# ELLIPTIC VARIATIONAL PROBLEMS

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**MSRI Lecture 1**

# Elliptic Variational Problems

- 1st variation and singular limits
- 2nd variation and stability

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- 2nd variation and stability
- Free boundary regularity $n = 3, 4$
- Higher critical points, topology
- Higher dimensions; $n \to \infty$ ?

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First variation $J'(u) = -2\Delta u + F'(u)$

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$$J(u + \varepsilon v) =$$

$$J(u) + \varepsilon \int [2\nabla u \cdot \nabla v + F'(u)v] + O(\varepsilon^2)$$

$$= J(u) + \varepsilon \langle -2\Delta u + F'(u), v \rangle + O(\varepsilon^2).$$

# CRITICAL POINTS

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## Euler-Lagrange eq'n:

$$2\Delta u = F'(u)$$

# Blow down limit

$$F(v) = 0,\ v \leq 0,\ F(v) = 1,\ v \geq a.$$

$$v_R(y) = \frac{1}{R} v(Ry) \ \longrightarrow \ v_\infty(y), \quad R \to \infty$$

# Blow down limit

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$$v_R(y) = \frac{1}{R} v(Ry) \longrightarrow v_\infty(y), \quad R \to \infty$$

$$\int_{B_R} [|\nabla v|^2 + F(v)] \frac{dx}{R^n}$$

$$\longrightarrow \int_{B_1} [|\nabla v_\infty|^2 + 1_{\{v_\infty > 0\}}] dy$$

# Alt-Caffarelli functional

$$J_\infty(v,\Omega) = \int_\Omega [|\nabla v|^2 + 1_{\{v>0\}}]\, dy$$

$$F_\infty(v) = 1_{\{v>0\}}$$

# Optimal Insulation

$$\text{Cost} = c_1 \int_R |\nabla u|^2 + c_2 \,\text{vol}(R)$$

Equilibrium temperature u:

$$\Delta u = 0 \quad \text{on} \quad R = \{T_0 < u < T_1\}$$

# Optimal Insulation

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Equilibrium temperature u:

$$\Delta u = 0 \quad \text{on} \quad R = \{T_0 < u < T_1\}$$

$$\partial u / \partial \nu = \text{const} \quad \text{on } \{u = T_0\}$$

# Global Minimizers

Functions $u : \mathbf{R}^n \to \mathbf{R}$ such that

$$J(u,\Omega) = \min_v J(v,\Omega) \quad \text{and all } \Omega \subset\subset \mathbf{R}^n,$$

min over all $v$ such that $v = u$ on $\partial\Omega$.

$$J(v,\Omega) = \int_\Omega (|\nabla v|^2 + 1_{\{v>0\}})\, dx$$

# Rescaled Euler-Lagrange Equation

$$u_R(x) := u(Rx)/R$$

$$2\Delta u_R = RF'(Ru_R) \longrightarrow \text{“}2\Delta u_\infty = \delta(u_\infty)\text{”}$$

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## Free boundary condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 1 \quad \text{on } \partial\{u > 0\}$$

One phase: $u^- \equiv 0$. Two phase: $u^- \not\equiv 0$.

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$$w_R(y) = v(Ry) \longrightarrow w_\infty(y), \quad R \to \infty$$

$$\int_{B_R} [|\nabla v|^2 + F(v)] \frac{dx}{R^{n-1}}$$
$$\longrightarrow c\, \mathrm{vol}_{n-1}(B_1 \cap \{|w_\infty| < a\})$$

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$J_\infty$ is the area of the $\pm a$ interface.

# Euler-Lagrange equation

If $F$ has "one hump", and

$$\int_{\mathbf{R}} F(w)\, dw = 1, \quad \text{then}$$

$$2\Delta w_R = R^2 F'(R w_R) \longrightarrow \text{“}2\Delta w_\infty = \delta'(w_\infty)\text{”}$$

$w_\infty = \pm a$ and $H = 0$ on the interface.

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What can minimal surface theory teach us?

