# Uniform rectifiability, bounded harmonic functions, and elliptic PDE's

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16 May 2017

# Rectifiability

We say that $E \subset \mathbb{R}^d$ is rectifiable if it is $\mathcal{H}^1$-a.e. contained in a countable union of curves of finite length.

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$E$ is $n$-AD-regular if

$$\mathcal{H}^n(B(x,r)\cap E) \approx r^n \quad \text{for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

$E$ is uniformly $n$-rectifiable if it is $n$-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform $n$-rectifiability is a quantitative version of $n$-rectifiability introduced by David and Semmes.

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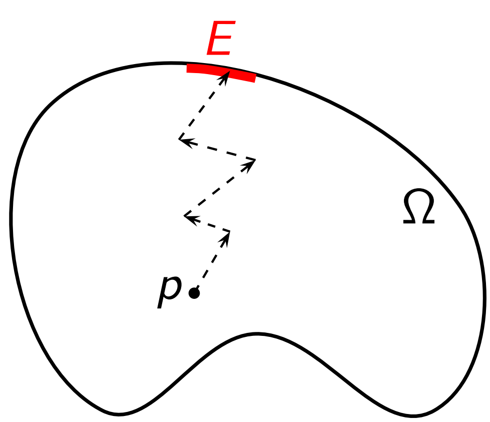
Probabilistic interpretation [Kakutani]:
When $\Omega$ is bounded, $\omega^p(E)$ is the probability that a particle with a Brownian movement leaving from $p \in \Omega$ escapes from $\Omega$ through $E$.



## Elliptic measure

We let $Lu = \mathrm{div}\, A\nabla u$ for $u \in W^{1,2}(\Omega)$, where $A$ is an elliptic matrix with real bounded coefficients.
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Quantitative properties of harmonic and elliptic measures, and connection to PDE's:
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A basic result:
If $\Omega$ is a Lipschitz domain, then $\omega \in A_\infty(\mu)$ (Dahlberg).

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- Zihui Zhao has shown that $\omega_L \in A_\infty(\mu)$ iff for any $u$ $L$-harmonic in $\Omega$, continuous in $\overline{\Omega}$, and any ball $B$ centered at $\partial\Omega$,

$$\int_{B\cap\Omega} |\nabla u|^2 \operatorname{dist}(x, \partial\Omega)\, dx \leq C\, \|u\|^2_{BMO(\mu)}\, r(B)^n.$$

(BMO solvability condition).

# Our main result in non-uniform domains

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- (c) should be understood as a substitute of $\omega \in A_\infty(\mu)$, which fails in general.

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- (d) Hofmann, Le, Martell and Nyström showed $\omega \in A_\infty^{weak}(\mu) \Rightarrow \partial\Omega$ is uniformly $n$-rectifiable, but (a) $\not\Rightarrow \omega \in A_\infty^{weak}(\mu)$.

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Up to now there was no characterization of uniform rectifiability in terms of harmonic measure.
But there was a characterization in terms of harmonic measure of big pieces of NTA domains by Bortz and Hofmann.

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- Corona decompositions are a basic tool in the work of David and Semmes.
- Connection with $\varepsilon$-approximability and work of Kenig, Kirchheim, Pipher and Toro.
- Condition (b) is related to the "area integral".
  We cannot replace $\|u\|_{L^\infty(\Omega)}$ by $\|u\|_{BMO(\mu)}$ with $u$ continuous in $\overline{\Omega}$.
  Related work by Hofmann and Le.

# Extension to elliptic operators

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The Carleson condition on $A$:
$$\int_{B\cap\Omega} \Bigg( \sup_{\substack{z_1,z_2\in B(y,M\delta_\Omega(y))\cap\Omega \\ \delta_\Omega(z_k)\geq\frac14\,\delta_\Omega(y)}} \frac{|A(z_1)-A(z_2)|}{|z_1-z_2|} \Bigg)\,dy \leq C\,r(B)^n,$$
for all balls $B$ centered at $\partial\Omega$, where $\delta_\Omega(z) = \mathrm{dist}(z,\partial\Omega)$.

