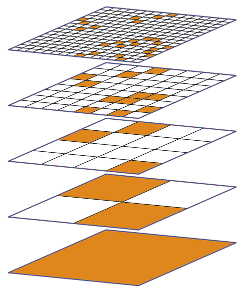

Sparse domination of singular integral operators

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Notation

- we work in Euclidean space $\mathbb{R}^d$, $d \geq 1$
- $Q \subset \mathbb{R}^d$ cube with center c_Q and sidelength ℓ_Q
- $s_Q = \log_2 \ell_Q$
- local norms

$$\langle f \rangle_{p,Q} := |Q|^{-\frac{1}{p}} \left\| f \mathbf{1}_Q \right\|_p, \qquad p \in (0, \infty]$$

- p-Hardy Littlewood maximal function

$$\mathrm{M}_p(f)(x) = \sup_{Q \subset \mathbb{R}^d} \langle f \rangle_{p,Q} \mathbf{1}_Q(x)$$

- Hölder tuple

$$\vec{t} = (t_1, \dots, t_m), \quad t_j \in [1, \infty], j = 1, \dots, m-1, \quad \sum_{j=1}^{m} \frac{1}{t_j} = 1$$

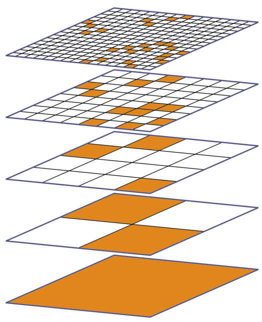
- $\lesssim$ implies positive absolute dimensional constant

Sparse collections and sparse forms

The collection of (dyadic) cubes $Q \in \mathcal{S}$ is η-sparse if $\forall Q \in \mathcal{S}$ there exists $E_Q \subset Q$ with

- $|E_Q| \geq \eta |Q|$
- $Q \neq Q' \in \mathcal{S} \implies E_Q \cap E_{Q'} = \emptyset$

namely if $\exists$ pairwise disjoint $\{E_Q : Q \in \mathcal{S}\}$ of major subsets



Positive sparse forms

If $\vec{p} = (p_1, \ldots, p_m) \in (0, \infty]^m$, $\vec{f} = (f_1, \ldots, f_m)$ define

$$\mathsf{PSF}_{\mathcal{S}}^{\vec{p}}(\vec{f}) := \sum_{Q \in \mathcal{S}} |Q| \prod_{j=1}^{m} \langle f_j \rangle_{p_j, Q}$$

associated to a sparse collection $\mathcal{S}$

Sparse domination and strong-type

If $\vec{p} = (p_1, \dots, \vec{p}_m)$ and Λ is an m-linear form, define

$$\|\Lambda\|_{\vec{p},\text{sparse}} := \inf \left\{ C : |\Lambda(\vec{f})| \leq C \sup_{\mathcal{S}} \mathrm{PSF}^{\vec{p}}_{\mathcal{S}}(\vec{f}) \text{ for all } \vec{f} = (f_1, \dots f_n) \right\}.$$

Lemma 1. If

- T is m-sublinear, $\langle T(f_1, \dots, f_{m-1}), \overline{f_m} \rangle = \Lambda(\vec{f})$
- $\vec{t}$ is a Hölder tuple with $\max\{1, p_j\} < t_j \leq \infty$, $j = 1, \dots, m$

then $\left\| T : L^{t_1} \times \cdots \times L^{t_{m-1}} \to L^{\frac{t_m}{t_m - 1}} \right\| \leq C_{\vec{t},\vec{p}} \|\Lambda\|_{\vec{p},\text{sparse}}$