**Def'n.** Area-minimizer $M \subset B_1 \subset \mathbf{R}^n$

$$\mathrm{vol}_{n-1}(M) \le \mathrm{vol}_{n-1}(N) \text{ all } N,\ \partial N = \partial M$$

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**Theorem.** (J. Simons 1967 + )
Area-minimizers are smooth for $n \leq 7$. But for $n = 8$,

$$x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2 \quad \text{minimizes.}$$

# Universal bounds, $n \leq 7$

Suppose $M$ is area-minimizing in $B_1$, $0 \in M$. After rotation, in $B_{1/100}$,

$$M = \{x_n = g(x')\}$$

$$|\nabla g| + |D^2 g| \leq 100$$

**Theorem.** Energy-minimizing free boundaries are smooth in $\mathbf{R}^n$, $n = 3, 4$.
(Caffarelli, J-, Kenig 2002; J-, Savin 2015)

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There are singular minimizing cones $n \geq 7$.
(De Silva, J-, 2006)

Dimensions $n = 5, 6$ are open.

# STABILITY

$$\frac{d^2}{d\varepsilon^2} J(u+\varepsilon v)|_{\varepsilon=0}$$

$$= \int (2|\nabla v|^2 + F''(u)v^2)\, dx \geq 0$$

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Linearization of $2\Delta u - F'(u) = 0$

$$\langle(-2\Delta + F''(u))v, v\rangle \geq 0$$

# SUBSOLUTION METHOD

**Positive subsolution $\iff$ instability.**

$$Lv = (-2\Delta + F''(u))v < 0; \quad v > 0 \quad \text{on } \Omega,$$

and $v = 0$ on $\partial\Omega$, then

$$0 > \int_\Omega (Lv)v = \int_\Omega (2|\nabla v|^2 + F''(u)v^2)$$

Stability for minimal hypersurface $M$:

$$\int_M f^2 |A|^2 \, d\text{vol} \leq \int_M |\nabla f|^2 \, d\text{vol}$$

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**Simons:** On cones

$$f = |A|\psi(|x|), \quad \psi \in C_0^\infty(\mathbf{R}_+)$$

Stability for minimal hypersurface $M$:

$$\int_M f^2 |A|^2 \, d\mathrm{vol} \le \int_M |\nabla f|^2 \, d\mathrm{vol}$$

Stability for energy-minimizing $u$:

$$\int_{\partial\Gamma} \phi^2 H \, d\sigma \le \int_\Gamma |\nabla\phi|^2 \, dx$$

$$(\Gamma = \{u > 0\})$$

Area instability $\iff$ Positive subsolution:

$$\Delta f > |A|^2 f, \quad f \geq 0, \quad f \in C_0^\infty(M)$$

Energy instability $\iff$ Pos. subsolution:

$$\Delta \varphi > 0 \text{ in } \Gamma; \quad \varphi_\nu \geq H\varphi \text{ in } \partial\Gamma; \quad \varphi \in C^\infty(\bar{\Gamma})$$

$$(\varphi \geq 0)$$

**Simons:** On cones

$$f = |A|\psi(|x|), \quad \psi \in C_0^\infty(\mathbf{R}_+)$$

**Energy functional analogue:**

$$\varphi = |D^2u|^\alpha \psi(|x|), \quad \psi \in C_0^\infty(\mathbf{R}_+)$$

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**Energy functional analogue:**

$$\varphi = |D^2 u|^\alpha \psi(|x|), \quad \psi \in C_0^\infty(\mathbf{R}_+)$$

Goal: $\varphi \geq 0$, $\Delta\varphi \geq 0$ on $\Gamma$; $\varphi_\nu \geq H\varphi$ on $\partial\Gamma$

## Plan

- 1st and 2nd variations
- Stability and subsolutions
- Free boundary regularity for $n = 3, 4$
- Higher critical points, topology
- Higher dimensions; $n \to \infty$ ?

