Recent developments in geometric multilinear operators – an overview

Anthony Carbery

University of Edinburgh &
Maxwell Institute for Mathematical Sciences

MSRI Harmonic Analysis Programme – Final Workshop
15th May 2017

Outline

1. Context
2. Multilinear duality
3. Multilinear Kakeya for k_j-planes and perturbed Brascamp–Lieb inequalities – work of Ruixiang Zhang
4. Discrete Analogues

Area

We'll be concerned with multilinear inequalities with a geometrical flavour such as

- Loomis–Whitney inequalities and their relatives
- Brascamp–Lieb inequalites and their relatives
- Multilinear Kakeya inequalities

We will focus on the sharp inequalities (no "ϵ-loss").

We'll work in the setting where curvature considerations do not come into play.

We'll not discuss any of the wide variety of applications...

Starting point: Guth's multilinear Kakeya theorem

A couple of the key ingredients:

- Guth's **universal** visibility theorem – the polynomial method
- Some elementary functional analysis

These, together with some other ingredients, give:

Theorem (Guth's endpoint multilinear Kakeya theorem)

Suppose that $\mathcal{T}_j$, $1 \leq j \leq d$ are finite transversal families of 1-tubes in $\mathbb{R}^d$. Then for a certain $C_d > 1$ we have

$$\int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1}(x) \cdots \sum_{T_d \in \mathcal{T}_d} \chi_{T_d}(x) \right)^{\frac{1}{d-1}} \mathrm{d}x$$

$$\leq C_d \prod_{j=1}^{d} (\#\mathcal{T}_j)^{\frac{1}{d-1}}$$

Affine-invariant formulation

Theorem (Bourgain–Guth variant)

Suppose that $\mathcal{T}_j$, $1 \leq j \leq d$ are finite families of 1-tubes in $\mathbb{R}^d$. Then for a certain $C_d > 1$ we have

$$\int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1}(x) \cdots \sum_{T_d \in \mathcal{T}_d} \chi_{T_d}(x) e(T_1) \wedge \cdots \wedge e(T_d) \right)^{\frac{1}{d-1}} \mathrm{d}x$$

$$\leq C_d \prod_{j=1}^{d} (\#\mathcal{T}_j)^{\frac{1}{d-1}}$$

Note that there is no longer an explicit transversality hypothesis; $e(T)$ is the unit vector in the direction of the axis of T.

The universal visibilty theorem

For a hypersurface Z in $\mathbb{R}^d$ we associate to it a fundamental convex body defined by

$$K(Z) := \{u \in \mathbb{B}^d : \int_Z |u \cdot n(x)| \mathrm{d}S(x) \leq 1\}.$$

Theorem (Guth)

For any nonnegative function M defined on the lattice of unit cubes Q in $\mathbb{R}^d$ there exists a polynomial p with zero set Z_p such that

$$\deg p \leq C_d \left(\sum_Q M(Q)^d \right)^{1/d}$$

and such that for each unit cube Q,

$$\mathrm{Vol}(K(Z_p \cap Q))^{-1/d} \geq CM(Q).$$

Comments

- (Strictly one needs a technical ϵ-mollification)
- Uses deep results from algebraic topology. (Simpler proof using only Borsuk–Ulam theorem (AC – S. Valdimarsson))
- Numerology: In the multilinear Kakeya theorem we have $d \times 1/(d-1) = d/(d-1)$ whose dual exponent corresponds to the L^d of the visibility theorem
- A tool to facilitate the introduction the polynomial method to and thence solve endpoint multilinear Kakeya...
- But perhaps it is better viewed as a **fundamental geometric property** of euclidean space?

NEXT: A high-level overview of the proof of the endpoint multilinear Kakeya theorem...more detail later

Elementary functional analysis

Suppose we are interested in inequalities of the form

$$\|\prod_{j=1}^{d}(S_j f_j)^{\alpha_j}\|_p \leq A\prod_{j=1}^{d}\left(\int f_j\right)^{\alpha_j}$$

where $\sum_{j=1}^{d}\alpha_j = 1$, S_j are positive linear operators and $f_j \geq 0$.