## The Riesz transform and uniform rectifiability

Let $\mu$ be a Borel measure in $\mathbb{R}^d$. The $n$-dimensional Riesz transform of $f \in L^1_{loc}(\mu)$ is $\mathcal{R}_\mu f(x) = \lim_{\varepsilon \searrow 0} \mathcal{R}_{\mu,\varepsilon} f(x)$, where

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### Theorem (Nazarov, T., Volberg, 2012)

*Let $E \subset \mathbb{R}^{n+1}$ n-AD-regular, and $\mu = \mathcal{H}^n_E$. Then E is uniformly n-rectifiable iff $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded.*

# Proof of (c) $\Rightarrow$ (a) for $L = \Delta$

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To this end, for $Q \in \mathcal{D}_\mu$ set

$$\mathcal{R}_Q\mu(x) = \chi_Q(x) \int_{\frac{1}{2}\ell(Q)<|x-y|\leq\ell(Q)} \frac{x-y}{|x-y|^{n+1}}\, d\mu(y),$$

and for $\mathcal{T} \in I$,

$$\mathcal{R}_\mathcal{T}\mu(x) = \sum_{Q\in\mathcal{T}} \mathcal{R}_Q\mu(x),$$

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We show:

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- By the packing condition and Carleson's theorem, it follows that $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded.

## Connection between harmonic measure and Riesz transform

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$$\mathcal{E}(x) = c_n \, \frac{1}{|x|^{n-1}}.$$

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Therefore, for $x \in \Omega$:

$$c\,\nabla_x G(x,p) = K(x-p) - \int K(x-y)\, d\omega^p(y).$$

That is, $$\mathcal{R}\omega^p(x) = K(x-p) - c\,\nabla_x G(x,p).$$

## $\mathcal{R}_{\mathcal{T}}$ is bounded in $L^2(\mu)$

Let $R = \mathrm{Root}(\mathcal{T})$. Let $x \in R$ and $x' \in \Omega$ such that $\mathrm{dist}(x', \partial\Omega) \approx \varepsilon$. Then use the identity

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By standard estimates for Green's function,

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$$\sup_{\ell(Q_x) \leq \varepsilon \leq \ell(R)} |\mathcal{R}_\varepsilon \omega^{p_R}(x)| \lesssim \frac{1}{\mu(R)},$$

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Approximating $\mu|_R$ by $\mu(R)\,\omega^{p_R}$ and applying some kind of $T1$ theorem, we deduce that $\mathcal{R}_{\mathcal{T}}$ is bounded in $L^2(\mu)$.

# The ACF formula for elliptic operators

## Theorem (AGMT)

*Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative L-subharmonic functions. Suppose that $A(x) = Id$ and that $u_1(x) = u_2(x) = 0$ and $u_1 \cdot u_2 \equiv 0$, $u_i$ Hölder continuous at $x$. Set*

$$J(x,r) = \left( \frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}} dy \right) \cdot \left( \frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}} dy \right).$$

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*Then $J(x,\cdot)$ is absolutely continuous and*

$$\frac{J'(x,r)}{J(x,r)} \geq -c\,\frac{w(x,r)}{r}, \qquad \text{for a.e. } 0<r<R,$$

*where*

$$w(x,r) = \sup_{y \in B(x,r)} |A(y) - A(x)|.$$

# Remarks about the ACF formula

- $u_i$ Hölder continuous at $x$ means that there exists $\alpha > 0$ such that

$$u_i(y) \leq C \left( \frac{|y - x|}{r} \right)^{\alpha} \|u\|_{\infty, B(x,r)},$$

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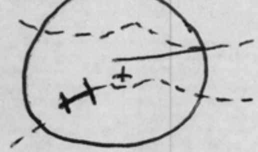
- For $L = \Delta$ we recover the classical Alt-Caffarelli-Friedman formula.
- There are less precise variants for parabolic equations and with weaker assumptions by Caffarelli-Jerison-Kenig, or by Matevosyan-Petrosyan.
- These formulas are a basic tool in free boundary problems.

# Thank you!

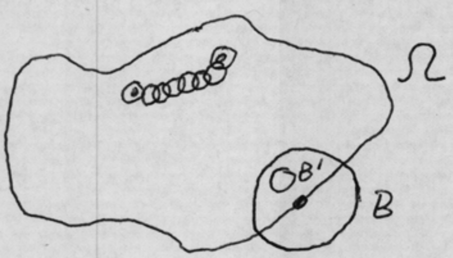
Uniform rectifiability, bounded harmonic functions and elliptic PDE's    Xavier Tolsa    16 May 2017