**WARM-UP:** Optimal $\alpha > 0$ for which

$$\Delta u = 0 \implies \Delta |D^k u|^\alpha \geq 0 \quad \text{(Cald.-Zyg)}$$

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$$w = |Du| \implies \Delta w^\alpha \geq 0, \quad \alpha = 1 - \frac{1}{n-1}$$

$$(\Delta(\log w) \geq 0, \ n = 2.)$$

$$w^2 = \sum_j u_j^2 \implies 2ww_k = 2\sum_j u_j u_{jk}$$

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$$w\Delta w + |\nabla w|^2 = |D^2 u|^2$$

$$D^2 u = \text{diag}[\lambda_1, \ldots, \lambda_n]$$

$$w_k = \lambda_k u_k / w \implies |\nabla w|^2 \leq \max_k \lambda_k^2$$

Since $\Delta u = 0$,

$$\lambda_n^2 = (\lambda_1 + \cdots + \lambda_{n-1})^2 \leq (n-1)(\lambda_1^2 + \cdots + \lambda_{n-1}^2)$$

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$$\begin{aligned} \lambda_n^2 \; &= \frac{1}{n}\lambda_n^2 + \frac{n-1}{n}\lambda_n^2 \; \le \; \frac{n-1}{n}(\lambda_1^2 + \cdots + \lambda_n^2) \\ &= \frac{n-1}{n}|D^2u|^2 \end{aligned}$$

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$$|\nabla w|^2 \le \max_k \lambda_k^2 \le \frac{n-1}{n}|D^2u|^2$$

$$w\Delta w + |\nabla w|^2 = |D^2 u|^2 \geq \frac{n}{n-1}|\nabla w|^2$$

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$$w\Delta w \;\geq\; \frac{1}{n-1}|\nabla w|^2$$

$$\Delta w^{\alpha} \geq 0, \quad \alpha = 1 - \frac{1}{n-1}.$$

# ELLIPTIC VARIATIONAL PROBLEMS

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**MSRI Lecture 2: Free Boundary Regularity**

# Regularity of free boundary energy minimizers

$n = 2$, Alt, Caffarelli, Friedman (ACF) 1984
$n = 3$, Caffarelli, Jerison, Kenig (CJK) 2002
$n = 4$, Jerison, Savin 2015

**Theorem** If $n \leq 4$ and $u$ minimizes

$$J(v) := \int_B [|\nabla v|^2 + 1_{v>0}]\, dx, \quad B \subset \mathbf{R}^n,$$

among all $v = u$ on $\partial B$, then

$$B \cap \partial\{u > 0\} \quad \text{is smooth;}$$

$$\Delta u = 0 \quad \text{in } \{u > 0\} \text{ and } \{u \leq 0\}^\circ;$$

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 1 \quad \text{on } B \cap \partial\{u > 0\}.$$

# Proof Outline

- Blow up limits exist (ACF monotonicity)
- Limits are cones (Weiss monotonicity)
- Two-phase limits are planar (ACF)
- Characterize one-phase limits
- Flat implies smooth (Caffarelli)

# Proof Outline

- Blow up limits exist (ACF monotonicity)
- Limits are cones (Weiss monotonicity)
- Two-phase limits are planar (ACF)
- One-phase limits are planar!
- Flat implies smooth (Caffarelli)

# One-phase Limit: Stable Cone $\Gamma$

$\Delta u = 0$, $u > 0$, $u(rx) = ru(x)$ in $\Gamma$.
$u = 0$ and $|\nabla u|^2 = 1$ on $\partial\Gamma$.

$$\int_{\partial\Gamma} H\varphi^2 d\sigma \leq \int_{\Gamma} |\nabla\varphi|^2 dx$$

By induction, $S^{n-1} \cap \partial\Gamma$ is smooth. Moreover, we will only need test functions $\varphi \geq 0$,

$$\varphi \in C_0^\infty(U \cap \bar{\Gamma}),$$

$$U = \{0 < a < |x| < b\}.$$

# Mean Curvature $H$

Rotate and translate so locally

$$\Gamma = \{x_n > g(x')\}, \quad \nabla g(0) = 0.$$

Then $u(x', g(x')) = 0$ implies for $i, j < n$,

$$u_j = 0, \quad u_{ij} = -u_n g_{ij} = -g_{ij} \quad \text{at } 0.$$

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$$\Delta u = 0 \implies u_{nn} = \sum_{j<n} g_{jj} =: -H.$$

