

$\boxed{\text{Question}}$: Which sets have the Kakeya property?

$\boxed{\text{Theorem}}$ (C., Héra, Laczkovich)

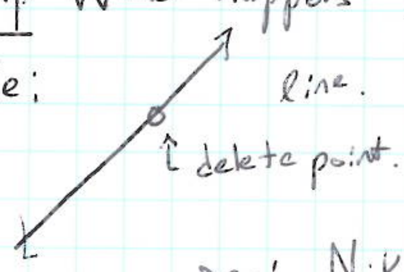
- If E closed $\wedge$ connected w/ K.P., then E $\subseteq$ line or circle.

- If E closed $\wedge$ has K.P., then the non-trivial non-connected components of E $\subseteq$ neighborhood of parallel lines or nghbh. of co-centric circles.

("proof of this theorem later")

$\boxed{\text{Question}}$: What happens if we're allowed to delete null set? (H^1 or smaller)

example:



can be used to any other position & cover 0 area.

see: Nikodym Sets.

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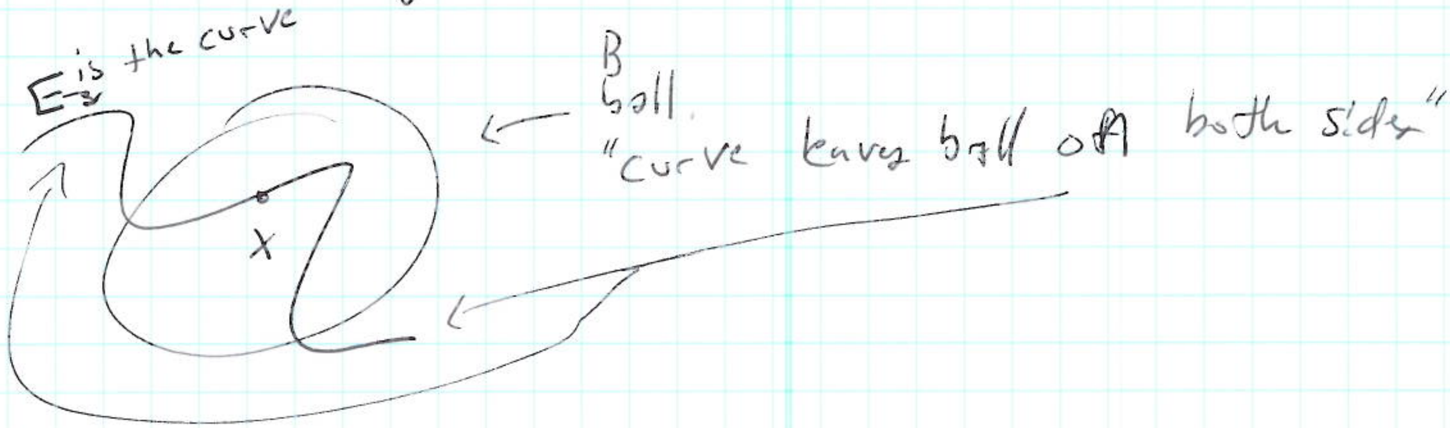
Claim 2: E is graph of convex function

E can be moved to any other position & cover 0 area.

Claim: Every rectifiable set o.k. if we are allowed to delete null set

"Proof is very technical, but we can look at the geometry"



E curve
B ball about x

α = isometry from Kakeya property.

$$A = (E \cap B) \cup \partial B$$

$|\beta x - x| = r$

Fix small distance. r Stop the movement when x is moved by this distance. Take a sequence of smaller & smaller movements. We get closer & closer to βx. Create continuous movement & move x arbitrarily close to βx.

Claim $\beta x \in E \Rightarrow \beta(\beta x) \in E$ & so on

This constructs a sequence of points in E.
If r smaller, we get a "denser" set. Let $r \downarrow 0$. Choose convergent subseq.

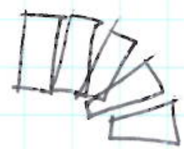
get a circle or line segment from all the limit points (in E) 3

Proof breaks down if we are allowed to puncture.

Constructions of Kakeya Sets:

① Fix N large. Take $\frac{1}{N}$ - rectangles (or triangles)

s.t. $\forall$ θ, a direction, $\exists$ rectangle w/ line angle θ



Total Area: $\frac{1}{N} \cdot N = 1$.