Proof. If $\|\Lambda\|_{\vec{p},\text{sparse}} = 1$, for all $\vec{f}$ we find $\mathcal{S}$ such that

$$|\Lambda(\vec{f})| \leq 2\mathrm{PSF}^{\vec{p}}_{\mathcal{S}}(\vec{f}) \leq 2\eta^{-1} \sum_{Q \in \mathcal{S}} |E_Q| \left(\prod_{j=1}^{m} \langle f_j \rangle_{p_j, Q} \right) \leq 2\eta^{-1} \sum_{Q \in \mathcal{S}} |E_Q| \left(\prod_{j=1}^{m} \inf_{E_Q} \mathrm{M}_{p_j} f_j \right)$$

$$\leq 2\eta^{-1} \int_{\mathbb{R}^d} \left(\prod_{j=1}^{m} \mathrm{M}_{p_j} f_j \right) \leq 2\eta^{-1} \prod_{j=1}^{m} \|\mathrm{M}_{p_j} f_j\|_{t_j} \leq C_{\vec{t},\vec{p}} \prod_{j=1}^{m} \|f_j\|_{t_j}.$$

Sparse domination and weak-type

LEMMA 2 (folklore? Culiuc-DP-Ou, arXiv:1610.01958). If

- T is $(m-1)$-sublinear, $\langle T(f_1,\ldots,f_{m-1}),\overline{f_m}\rangle = \Lambda(\vec f)$
- $\vec p$ is a tuple with $\min p_j \geq 1$, $\displaystyle\sum_{j=1}^{m} \frac{1}{p_j} > 1$.

Then

$$\|T : L^{p_1} \times \cdots \times L^{p_{m-1}} \to L^{q,\infty}\| \lesssim \|\Lambda\|_{\vec p,\text{sparse}}, \qquad \frac{1}{q} = 1 - \sum_{j=1}^{m-1} \frac{1}{p_j}.$$

REM. In fact

$$|\Lambda(\vec f)| \lesssim \|\Lambda\|_{\vec p,\text{sparse}} \langle \mathrm{M}_{p_1,\ldots p_{m-1}}(f_1,\ldots,f_{m-1}), \mathrm{M}_{p_m} f_m\rangle$$

whence Lemma 2 and weighted variants (observation by K. Li)

A relevant example: $\left\|T : L^1 \to L^{1,\infty}\right\| \lesssim C\|T\|_{(1,p),\text{sparse}}$ whenever $1 \leq p < \infty$.

Sparse and sharp weighted norm inequalities

(sample) LEMMA 3 (self contained proof: K. Moen, Arch. Math. (Basel)).
$\Big\| T : L^t(w) \to L^t(w) \Big\| \lesssim \|\langle T(\cdot), \cdot\rangle\|_{(1,1),\text{sparse}} [w]_{A_t}^{\max\{1, \frac{1}{t-1}\}}$, where

$$[w]_{A_t} := \sup_Q \left(\langle w^{\frac{1}{t}} \rangle_{t,Q} \langle w^{-\frac{1}{t}} \rangle_{t',Q} \right)^t \qquad 1 < t < \infty.$$

Proof for $p = 2$. If $\mathcal{S}$ sparse, $Q \in \mathcal{S}$, then $|Q| \lesssim |E_Q| = \int_{E_Q} w^{\frac12} v^{\frac12} \le \Big(w(E_Q) v(E_Q) \Big)^{\frac12}$

Write $v = w^{-1}$. Using $\|T\|_{L^2(w)} = \sup \Big\{ |\langle T(fw), gv\rangle| : \|f\|_{L^2(w)} = \|g\|_{L^2(v)} = 1 \Big\}$,

$$|\langle T(fw), gw^{-1}\rangle| \lesssim \|T\|_{(1,1),\text{sparse}} \sum_{Q \in \mathcal{S}} |Q| \langle fw\rangle_{1,Q} \langle gw^{-1}\rangle_{1,Q}$$

$$\lesssim \sum_{Q \in \mathcal{S}} \Big(\langle w\rangle_{1,Q} \langle v\rangle_{1,Q} \Big) \left(w(E_Q)^{\frac12} \frac{\langle fw\rangle_{1,Q}}{\langle w\rangle_{1,Q}} \right) \left(v(E_Q)^{\frac12} \frac{\langle gv\rangle_{1,Q}}{\langle v\rangle_{1,Q}} \right)$$

$$\lesssim [w]_{A_2} \left(\int (\mathrm{M}_w f)^2 \, \mathrm{d}w \right)^{\frac12} \left(\int (\mathrm{M}_v g)^2 \, \mathrm{d}v \right)^{\frac12} \lesssim [w]_{A_2}.$$