$\Delta|\nabla u|^2 \geq 0$ and $|\nabla u|^2$ bounded in $\Gamma$ imply

$$|\nabla u|^2 \leq 1 \quad \text{(Maximum principle)}$$

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$$|\nabla u|^2 \leq 1 \quad \text{(Maximum principle)}$$

$$\partial_n |\nabla u|^2 < 0 \quad \text{(Hopf lemma)}$$

$$\partial_n |\nabla u|^2 = \sum_j 2u_j u_{nj} = 2u_{nn} = -2H.$$

**Hence** $-u_{\nu\nu} = H > 0$.

$$u_\varepsilon = (u - \varepsilon\varphi)^+, \ S_\varepsilon = \{0 < u < \varepsilon\varphi\}.$$

$$J(u_\varepsilon) = J(u) - 2\varepsilon \int_\Gamma \nabla u \cdot \nabla \varphi + \varepsilon^2 \int_\Gamma |\nabla \varphi|^2$$
$$- \int_{S_\varepsilon} [|\nabla u|^2 + 1] + 2\varepsilon \int_{S_\varepsilon} \nabla u \cdot \nabla \varphi + O(\varepsilon^3).$$

$$u_\varepsilon = (u - \varepsilon\varphi)^+, \ S_\varepsilon = \{0 < u < \varepsilon\varphi\}.$$

$$J(u_\varepsilon) = J(u) + 2\varepsilon \int_{\partial\Gamma} \varphi \, d\sigma + \varepsilon^2 \int_\Gamma |\nabla\varphi|^2$$
$$- \int_{S_\varepsilon} [|\nabla u|^2 + 1] + 2\varepsilon \int_{S_\varepsilon} \nabla u \cdot \nabla\varphi + O(\varepsilon^3).$$

$$u_\varepsilon = (u - \varepsilon\varphi)^+, \; S_\varepsilon = \{0 < u < \varepsilon\varphi\}.$$

$$J(u_\varepsilon) = J(u) + 2\varepsilon \int_{\partial\Gamma} \varphi \, d\sigma + \varepsilon^2 \int_\Gamma |\nabla\varphi|^2$$

$$- \int_{S_\varepsilon} [|\nabla u|^2 + 1] + 2\varepsilon^2 \int_{\partial\Gamma} \varphi\varphi_\nu \, d\sigma + O(\varepsilon^3).$$

$x_t = x_0 - t\nu$ ($\nu$ inner unit normal)

$$dx = (1 + tH + O(t^2))dtd\sigma,$$
$$|\nabla u(x_t)|^2 = 1 - 2tH + O(t^2).$$

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$$S_\varepsilon : 0 < u < \varepsilon\varphi \iff 0 < t < \bar{t},$$
$$\bar{t} = \varepsilon\varphi + \varepsilon^2 \left( \frac{H}{2}\varphi^2 - \varphi\varphi_\nu \right) + O(\varepsilon^3).$$

# SUMMARY

If $u$ is a critical point and $u_\varepsilon = (u - \varepsilon\varphi)^+$,

$$J(u_\varepsilon) - J(u) = \varepsilon^2 \int_\Gamma |\nabla\varphi|^2 - \varepsilon^2 \int_{\partial\Gamma} H\varphi^2 d\sigma + O(\varepsilon^3)$$

**SUBSOLUTION:** $U = \{a < |x| < b\}$.
$\varphi \in C_0^\infty(U)$, $\varphi \geq 0$.

$$\Delta\varphi \geq 0 \text{ in } U \cap \Gamma, \quad \varphi_\nu \geq -H\varphi \text{ on } \partial\Gamma.$$