Suppose that for all nonnegative $G \in L^{p'}$ we have that there exist $G_1, \dots, G_d$ such that

$$G(x) \leq G_1(x)^{\alpha_1} \dots G_d(x)^{\alpha_d}$$

and such that for all j

$$\|S_j^* G_j\|_\infty \leq A\|G\|_{p'}$$

which (if S_j is not bounded) can be interpreted as

$$\int_X S_j f(x) G_j(x)\mathrm{d}x \leq A\|G\|_{p'}\|f\|_1.$$

Then our desired inequality is true. PREPOSTEROUS!!

Proof

Take $f_j \in L^1$ for $j = 1, \ldots, d$ and $G \in L^{p'}$ with $\|G\|_{p'} \le 1$. Then

$$\int_X G(x) \prod_{j=1}^{d} (S_j f_j)^{\alpha_j} \mathrm{d}x \le \int \prod_{j=1}^{d} g_j(x)^{\alpha_j} \prod_{j=1}^{d} S_j f_j(x)^{\alpha_j} \mathrm{d}x$$

$$= \int \prod_{j=1}^{d} (g_j(x) S_j f_j(x))^{\alpha_j} \mathrm{d}x \le \prod_{j=1}^{d} \left(\int g_j(x) S_j f_j(x) \mathrm{d}x \right)^{\alpha_j}$$

$$\le \prod_{j=1}^{d} \left(A \|G\|_{p'} \|f_j\|_1 \right)^{\alpha_j} \le A \prod_{j=1}^{d} \|f_j\|_1^{\alpha_j}.$$

Now take the supremum over all G. (Note that we actually need only

$$\prod_{j=1}^{d} \|S_j^* G_j\|_\infty^{\alpha_j} \le A \|G\|_{p'}$$

rather than an estimate on each multiplicand separately.)

Gathering together

We can recast the endpoint multilinear Kakeya inequality in the form

$$\|\prod_{j=1}^{d}(S_j f_j)^{\alpha_j}\|_p \leq A\prod_{j=1}^{d}\left(\int f_j\right)^{\alpha_j}$$

with each $\alpha_j = 1/d$, $p = d/(d-1)$ and the S_j are constant at scale 1.

Then the visibilty theorem – given $M = G$, existence of a p such that

$$\deg p \leq C_d \left(\sum_Q M(Q)^d\right)^{1/d}$$

and such that for each unit cube Q,

$$\mathrm{Vol}(K(Z_p \cap Q))^{-1/d} \geq CM(Q)$$

– is used to construct the desired $M_j = G_j$ to feed the functional analysis.

An equivalent dual form of Young's inequality

Young's inequality in the form

$$\|\prod_{j=1}^{3} f_j(x \cdot v_j)^{\alpha_j}\|_{L^2(\mathbb{R}^2)} \leq B \prod_{j=1}^{3} \left(\int f_j\right)^{\alpha_j}$$

for $0 \leq \alpha_j \leq 1/2$, $\sum \alpha_j = 1$ is equivalent to the following statement:

For all $M \in L^2(\mathbb{R}^2)$ there exist M_1, M_2, M_3 such that

$$M(x) \leq M_1(x)^{\alpha_1} M_2(x)^{\alpha_2} M_3(x)^{\alpha_3}$$

and, for $1 \leq j \leq 3$,

$$\sup_{s \in \mathbb{R}} \int_{\mathbb{R}} M_j(sv_j + tv_j^{\perp}) \mathrm{d}t \leq B\|M\|_2.$$

The "B" in both instances is the same.

Exercise. Find a factorisation of $M \in L^2(\mathbb{R}^2)$ of this form which yields the sharp Beckner constant $B(\{v_j\}, \{\alpha_j\})$ for the problem.