The endpoint maximal truncation approach

Thm (Lerner'13; Conde-Rey, Lerner-Nazarov; Lacey'15; Lerner'15)

Let T CZO. Then $\exists \mathcal{S} = \mathcal{S}(T, f)$ such that pointwise

$$|Tf| \lesssim \sum_{Q \in \mathcal{S}} \langle f \rangle_{Q,1} \mathbf{1}_Q$$

The maximal operator approach: define

$$\mathcal{M}_T f := \sup_{Q \subset \mathbb{R} \text{ interval}} \left\| T(f \mathbf{1}_{\mathbb{R}\setminus 3Q}) \right\|_{L^\infty(Q)} \mathbf{1}_Q$$

- $\mathcal{M}_T$ dominates T: $|Tf| \lesssim \|T\|_{1\to 1,\infty} |f| + \mathcal{M}_T f$
- $\mathcal{M}_T$ is controlled by $T^\star$: $\mathcal{M}_T f \lesssim \mathrm{M}_1 f + T^\star f$

Using the resulting weak type (1,1) of $\mathcal{M}_T \rightsquigarrow$ sparse.

A non-exhaustive list of related works

Non-kernel and rough SIO

- Bernicot-Frey-Petermichl, *Sharp weighted norm estimates beyond CZ theory*, arXiv:1510.00973, Analysis and PDE.
- Hytönen-Roncal-Tapiola, *Quantitative weighted estimates for rough homogeneous singular integrals*, arxiv:1510.05789, Israel J. Math.
- Benea-Bernicot-Luque, *Sparse bilinear forms for Bochner Riesz multipliers and applications*, arXiv:1605.06401, Transactions LMS

Non-homogeneous singular integrals

- Conde and Parcet, *Nondoubling Calderón-Zygmund theory -a dyadic approach*, arXiv:1604.03711.
- Volberg and Zorin-Kranich, *Sparse domination on non-homogeneous spaces with an application to A_p weights*, arXiv:1606.03340, to appear, RMI

Vector-valued singular integrals

- Nazarov-Petermichl-Treil-Volberg, *Convex body domination and weighted estimates with matrix weights*, arXiv:1701.01907.

... out of more than 30 ArXiv postings in 18 months

The main idea in a simple setting: martingale transform

$$\Lambda_Q(f_1,f_2) := \sum_{I\in\mathcal{D}: I\subset Q} \varepsilon_I \langle f_1,h_I\rangle\langle f_2,h_I\rangle, \qquad \|\Lambda_Q\|_{(1,1)\text{sparse}} \le 2^6 \sup_{I\in\mathcal{D}}|\varepsilon_I|$$

Proof. Set $\mathcal{Q} := \left\{\text{max. dyadic } L\subset Q \text{ with } \langle f_j\rangle_{1,L} > 2^2\langle f_j\rangle_{1,Q} \text{ for some } j=1,2\right\}$,

$$F_Q := \bigcup_{L\in\mathcal{Q}} L, \qquad \Lambda_{\mathcal{Q}}(f_1,f_2) := \sum_{I\subset Q,\, I\not\subset F_Q} \varepsilon_I\langle f_1,h_I\rangle\langle f_2,h_I\rangle$$