**SUBSOLUTION:** $U = \{a < |x| < b\}$.
$\varphi \in C_0^\infty(U)$, $\varphi \geq 0$.

$$\Delta\varphi \geq 0 \text{ in } U \cap \Gamma, \quad \varphi_\nu \geq -H\varphi \text{ on } \partial\Gamma.$$

$$\int_{U\cap\Gamma} |\nabla\varphi|^2 = -\int_{U\cap\Gamma} \varphi\Delta\varphi - \int_{\partial\Gamma} \varphi\varphi_\nu \, d\sigma$$
$$\leq \int_{\partial\Gamma} H\varphi^2 \, d\sigma.$$

For instability, we need strictness somewhere.

$$w^2 := |D^2 u|^2 = \sum_{k\ell} u_{k\ell}^2 .$$

$$\varphi = w^\alpha \psi(|x|), \quad 0 \le \psi \in C_0^\infty(\mathbf{R}_+)$$

If $n = 3$,

$$(w^\alpha)_\nu = -2\alpha H w^\alpha \implies \varphi_\nu = -2\alpha H \varphi$$

Hence $\varphi_\nu \ge -H\varphi$ for $\alpha \le 1/2$.

Rotate so $D^2 g(x_0)$ is diagonal.
Differentiating $|\nabla u(x', g(x'))|^2 = 1$ gives

$$u_{in} = 0, \quad u_{ijn} = 0 \quad i, j < n, \, i \neq j,$$

$$u_{iin} = u_{nn} u_{ii} - u_{ii}^2, \quad u_{nnn} = \sum u_{kk}^2 .$$

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$$\partial_n \sum u_{k\ell}^2 = 2 \sum u_{k\ell} u_{k\ell n} = 2 \sum u_{kkn} u_{kk} .$$

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Differentiating $|\nabla u(x', g(x'))|^2 = 1$ gives

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$$u_{iin} = \lambda_n \lambda_i - \lambda_i^2, \quad u_{nnn} = \sum \lambda_k^2 .$$

$$\partial_n (w^2) = -4Hw^2 - 2 \sum \lambda_k^3 \;\; (= -4Hw^2, \; n = 3) .$$

$$\varphi = w^{\alpha}\psi(|x|), \quad 0 \le \psi \in C_0^{\infty}(\mathbf{R}_+)$$

If $n = 3$, there is $\psi$ such that $\Delta\varphi \ge 0$ iff

$$\Delta(w^{\alpha}) \ge (\alpha - 1/2)^2 w^{\alpha}/|x|^2$$

$$\varphi = w^{\alpha}\psi(|x|), \quad 0 \le \psi \in C_0^{\infty}(\mathbf{R}_+)$$

If $n = 3$, there is $\psi$ such that $\Delta\varphi \ge 0$ iff

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$$\Delta(w^{\alpha}) \ge (\alpha + 1)\alpha\, w^{\alpha}/|x|^2, \quad \alpha > 0.$$

Hence $\varphi$ is a subsolution iff $1/8 \le \alpha \le 1/2$.

Compare our inequality for $w = |D^2 u|$,

$$\Delta(w^\alpha) \geq \alpha(\alpha+1)\frac{w^\alpha}{|x|^2},$$

with Simons's inequality for $w = |A|$:

$$(\Delta + |A|^2)w \geq 2\frac{w}{|x|^2}.$$

$\Delta$, Laplace-Beltrami; $A$, 2nd fund. form

**Lemma** If $u$ is harmonic, $w = |D^2 u|$, then

$$w \Delta w \geq \frac{2}{n} |\nabla w|^2 .$$

If, in addition, $u(rx) = ru(x)$, then

$$w \Delta w \geq \frac{2}{n-1} |\nabla w|^2 + 2 \frac{n-2}{n-1} \frac{w^2}{|x|^2} .$$