$H^n$ - Hausdorff measure

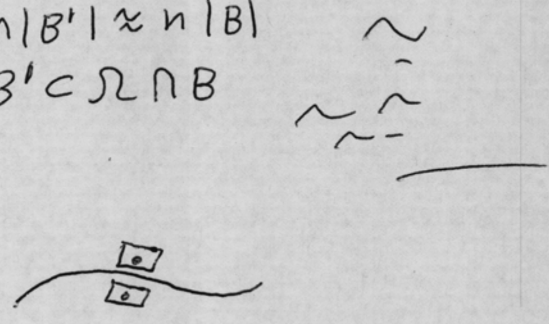
$H^n(\partial\Omega) < \infty$

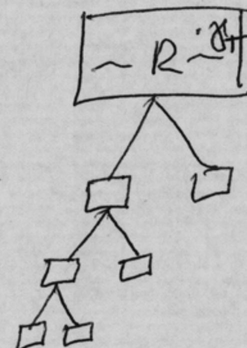
$H^n|_{\partial\Omega}$ n-AD regular, $H^n(B(x,r) \cap E) \approx r^n \quad \forall x \in E, 0 < r \leq \operatorname{diam}(E)$




$n|B'| \approx n|B|$

$B' \subset \Omega \cap B$






<u>Theorem</u>: Let $E \subset \mathbb{R}^{n+1}$ be closed, n-AD-regular, and $\Omega = \mathbb{R}^{n+1} \setminus E$

TFAE: a) $\partial\Omega$ is uniformly n-rectifiable

b) There is $C > 0$ s.t. if $u$ is a bounded harmonic function on $\Omega$ and $B$ is a ball centered at $\partial\Omega$,

$$\int_B |\nabla u(x)|^2 \operatorname{dist}(x, \partial\Omega)\, dx \leq C \|u\|^2_{L^\infty(\Omega)} r(B)^n$$

c) There is a corona decomposition of $\mu = H^n|_{\partial\Omega}$ in terms of harmonic measure.

<u>(b) ⇒ (c)</u> Build trees $T \in I$ imposing stopping conditions.

Given $R \in \mathcal{D}_\mu$ **set**: (1) $Q \in HD(R)$ if $\frac{\omega^{p_R}(Q)}{\omega^{p_R}(R)} \geq A \frac{\mu(Q)}{\mu(R)}$, $A >> 1$

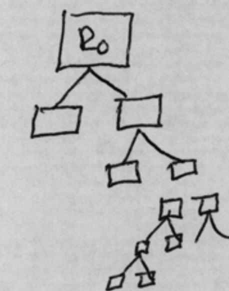
$Q \subset R$ maximal

(2) $Q \in LD(R)$ if " $\leq \delta$ , " , $\delta << 1$

Xavier Tolsa 16 May 2017

$\mathrm{Stop}(R) = LD(R) \cup HD(R)$

$\mathrm{Tree}(R)$ = family of cubes from $D_\mu(R)$ not contained strictly in any cube from $\mathrm{Stop}(R)$




If $\mathrm{supp}\,\mu = R_0 \in D_\mu$,

Let $T_0 = \mathrm{Tree}(R_0)$

Next roots of trees are the sons of cubes from $\mathrm{Stop}(R_0)$

$$\tilde{\mu}_R(x) = \int_R \frac{1}{|x-y|^{n-1}}\, d\mu(y)$$

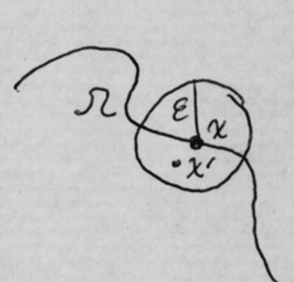
$$(1) \Rightarrow \mu\Big(\bigcup_{Q \in HD(R)} Q\Big) \le \frac{1}{A}\mu(R)$$

$$(2) \Rightarrow \omega^{p_R}\Big(\bigcup_{Q \in LD(R)} Q\Big) \le \delta\, \omega^{p_R}(R)$$

$$M_R(x) = \int_{R \setminus LD(R)} \tilde{\mu}_R \, d\omega^x$$

$\mathrm{supp}(\mu_R|_{\partial\Omega}) = R \setminus LD(R)$

$$M = \sum_{R \in \mathrm{Roots}} \lambda_R M_R$$




$$\|M\|_{\infty,\partial\Omega} = \sup_{R \in \mathrm{Root}_1} |\lambda_R|\, \|\mu_R\|_\infty$$

$$\int |\nabla^2 g(p_{R_1}, \cdot)|\, g(\cdot, p_{R_2})\, dx$$