- $L\in\mathcal{Q}$ are pairwise disjoint and $\left|Q\setminus F_Q\right| \ge \frac{1}{2}|Q|$
- $f_j = g_j + b_j, \quad \langle g_j\rangle_{2,Q} \le 2^3\langle f_j\rangle_{1,Q}, \quad b_j := \sum_{L\in\mathcal{Q}} b_{jL}, \quad \operatorname{supp} b_{jL}\subset L, \int b_{jL} = 0$

$$\begin{aligned}|\Lambda_Q(f_1,f_2)| &\le |\Lambda_{\mathcal{Q}}(f_1,f_2)| + \sum_{L\in\mathcal{Q}}|\Lambda_L(f_1,f_2)| = |\Lambda_{\mathcal{Q}}(g_1,g_2)| + \sum_{L\in\mathcal{Q}}|\Lambda_L(f_1,f_2)|\\ &\le |Q|\langle g_1\rangle_{2,Q}\langle g_2\rangle_{2,Q} + \sum_{L\in\mathcal{Q}}|\Lambda_L(f_1,f_2)|\\ &\le 2^6|Q|\langle f_1\rangle_{1,Q}\langle f_2\rangle_{1,Q} + \sum_{L\in\mathcal{Q}}|\Lambda_L(f_1,f_2)| \qquad \text{...ITERATE}\end{aligned}$$

Uniform sparse domination via dyadic shifts

L^2-bounded dyadic shifts:

$$\mathbb{S}^{\varrho}_{\boldsymbol{\varepsilon}}(f_1, f_2) = \sum_{Q \in \mathcal{D}} \varepsilon_Q S_Q(f_1, f_2), \quad S_Q(f_1, f_2) := \int_{Q\times Q} K_Q(x_1, x_2) f_1(x_1) f_2(x_2)\,\mathrm{d}x_1 \mathrm{d}x_2$$

(A1) $\sup_{\boldsymbol{\varepsilon}=\{\varepsilon_Q\}\in \mathbb{D}^{\mathcal{D}}} |\mathbb{S}^{\varrho}_{Q}(f_1, f_2)| \le \|f_1\|_2 \|f_2\|_2$

(A2) if $L \in \mathcal{D}, L \subset Q$ and $s_L < s_Q - \varrho$ then $K_Q(\cdot, x_2)$ is constant on R for all $x_2 \in Q$, and symmetric assumption with x_1, x_2 interchanged.

Theorem (Culiuc-DP-Ou, arXiv:1610.01958).

$$\sup_{\boldsymbol{\varepsilon}} \|\mathbb{S}^{\varrho}_{\boldsymbol{\varepsilon}}\|_{(1,1),\text{sparse}} \lesssim \varrho.$$

Rem. Hytönen representation theorem + convex hull $\rightsquigarrow$ $\|T\|_{(1,1),\text{sparse}} \lesssim 1$ for T CZO

Rem. Easy extension to $\mathbb{R}^n$-valued f_j, with $\langle f_j \rangle_{1,Q}$ replaced by *convex bodies*

$$\langle f \rangle_{1,Q} := \left\{ \frac{1}{|Q|} \int_Q f \varphi \,\mathrm{d}x : \|\varphi\|_\infty \le 1 \right\} \subset \mathbb{R}^n$$

variation on theme by Nazarov-Petermichl-Treil-Volberg.

Weighted inequalities for rough singular integrals

Rough homogeneous SI

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^d} f(x-y)\Omega\left(\frac{y}{|y|}\right)\frac{\mathrm{d}y}{|y|^d}, \qquad \Omega \in L^q(S^{d-1}), \int_{S^{d-1}} \Omega = 0.$$

A very brief history:

- for $p, q > 1$, $T_\Omega, T_\Omega^\star : L^p \to L^p$; method of rotations, C-Z, 60 years ago.
- $T_\Omega : L^1 \to L^{1,\infty}$; Hofmann, Christ-Rubio '88 in $d \leq 7$; Seeger, JAMS, '96.
- $T_\Omega^\star : L^1 \to L^{1,\infty}$ is not known even if $q = \infty$ and might be false
- $T_\Omega : L^t(w) \to L^t(w)$ if $w \in A_t, q = \infty$
  (Duoandikoetxea-Rubio de Francia'86, Watson'90, Duoandikoetxea'93)