# Bochner identities

$$w^2 := \sum u_{k\ell}^2, \quad \Delta u = 0.$$

$$w\Delta w + |\nabla w|^2 = \Delta(w^2) = 2\sum u_{k\ell m}^2.$$

# Bochner identities

$$w^2 := \sum u_{k\ell}^2, \quad \Delta u = 0.$$

$$w\Delta w + |\nabla w|^2 = \Delta(w^2) = 2\sum u_{k\ell m}^2 .$$

$$w_m = \sum_{k\ell} u_{k\ell} u_{k\ell m}/w = \sum_{k} \lambda_k u_{kkm}/w .$$

For $n = 4$, $\alpha \geq 1/3$ works:

$$\Delta w^{1/3} \geq \frac{4}{9} \frac{w^{1/3}}{|x|^2}$$

yields $\Delta\varphi \geq 0$ for appropriate $\psi(|x|)$.

For $n = 4$, $\alpha \geq 1/3$ works:

$$\Delta w^{1/3} \geq \frac{4}{9}\frac{w^{1/3}}{|x|^2}$$

yields $\Delta\varphi \geq 0$ for appropriate $\psi(|x|)$.
Unfortunately,

$$\varphi_\nu \geq -H\varphi \quad \text{fails!}$$

It's even worse for $n \geq 5$: no multiple of $H\varphi$ is a lower bound for $\varphi_\nu$.

**n=4:** $\phi = w^{1/3}\psi(|x|)$, **new** $w$

$$w = \left( \sum_{\lambda_j > 0} \lambda_j^2 + 4 \sum_{\lambda_k < 0} \lambda_k^2 \right)^{1/2}$$

**Lemma.** Suppose $\Delta u = 0$ and $u(rx) = ru(x)$. Let

$$w = S(\lambda_1, \dots, \lambda_n)$$

with $S$ symmetric, convex and homogeneous degree 1, and $\lambda_j$ the eigenvalues of $D^2 u$. Then

$$w\Delta w + |\nabla w|^2 \geq \frac{2}{n-1}|\nabla w|^2 + \frac{2(n-2)}{n-1}\frac{w^2}{|x|^2}$$

Eigenvalues of

$$\begin{pmatrix} \lambda_1 & \varepsilon \\ \varepsilon & \lambda_2 \end{pmatrix} \qquad \lambda_1 \neq \lambda_2$$

$$\lambda_1 + \frac{\varepsilon^2}{\lambda_1 - \lambda_2} + O(\varepsilon^4); \quad \lambda_2 + \frac{\varepsilon^2}{\lambda_2 - \lambda_1} + O(\varepsilon^4).$$

Eigenvalues of

$$\begin{pmatrix} \lambda_1 & \varepsilon \\ \varepsilon & \lambda_2 \end{pmatrix} \qquad \lambda_1 \neq \lambda_2$$

$$\lambda_1 + \frac{\varepsilon^2}{\lambda_1 - \lambda_2} + O(\varepsilon^4); \quad \lambda_2 + \frac{\varepsilon^2}{\lambda_2 - \lambda_1} + O(\varepsilon^4).$$

$$(\text{and} \quad \lambda \pm \varepsilon, \quad \text{if} \ \lambda_1 = \lambda_2 = \lambda)$$

# Open Questions.

Suppose that $u : \Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^n$, and

$$\Delta u = f(u).$$

**What do level sets of $u$ look like?**

In Lecture 3, we will discuss $n = 2$, $n = 3$, and $n \to \infty$ by analogy with minimal surfaces.

# Higher critical points, Isoperimetric sets

David Jerison (MIT)

MSRI, Lecture 3, Jan 2017

# What else do we learn from minimal surface theory?

Theorems of Colding-Minicozzi

## C-M Removable Singularities Theorem

$\forall \delta > 0$, $\exists C$, such that every minimal annulus

$$M \subset B_1 \backslash B_\varepsilon; \quad \partial M = \text{two loops in } \ \partial B_\varepsilon \cup \partial B_1,$$

satisfies

$$M \text{ is a } \delta\text{-Lipschitz graph in } \ B_{1/C} \backslash B_{C\varepsilon}$$

(Proof inserts stable surfaces)

**Thm 1. (J-, Kamburov)** Let $A^+ := \{u > 0\}$ be simply-connected in

$$\text{annulus} \quad A = \{x \in \mathbf{R}^2 : \varepsilon < |x| < 1\},$$

and assume that the two strands of $F$ connecting $\partial D_1$ to $\partial D_\varepsilon$ don't get close to each other.