Duality theorem

Theorem (AC, S. Valdimarsson)

Suppose $\sum_{j=1}^{d} \alpha_j = 1$ *and* S_j *are positive linear operators. Then*

$$\|\prod_{j=1}^{d}(S_j f_j)^{\alpha_j}\|_p \leq A \prod_{j=1}^{d}\left(\int f_j\right)^{\alpha_j}$$

holds for all $f_j \geq 0$ *if* and only if *for all nonnegative* $G \in L^{p'}$ *we have that there exist* $G_1, \ldots, G_d$ *such that the sub-factorisation*

$$G(x) \leq G_1(x)^{\alpha_1} \ldots G_d(x)^{\alpha_d}$$

holds, and such that for $1 \leq j \leq d$*,*

$$\int_X S_j f(x) G_j(x) \mathrm{d}x \leq A\|G\|_{p'}\|f\|_1.$$

Remarks

- When $d = 1$ this is ordinary linear duality
- Techniques: solving a convex optimisation problem – $(L^\infty)^*$ and finitely additive measure theory of Hewitt and Yosida – non-constructive
- Despite the audacity of Guth's approach to multilinear Kakeya, there is an inevitability to it!
- More important than the result itself are the perspectives and insights it brings to bear in the situations in which it applies...
- What does it tell us about Loomis–Whitney, Young's inequality and Brascamp–Lieb? Can we analyse these inequalities from this perspective?

Combinatorial Loomis–Whitney inequalities

What does multilinear duality and factorisation tell us about the class of Loomis–Whitney inequalities?

The perspective is clearest when we consider the class of “Combinatorial” Loomis–Whitney inequalities.

It provides a unified approach to various sub-classes of Loomis–Whitney type inequalities – classical, affine invariant, projections on subspaces of various dimensions (Calderón, Finner), “nonlinear” Loomis–Whitney inequalites of Bennett–C– Wright, Bennett & Bez, Bejenaru–Herr–Tataru, Koch & Steinerberger....

It also gives sharp constants (when these make sense) – under minimal regularity hypotheses in the case of “nonlinear” Loomis–Whitney.

A Combinatorial Loomis–Whitney inequality

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that for $1 \leq j \leq d$ we have a family $\mathcal{R}_j$ consisting of disjoint subsets $R_j \in \mathcal{M}$. Catalogue the nonempty $R_1 \cap \cdots \cap R_d$ as $Q \in \mathcal{Q}$.

Theorem

Suppose for that every $Q, Q' \in \mathcal{Q}$ there are at most K chains $Q = Q_0, Q_1, \ldots, Q_d = Q'$ such that $R_i(Q_{i-1}) = R_i(Q_i)$ for $1 \leq i \leq d$. Then

$$\int_X \left(\sum_{R_1 \in \mathcal{R}_1} \cdots \sum_{R_d \in \mathcal{R}_d} c_{R_1} \chi_{R_1} \ldots c_{R_d} \chi_{R_d} \mu(R_1 \cap \cdots \cap R_d)^{-(d-1)} \right)^{1/(d-1)} \mathrm{d}\mu$$

$$\leq K^{1/(d-1)} \left(\sum_{R_1 \in \mathcal{R}_1} c_{R_1} \right)^{1/(d-1)} \ldots \left(\sum_{R_d \in \mathcal{R}_d} c_{R_d} \right)^{1/(d-1)} .$$

Proof

All we need to do is to write down a suitable factorisation for an arbitrary M defined on $\mathcal{Q}$.

For $Q, Q' \in \mathcal{Q}$ and $1 \leq j \leq d$, let $\lambda_j(Q', Q)$ be the number j-connections of Q' with Q, i.e. the number of chains $Q = Q_0, Q_1, \ldots, Q_j = Q'$ such that $R_i(Q_{i-1}) = R_i(Q_i)$ for $1 \leq i \leq j$. Let $\lambda_0(Q', Q) = 1$ if $Q' = Q$ and $\lambda_0(Q', Q) = 0$ otherwise.