"classic example"

② $(a,b) \in \mathbb{R}^2 \longrightarrow y = ax+b$.

Dual* "for every a there is a b"

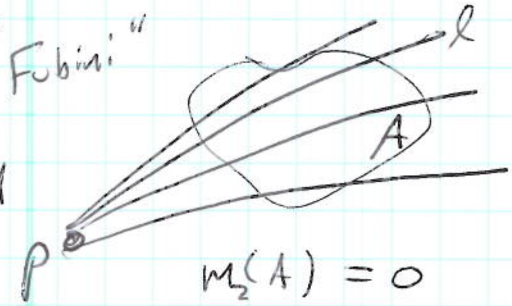
"Useful when studying curves"

Besicovitch Set

① Measure zero & ② contains line in each direction.

Dual Set points.

"Fubini"



①* in a.e. direction, the projection is null

②* full projection to X-Axis

$m_2(A) = 0$

$\Longleftrightarrow m_1(A \cap \ell) = 0$ for a.e. ℓ.

Projection Thm $H^1(E) < \infty$ then

a.e. projection null $\Longleftrightarrow$ E meets every line in a null set

Kakeya
Besicovitch
Nikodym — for a.e. $x \in \mathbb{R}^2$, $\exists$ copy E_x $\ni$ $\bigcup E_x \setminus \{x\}$ has m. $= 0$.

K.P. — E line segment or arc

Hera Laczkovich — circle arcs

CHANG, C. — shorter than ½-circle $\Rightarrow$ strong K.P.

If we are allowed to delete $\mathcal{H}^1$-null set, then K, B, N true.
for all rectifiable sets.

<u>Stein</u> — spheres in $\mathbb{R}^n$, $n \geq 3$ centers have pos. measure
$\Rightarrow$ union (+) measure.

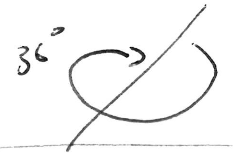
<u>Mitsis</u> — set of centers dim > 1

"we must be careful w/ Nikodym business"

<u>Bourgain, Marstrand</u> — $n = 2$

<u>Wolff</u> — $n = 2$, centers have dim > 1, curves not just circles.

<u>Theorem</u> $\exists$ cont. movement of a line
that rotates it $360°$, $\exists$ $x_t \in l_t$
s.t. $\bigcup l_t \setminus \{x_t\}$ has measure zero.



"without puncture, we cover all $\mathbb{R}^2$"
"~~So punctures must cover a.e. $\mathbb{R}^2$~~"
"go thru. implies the punctures cover almost everything"

line — 1 pt.
parabola — 2 pts.
shift circle, enough to delete 2 pts.

Csornyei
TALK II
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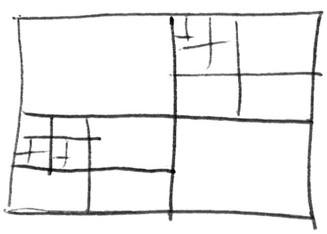
Constructions

1

$\to$ move them together $\log N$ overlap $\to$ $\frac{1}{\log N}$ area

2 $y = ax + b$ line $\longrightarrow$ $(a, b) \in \mathbb{R}^2$

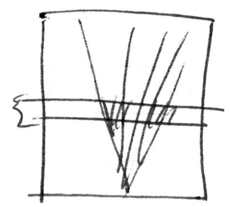


E has full proj. on x-axis
$\Rightarrow$ dual set of lines contain line in each direction.

$H^1(E) < \infty$
proj. thm. dual set of lines in B-set
$\Leftrightarrow$ E meets every C^1 curve in 0 length

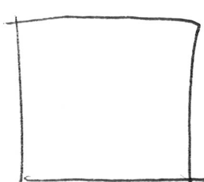
3 Baire Category Thm.

Körner short paper.



All compact union lines.

dual



Consider all compact sets w/ full proj. onto the x-axis w/ H-metric

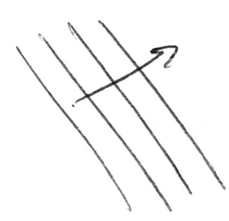
A typical one has zero projection in any other direction.

KELETI, NAGY, SHMERKIN

$\exists$ 1-d set in $\mathbb{R}^2$ that contains an axis parallel square around each pt.

THORNTON

Chang, C, Hera, Keleti



pick pt. each line
dual
projection at least d-1