Then $\forall \delta > 0$, $\exists C$ such that

$$F \quad \text{is a } \delta\text{-Lipschitz graph on } C\varepsilon < |x| < 1/C$$

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Significance: rules out spirals.

In particular,

**Flat implies Lipschitz**

is valid for 2-dimensional, simply-connected phases with a small hole.

**C-M** Dichotomy for embedded minimal annuli.

a) curvature is bounded, or

b) near points of high curvature,

they resemble catenoids:

**C-M** Dichotomy for embedded minimal annuli.

a) curvature is bounded, or

b) near points of high curvature,

they resemble catenoids:

$$\text{If} \quad M \subset B_1 \subset \mathbf{R}^3 \quad (\partial M = \text{two loops in } \partial B_1)$$

has neck size $\varepsilon > 0$ near the origin, then

$$M \approx \text{standard catenoid in } |x| < \sqrt{\varepsilon}.$$

**Thm 2. J-, Kamburov** There is $c > 0$ such that if $0 \in \partial D^+$ and $D^+$ is simply-connected, then either

$B_c(0) \cap \partial D^+$ has one strand of bounded curvature

or it resembles a piece of a double hairpin HHP

## Hauswirth, Hélein, Pacard 2012

$$S = \{\zeta = \xi + i\eta : |\eta| \le \pi/2\}$$

$$\Omega := \varphi(S), \qquad \varphi(\zeta) = i(\zeta + \sinh \zeta).$$

$$H(z) = \operatorname{Re} \cosh(\varphi^{-1}(z)) = (\cosh \xi)(\cos \eta)$$

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$$H(z) = \mathrm{Re}\cosh(\varphi^{-1}(z)) = (\cosh\xi)(\cos\eta)$$

$$H_a(z) = aH(z/a); \qquad \Omega_a = a\Omega$$

**Thm 3 (J-, Kamburov, Rigidity).** $\forall \delta > 0$, $\exists c > 0$ such that if "neck size" is $\varepsilon > 0$ near 0, then

$$\text{there is} \quad a \approx \varepsilon, \quad \psi : \Omega_a \cap B_c(0) \to D^+$$

**Near isometry:** If $|z| < c$, then

$$|\psi'(z) - 1| \leq \delta, \; z \in \Omega_a; \quad |\psi'(z)| = 1, \; z \in \partial\Omega_a$$

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**Curvature bounds:** If $|z| < c$, then

$$|\psi''(z)| \leq \delta, \ z \in \Omega_a; \quad |\kappa(\psi(z)) - \kappa_a(z)| \leq \delta; \ z \in \partial\Omega_a$$

**Colding-Minicozzi:** Embedded minimal annulus

$$M \subset B_1 \subset \mathbf{R}^3 \quad \text{with neck size } \varepsilon > 0,$$

satisfies $M \approx$ standard catenoid, $|x| < \sqrt{\varepsilon}$

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Our rigidity extends the range to unit scale

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$$\sqrt{\varepsilon} \le |x| < 1/C.$$

Traizet correspondence: free boundary solutions $\longleftrightarrow$ minimal surfaces with reflection symmetry

The two theorems overlap in an easy, but illustrative special case.