Then we have a telescoping product

$$M(Q)^d = \prod_{j=1}^{d} \frac{\sum_{q\in\mathcal{Q}} \lambda_{j-1}(q, Q)M(q)^d}{\sum_{q\in\mathcal{Q}} \lambda_j(q, Q)M(q)^d} \times \sum_{q\in\mathcal{Q}} \lambda_d(q, Q)M(q)^d.$$

For $1 \leq j \leq d$ the j'th multiplicand S_j satisfies $\sum_{Q\,:\,R_j(Q)=R_j} S_j(Q) = 1$ by construction, and the last term is at most $K \sum_Q M(Q)^d$ by hypothesis.

Factorisation and Brascamp–Lieb

It seems challenging to find factorisations yielding the sharp constants in Brascamp–Lieb inequalities (given by evaluation on gaussians).

However for the problem of **boundedness** of the multilinear form, we can give an algorithm for the factorisation.

Suppose that $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ are linear surjections for $1 \le j \le d$. Brascamp–Lieb inequalities take the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^{d} f_j(B_j x)^{p_j} \mathrm{d}x \le \mathbf{BL}(B_j, p_j) \left(\int f_1\right)^{p_1} \cdots \left(\int f_d\right)^{p_d}.$$

Let $\mathcal{V}$ be the lattice of vector subspaces of $\mathbb{R}^n$ generated by the subspaces $\ker B_j$. Then (BCCT) $\mathbf{BL}(B_j, p_j)$ is finite if and only if $\sum_{j=1}^{d} p_j n_j = n$, and for all $V \in \mathcal{V}$ we have

$$\dim V \le \sum_{j=1}^{d} p_j \dim B_j V.$$

Factorisation and Brascamp–Lieb, cont'd

The proof of this result is functorial: for given $\{B_j\}$ let the Brascamp–Lieb polyhedron be those $(p_1, \ldots, p_d)$ satisfying the scaling condition $\sum_{j=1}^{d} p_j n_j = n$ and all the (necessary) conditions

$$\dim V \leq \sum_{j=1}^{d} p_j \dim B_j V.$$

Then *either* $(p_1, \ldots, p_d)$ is in the "interior" of the polytope – in which case interpolation techniques apply – *or* it is an extreme point, in which case there is some proper **critical subspace** $V \in \mathcal{V}$ such that $\dim V = \sum_{j=1}^{d} p_j \dim B_j V$. In this case the problem decomposes ("factorises") into two B–L subproblems over V and the quotient space $\mathbb{R}^n / V$ respectively. When this process stops we invoke a trivial identity.

Factorisation works perfectly with respect to interpolation (simply take log-convex combinations of "endpoint" factorisations) and with respect to decomposition over critical subspaces. The trivial identity corresponds to the trivial factorisation.

Multilinear Kakeya again

Recall Guth's endpoint multilinear Kakeya theorem:

Theorem (Bourgain–Guth variant)

Suppose that $\mathcal{T}_j$, $1 \le j \le d$ are finite families of 1-tubes in $\mathbb{R}^d$. Then for a certain $C_d > 1$ we have

$$\int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1}(x) \cdots \sum_{T_d \in \mathcal{T}_d} \chi_{T_d}(x) e(T_1) \wedge \cdots \wedge e(T_d) \right)^{\frac{1}{d-1}} \mathrm{d}x$$

$$\le C_d \prod_{j=1}^{d} (\#\mathcal{T}_j)^{\frac{1}{d-1}}$$

What happens if we replace d families of 1-tubes by d families of 1-neighbourhoods of k_j-planes where $\sum_{j=1}^{d} k_j = n$? (We will be working on $\mathbb{R}^n$ and d will be the degree of multilinearity.)

Multilinear duality paradigm in Guth's result

Following the multilinear duality paradigm, Guth's theorem follows immediately from the following factorisation property:

For all nonnegative functions M defined on the lattice of unit cubes $\mathcal{Q}$ in $\mathbb{R}^d$, for all finite families of 1-tubes $\mathcal{T}_j$, there exist $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}_+$ such that for each cube Q and each $T_j \in \mathcal{T}_j$ with $T_j \cap Q \neq \emptyset$ we have

$$M(Q) \leq C_d \left(S_1(Q, T_1) \ldots S_d(Q, T_d)\right)^{1/d} |e(T_1) \wedge \cdots \wedge e(T_d)|^{-1/d},$$

and such that for all j, for all $T_j \in \mathcal{T}_j$,

$$\sum_{Q : Q \cap T_j \neq \emptyset} S_j(Q, T_j) \leq C_d \left(\sum_{Q \in \mathcal{Q}} M(Q)^d \right)^{1/d}.$$