Quantitative weighted inequalities: when $q = \infty$

- $\|T_\Omega\|_{L^t(w)} \leq C_t [w]_{A_t}^{2\max\{1, \frac{1}{t-1}\}}$: Hytonen-Roncal-Tapiola'15, IJM
- $\|T_\Omega\|_{L^t(w)} \leq C_{t,\tau} [w]_{A_\tau}^2$ when $1 < \tau < t < \infty$: Perez-Rivera-Roncal '16, IJM

Sparse domination of T_Ω and corollaries

Theorem A, Conde-Culiuc-DP-Ou, arXiv:1612.09201

- $\|T_\Omega\|_{(1,q'),\mathrm{sparse}} \lesssim q\|\Omega\|_{L^{q,1}\log L}$ for all $1 < q < \infty$;
- $\|T_\Omega\|_{(1,p),\mathrm{sparse}} \lesssim \dfrac{p}{p-1}\|\Omega\|_\infty$ for all $1 < p < \infty$.

- $$\|T_\Omega\|_{L^t(w)} \le \begin{cases} C_{t,q}\|\Omega\|_{L^{q,1}\log L}[w]_{A_{\frac{t}{q'}}}^{\max\left\{1,\frac{1}{t-q'}\right\}} & q' < t < \infty \qquad \text{(SHARP)} \\ C_t\|\Omega\|_\infty [w]_{A_t}^{\frac{t}{t-1}} & 1 < t < \infty \end{cases}$$

  (applying K. Li, Coll. Math. 2017)

- $\|T_\Omega f\|_{L^t(w)} \le C_t\|\Omega\|_\infty\|f\|_{L^t(\mathrm{M}^{\lfloor t\rfloor+1}w)}$ $\quad 1 < t < \infty$ $\quad$ (SHARP)

  (applying D. Beltran, Rev. Mat. Iberoam. 2014)

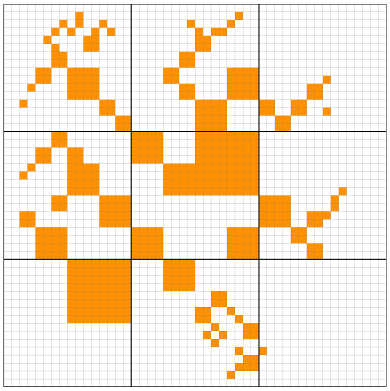
- $\|T_\Omega f\|_{L^t(w)} \lesssim [w]_{A_\infty}^2\|\Omega\|_\infty\|\mathrm{M}f\|_{L^t(w)}$ $\quad 0 < t < \infty$

  due to K. Li-Perez-Roncal-Rivera Rios, arXiv 2017)

Stopping collections and norms

$\mathcal{Q}$ is a *stopping collection* of cubes with top Q if

- $\displaystyle\bigcup_{L\in\mathcal{Q}} L =: \mathrm{sh}\mathcal{Q} \subset 3Q$;
- $L \cap L' \neq \emptyset \implies L = L'$ (disjoint)
- $|s_L - s_{L'}| \geq 8 \implies 7L \cap 7L' = \varnothing$ (Whitney)



$$\cdot\ \|h\|_{\mathcal{Y}_p(\mathcal{Q})} := \begin{cases} \sup\limits_{L\in\mathcal{Q}} \sup\limits_{I \gtrapprox L} \langle h\rangle_{p,I} & p<\infty \\ \|h\|_\infty & p = \infty \end{cases}$$

$$\cdot\ \mathcal{Y}_p(\mathcal{Q}) := \left\{ h : \mathrm{supp} h \subset 3Q, \|h\|_{\mathcal{Y}_p(\mathcal{Q})} < \infty \right\},$$

$$\cdot\ \mathcal{X}_p(\mathcal{Q}) := \left\{ b \in \mathcal{Y}_p(\mathcal{Q}) : b = \sum_{L\in\mathcal{Q}} b_L,\ \mathrm{supp} b_L \subset L \right\}$$

$$\cdot\ \dot{\mathcal{X}}_p(\mathcal{Q}) := \left\{ b \in \mathcal{X}_p(\mathcal{Q}) : \int_L b_L = 0 \quad \forall L \in \mathcal{Q} \right\}$$