## TRAIZET CORRESPONDENCE

$$dX_1 + i dX_2 = \frac{1}{2} d\bar{z} - 2\left(\frac{\partial u}{\partial z}\right)^2 dz$$

$$\boxed{z \mapsto (X_1, X_2, \pm u(z))}$$

The image is an immersed minimal surface with symmetry $x_3 \leftrightarrow -x_3$. Moreover,

$$|\nabla u| < 1 \iff \text{embedded}$$

## Open Questions.

Suppose that $u : \mathbf{R}^3 \to \mathbf{R}$, $f \in C_0^\infty(\mathbf{R})$,

$$\Delta u = f(u).$$

**What do the level sets of $u$ look like?**

## Open Questions.

Suppose that $u : \mathbf{R}^3 \to \mathbf{R}$, $f \in C_0^\infty(\mathbf{R})$,

$$\Delta u = f(u).$$

**What do the level sets of $u$ look like?**

**Conjecture 1.** If $\{u > 0\}$ and $\{u \leq 0\}$ are contractible, then $u = w(a \cdot x)$.

**Conjecture 2.** (De Giorgi/Calabi-Yau type): Suppose $\Delta u = f(u)$ in $\mathbf{R}^3$, $\nabla u \neq 0$, and/or $u$ has finite topology level sets. If one level set is contained in a half space,

$$\{u = 0\} \subset \{x_3 > 0\},$$

then

$$u(x) = w(x_3)$$

# What estimates work for all $n$?

## What estimates work for all $n$?

## Bombieri-Giusti/Almgren, De Giorgi Quantitative connectivity:

$$\left( \int_{B(cr)} |f - \bar{f}|^p \, d\sigma \right)^{1/p} \le C \int_{B(r)} |\nabla f| \, d\sigma$$

**Bombieri-Giusti** 1972
**Scale-invariant Harnack:**

$$\sup_{B(r)} u \leq C \inf_{B(r)} u$$

for positive solutions to Laplace-Beltrami

$$\Delta u = 0.$$

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**Scale-invariant Harnack:**

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for positive solutions to Laplace-Beltrami

$$\Delta u = 0.$$

**Corollary.** A global area-minimizing surface in a half-space is a hyperplane. (Miranda)

**Conjecture 3.** Isoperimetric subsets of symmetric convex bodies are contractible: bounded by smooth graphs in all dimensions.

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Neumann boundary condition implies no Simons cone!

**Sternberg-Zumbrun** 1990-92

**Connectivity in isoperimetric case**

$$\left(\int_{B(r)} |f|^2 \, d\sigma\right)^{1/p} \leq C(r) \int_{B(r)} |\nabla f|^2 \, d\sigma,$$

$f \in C_0^\infty(B(r))$, using stability.

**G. David, DJ** (work in progress)

**Another version of connectivity:**

Intrinsic distance $\approx$ extrinsic distance

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**Another version of connectivity:**
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The embedding is proper = key step in CM proof of Calabi-Yau conjecture.

# Why does connectivity help?

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**Bombieri, De Giorgi, Miranda** 1969:
The $L^\infty$ bound on a minimal graph implies a Lipschitz bound (hence a $C^\infty$ bound).

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**Bombieri, De Giorgi, Miranda** 1969:
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2006 proof using Harnack by De Silva, J-.
Also valid in free boundary setting

**Conjecture 4.** There is $c < 1$ such that if $\Omega$ is convex, symmetric and $E \subset \Omega$ has least perimeter for $|E| = |\Omega|/2$, then

$$\Omega \cap \partial E \subset \{-a < x_n < a\},$$

$$|\Omega \cap \{-a < x_n < a\}| \leq c|\Omega|.$$

**C. Borell** 1975. Isoperimetric subsets of gauss space are half spaces for all $n$.

**S. G. Bobkov** 1999. Isoperimetric subsets for log-concave densities on the real line are half lines.

**Hot spots conjecture of J. Rauch.**

The hottest spot of an insulated region tends to the boundary as $t \to \infty$.

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The hottest spot of an insulated region tends to the boundary as $t \to \infty$.

My favorite version: The first nontrivial Neumann eigenfunction of a symmetric convex domain is monotone in some direction.

**KLS Conjecture** A least perimeter bisector of a convex set has area comparable to the best bisecting hypersurface.

December 2016 progress by Lee and Vempala

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Our Conjecture 3 is that the extremal interface is a Lipschitz graph, that is, it resembles a hyperplane.