Defining the S_j's – directional surface area

Given a hypersurface Z we define the **directional surface area** of Z in the direction $e \in \mathbb{S}^{d-1}$ as

$$\mathrm{surf}_e(Z) = \int_Z |e \cdot n(x)| \mathrm{d}S(x).$$

Given M defined on $\mathcal{Q}$, we take Guth's polynomial hypersurface Z_p

$$\deg p \lesssim \left(\sum_Q M(Q)^d \right)^{1/d}$$

and, for all Q,

$$\mathrm{Vol}(K(Z_p \cap Q))^{-1/d} \geq CM(Q),$$

and we define $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}_+$ by

$$S_j(Q, T) = \mathrm{surf}_{e(T)}(Z_p \cap Q).$$

(A small lie – for technical reasons which we shall supress that everything needs to be scaled up by a catalytic parameter...)

Convex geometry

Some easy convex geometry (under the mild "catalytic" assumption):

Definition chasing shows that for any hypersurface Z, and any 1-tubes T_j, the convex set $K(Z \cap Q)$ contains each of the vectors

$$\frac{e(T_j)}{\mathrm{surf}_{e(T_j)}(Z \cap Q)}$$

and hence the parallepiped spanned by them.

Therefore we have that for any 1-tubes T_j with $T_j \cap Q \neq \emptyset$,

$$\mathrm{Vol}(K(Z_p \cap Q)) \gtrsim \frac{|e(T_1) \wedge \cdots \wedge e(T_d)|}{S_1(Q, T_1) \ldots S_d(Q, T_d)},$$

which, together with $\mathrm{Vol}(K(Z_p \cap Q))^{-1/d} \geq CM(Q)$ gives

$$M(Q) \lesssim (S_1(Q, T_1) \ldots S_d(Q, T_d))^{1/d} \, |e(T_1) \wedge \cdots \wedge e(T_d)|^{-1/d}$$

upon rearrangement.

There are two surpises to come a bit later in this context...

The degree estimate

We have that for any tube T,

$$\sum_{Q\,:\,Q\cap T\neq\emptyset} \mathrm{surf}_{e(T)}(Z_p\cap Q) \sim \int_{Z_p\cap T} |e(T)\cdot n(x)|\,\mathrm{d}S(x)$$

$$= \int_{\mathbb{B}^{d-1}} \#(Z_p\cap \ell_y)\,\mathrm{d}y \lesssim \deg p$$

where the ℓ_y are lines parallel to $e(T)$ foliating T, since, for almost every y we have $\#(Z_p\cap\ell_y) \leq \deg p$.

Hence for all j, for all $T_j\in\mathcal{T}_j$,

$$\sum_{Q\,:\,Q\cap T_j\neq\emptyset} S_j(Q,T_j) \leq C_d\left(\sum_{Q\in\mathcal{Q}} M(Q)^d\right)^{1/d}.$$

Note here the complementarity of the dimensions involved: tubes are 1-neighbouroods of lines – which are 1-dimensional; a hypersurface is $(d-1)$-dimensional and $1+(d-1)=d$.

Zhang's k_j-plane multilinear Kakeya theorem

Let $\mathcal{T}_j$ (for $1 \le j \le d$) be families of 1-neighbourhoods of k_j-planes such that $\sum_{j=1}^d k_j = n$.

Theorem (Ruixiang Zhang, 2016)

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1}(x) \cdots \sum_{T_d \in \mathcal{T}_d} \chi_{T_d}(x) E(T_1) \wedge \cdots \wedge E(T_d) \right)^{\frac{1}{d-1}} \mathrm{d}x$$

$$\le C_{d,n} \prod_{j=1}^{d} (\#\mathcal{T}_j)^{\frac{1}{d-1}}$$

[(BCT): when $q > 1/(d-1)$ in the transversal case, via heat-flow.]

The index $1/(d-1)$ in the context of a d-linear inequality strongly suggests factorisation of and a focus on functions in $L^d(\mathbb{R}^n)$, (cf. Combinatorial Loomis–Whitney,) not $L^n(\mathbb{R}^n)$ which is the setting for Guth's visibility theorem.

Factorisation for k_j-plane multilinear Kakeya

We need: For all functions G defined on the lattice of unit cubes Q in $\mathbb{R}^n$, there exist $S_1(Q, T_1), \dots, S_d(Q, T_d)$ for $T_j \in \mathcal{T}_j$ such that for each cube Q and each $T_j \in \mathcal{T}_j$ with $T_j \cap Q \neq \emptyset$; and for all j,all T_j we have

$$G(Q) \leq C_d \left(S_1(Q, T_1) \dots S_d(Q, T_d)\right)^{1/d} |T_1 \wedge \dots \wedge T_d|^{-1/d};$$

$$\sum_{Q\,:\,Q \cap T_j \neq \emptyset} S_j(Q, T_j) \leq C_d \left(\sum_Q G(Q)^d\right)^{k_j/n}.$$

Change of notation: $M(Q)^n := G(Q)^d$: need for for all M, Q, $T_j \in \mathcal{T}_j$ there exist $S_j(Q, T_j)$ such that for all $T_j \in \mathcal{T}_j$ such that $T_j \cap Q \neq \emptyset$,

$$M(Q) \leq C_n \left(S_1(Q, T_1) \dots S_d(Q, T_d)\right)^{1/n} |T_1 \wedge \dots \wedge T_d|^{-1/n}$$

and, for all j, for all $T_j \in \mathcal{T}_j$,

$$\sum_{Q: Q \cap T_j \neq \emptyset} S_j(Q, T_j) \lesssim \left(\sum_Q M(Q)^n\right)^{k_j/n}.$$

What are the S_j's? – I

When $k_j = 1$ for all j we have seen that a good choice is

$$S_j(Q,T) = \int_{Z\cap Q} |\mathbf{e}(T)\cdot \mathbf{n}(x)|\,\mathrm{d}S(x) = \int_{Z\cap Q} |\mathbf{e}(T)^{\perp} \wedge \mathbf{n}(x)|\,\mathrm{d}S(x)$$

where Z is Guth's hypersurface obtained from the polynomial method.

Analogues of these S_j for the general case are not so clear...

Dimensional considerations lead us to expect that now we are working with k_j-planes rather than lines, S_j should have something to do with the $(n - k_j)$-dimensional (directional) surface areas of an algebraic variety of codimension k_j.

Even if we are fortunate enough to be working in the case of equal $k_j = k$, with $kd = n$, there is no natural $k(d-1)$-dimensional algebraic variety lurking – Guth's hypersurface is squarely of dimension $n - 1$ and so looks to be useless. If the k_j's are different, matters look even more hopeless.

What are the S_j's? – II

Zhang introduces quantities S_j which seem to have **no** direct relationship with k_j-dimensional surface area.

Indeed there are *no* algebraic surfaces of lower dimension featuring explicitly, and instead he works, somewhat against intuition, once again with the Guth polynomial **hypersurface** Z associated to the function M which we are aiming to factorise.

For T_j a 1-neighbourhood of a k_j-plane, Zhang defines

$$S_j(Q, T_j) := \int_{Z\cap Q} \cdots \int_{Z\cap Q} |E(T_j)^{\perp} \wedge \mathbf{n}(x_1) \wedge \cdots \wedge \mathbf{n}(x_{k_j})| \mathrm{d}S(x_1) \ldots \mathrm{d}S(x_{k_j}).$$

A quick dimension count shows that this makes sense, and it agrees with directional surface area when $k_j = 1$.

Once this definition has been made it is not *so* hard to establish the desired estimates leading to the factorisation and thus the k_j-plane multilinear Kakeya theorem. Indeed...

Bézout's theorem

In fact, for any 1-neighbourhood T of any k_j-plane H we have

$$\sum_{Q\,:\,Q\cap T\neq\emptyset} S_j(Q,T)$$

$$= \sum_{Q\,:\,Q\cap T\neq\emptyset} \int_{Z_p\cap T} \cdots \int_{Z_p\cap T} |E(T)^\perp \wedge \mathbf{n}(x_1)\wedge\cdots\wedge\mathbf{n}(x_{k_j})| \mathrm{d}S(x_1)\ldots\mathrm{d}S(x_{k_j})$$

$$\approx \int_{\mathbb{B}^n}\cdots\int_{\mathbb{B}^n} \#\{H\cap(Z_p - v_1)\cap\cdots\cap(Z_p - v_{k_j})\}\mathrm{d}v_1\ldots\mathrm{d}v_{k_j}$$

$$\lesssim (\deg p)^{k_j} \lesssim \left(\sum_Q M(Q)^n\right)^{k_j/n}$$

by first doing some integral calculus, and then applying Bézout's theorem, and finally the estimate on $\deg p$ from Guth's theorem.

Note that, magically, the hypotheses of Bézout's theorem verify themselves automatically!

Convex geometry again

The other condition we need is that for all $T_j \in \mathcal{T}_j$ with $T_j \cap Q \neq \emptyset$,

$$M(Q) \leq C_n (S_1(Q, T_1) \dots S_d(Q, T_d))^{1/n} |E(T_1) \wedge \cdots \wedge E(T_d)|^{-1/n}.$$

We have $M(Q) \lesssim \mathrm{Vol}(K(Z_p \cap Q))^{-1/n}$ by Guth's theorem, so we need

$$\mathrm{Vol}(K(Z_p \cap Q)) \gtrsim \frac{|T_1 \wedge \cdots \wedge T_d|}{S_1(Q, T_1) \dots S_d(Q, T_d)}$$

for all $T_1, \dots, T_d$ with $T_j \cap Q \neq \emptyset$ in order to complete the argument.

We can establish this by doing some (slightly more complicated) convex geometry as before (modulo a cataytic assumption).

A surprise: this is *also* a direct consequence (with a little convex geometry) of the affine-invariant generalised L–W inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{d} f_j(P_j x)^{\frac{1}{d-1}} \mathrm{d}x \leq (E(T_1) \wedge \cdots \wedge E(T_d))^{\frac{-1}{d-1}} \left(\int f_1 \right)^{\frac{1}{d-1}} \cdots \left(\int f_d \right)^{\frac{1}{d-1}}$$

where P_j is projection with kernel parallel to T_j.

Multilinear Kakeya–Brascamp–Lieb inequalities

Summarising, Guth's visibility theorem plus generalised affine-invariant Loomis–Whitney implies multilinear k_j-plane Kakeya!

Likewise, if we have Brascamp–Lieb inequalities with *fixed* indices $\mathbf{p}$ for a **family** of linear surjections $\mathbf{B} \in \mathcal{B}$ with uniform bounds

$$\sup_{\mathbf{B}\in\mathcal{B}} \mathbf{BL}(\mathbf{B}, \mathbf{p}) \leq A,$$

the same argument gives a **multilinear Kakeya type Brascamp–Lieb inequality**, where the various B_j are allowed come from any of the corresponding components of members of the family $\mathcal{B}$.

So, with the benefit of hindsight, Guth's theorem can be viewed as a **universal machine** or a **black box** which takes any uniform family of Brascamp–Lieb inequalities on $\mathbb{R}^n$ as input, and churns out multilinear Kakeya–Brascamp–Lieb inequalities as output.

Everything applies to k_j-dimensional algebraic varieties too...

Multijoints

This is a counterpart of the multilinear Kakeya scenario where we replace 1*-neighbourhoods of lines* by *lines*.

We have a d-dimensional vector space $\mathbb{F}^d$ over a field $\mathbb{F}$, and families $\mathcal{L}_1, \ldots, \mathcal{L}_d$, each $\mathcal{L}_j$ each consisting of finitely many lines L_j. A **multijoint** is a point $x \in \mathbb{F}^d$ such that for each j there is an $L_j \in \mathcal{L}_j$ such that $x \in L_j$ for all j and such that the directions $e(L_1), \ldots, e(L_d)$ span $\mathbb{F}^d$. The **multiplicity** $N(x)$ of the multijoint x is given by

$$\#\{(L_1, \ldots, L_d) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d \; : ((L_1, \ldots, L_d) \text{ forms a multijoint at } x\}$$

and is denoted by $N(x)$.

In direct analogy with the multilinear Kakeya problem in $\mathbb{R}^d$ we can ask whether we have

$$\sum_{x \in \mathbb{F}^d} N(x)^{1/(d-1)} \lesssim (\#\mathcal{L}_1)^{1/(d-1)} \ldots (\#\mathcal{L}_d)^{1/(d-1)}.$$

Note that we allow "repeats" in $\mathcal{L}_j$ and $\#\mathcal{L}_j$ counts according to multiplicities.

Joints vs. multijoints

$$\sum_{x \in \mathbb{F}^d} N(x)^{1/(d-1)} \lesssim (\#\mathcal{L}_1)^{1/(d-1)} \dots (\#\mathcal{L}_d)^{1/(d-1)}.$$

Specialising to $\mathcal{L}_j = \mathcal{L}$ for all j we get the better-known **joints problem with multiplicities**; at this level the problems are equivalent. On the other hand, partial progress on one problem does not necessarily imply partial progress on the other. Originally posed by Sharir et al for simple counting of joints (rather than multijoints) in $\mathbb{R}^d$ without multiplicities.

- Simple joint counting: Sharir & coauthors; Bennett–C–Tao; Guth–Katz ($\mathbb{R}^3$); Quildodrán and Kaplan, Sharir & Shustin ($\mathbb{R}^d$); eventually all $\mathbb{F}^d$
- Joints with multiplicities: Iliopoulou ($\mathbb{R}^3$); Hablicsek ($\mathbb{F}^d$ under a generic hypothesis)
- Simple multijoints (without multiplicities): Iliopoulou ($\mathbb{R}^d$ and $\mathbb{F}^3$), (also C& Valdimarsson for a weak result in $\mathbb{F}^d$)
- Multijoints with multiplicities: Iliopoulou ($\mathbb{R}^3$)

Recent results

Theorem (Ruixiang Zhang, 2016)

The joints and multijoints problems (with multiplicities) have an affirmative solution over arbitrary fields.

There is the question of replacing lines by algebraic curves:

Theorem (AC & M. Iliopoulou, Work in progress, 2017)

An affirmative solution with polynomial or algebraic curves.

Lines $\rightsquigarrow$ k_j-planes? $\mathbb{R}^d$, Yes, Yang (2016) with extra ϵ powers.

Theorem (AC & M. Iliopoulou, Work in progress, 2017)

If $k_j = 1$ for $1 \leq j \leq d-1$ and $k_d = k$, $(k + d - 1 = n)$, and if $\#\mathcal{L}_j = L$ for $1 \leq j \leq d-1$, then

$$\#J \lesssim \prod_{j=1}^{d} \left(\#\mathcal{L}_j\right)^{1/(d-1)}.$$

Techniques and challenges?

Yang uses polynomial partitioning.

In other cases the first challenge is to find suitable analogues of the continuous quantities $S_j(Q, T)$. Zhang has done this when all the $k_j = 1$ by using local Taylor expansions with respect to special axes at every multijoint which are aligned to the directions of lines forming the multijoint.

Less clear in general – do we work with a single polynomial or a -tuple of polynomials?

The next challenge is to find ways of applying Bézout's theorem, in particular of veryfing its hypothesis – recall that in the continuous case this happened automatically as Zhang was working with averages... not at all clear how to bring this in in the discrete setting.

Challenges outside the discrete setting...? Non-linear Brascamp–Lieb outside the algebraic setting...?

Recent developments in some multilinear problems

Anthony Carbery   15 May 2017