An abstract sparse domination theorem

Let T be an operator on $\mathbb{R}^d$ with uniformly bounded truncations:

- $\langle Tf_1, f_2\rangle = \lim_{s\to\infty, \sigma\to-\infty} \Lambda^s_\sigma(f_1, f_2), \qquad \sup_{\sigma<s} \|\Lambda^s_\sigma\|_{L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)} \leq 1$
- $\Lambda^s(f_1\mathbf{1}_Q, f_2) = \Lambda^s(f_1\mathbf{1}_Q, f_2\mathbf{1}_{3Q}), \qquad \ell(Q) = 2^s$
- $|\Lambda^s_{s-1}(f_1\mathbf{1}_Q, f_2)| \leq |Q|\langle f_1\rangle_{p_1,Q}\langle f_2\rangle_{p_2,3Q}$ for some $1 \leq p_1, p_2 < \infty$.

For a stopping collection $L \in \mathcal{Q}$ with top Q define

$$\Lambda_{\mathcal{Q}}(h_1, h_2) := \frac{1}{|Q|}\left[\Lambda^{s_Q}(h_1\mathbf{1}_Q, h_2) - \sum_{L\in\mathcal{Q}} \Lambda^{s_L}(h_1\mathbf{1}_L, h_2)\right] = \Lambda_{\mathcal{Q}}(h_1\mathbf{1}_Q, h_2\mathbf{1}_{3Q}).$$

Rem. Uniform L^2-bound of $\Lambda^s_\sigma \implies |\Lambda_{\mathcal{Q}}(h_1, h_2)| \lesssim \|h_1\|_{\mathcal{Y}_2(\mathcal{Q})}\|h_2\|_{\mathcal{Y}_2(\mathcal{Q})}$

An abstract (continuous) sparse domination theorem

THEOREM C, Conde-Culiuc-DP-Ou, arXiv:1612.09201

Assume in addition that

$$|\Lambda_{\mathcal{Q}}(b,h)| \le C_{\mathsf{L}} \|b\|_{\dot{\mathcal{X}}_{p_1}(\mathcal{Q})} \|h\|_{\mathcal{Y}_{p_2}(\mathcal{Q})},$$
$$|\Lambda_{\mathcal{Q}}(h,b)| \le C_{\mathsf{L}} \|h\|_{\mathcal{Y}_{\infty}(\mathcal{Q})} \|b\|_{\dot{\mathcal{X}}_{p_2}(\mathcal{Q})},$$

uniformly over all stopping collections $\mathcal{Q}$. Then

$$\|\Lambda\|_{(p_1,p_2),\mathsf{sparse}} \lesssim C_{\mathsf{L}}.$$

Verifying condition (L)

Rem. If $b \in \mathcal{X}_p(Q)$ then

$$\Lambda_Q(b,h) = \sum_{j\geq 1} \Lambda^j(b,h), \qquad \Lambda^j(b,h) := \frac{1}{|Q|} \sum_{s \leq s_Q} \sum_{L \in Q: s_L = s-j-3} \left\langle K_s * b_L, \overline{h} \right\rangle$$

If $b \in \dot{\mathcal{X}}_1(Q)$ (aka mean zero)

(1) K ω-smooth CZ kernel: $|\Lambda^j(b,h)| \lesssim \omega(2^{-j})|Q| \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_1}$

(2) RH, $\|\Omega\|_\infty = 1$, trivial estimate: $|\Lambda^j(b,h)| \lesssim \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_1}$

(3) RH, $\|\Omega\|_\infty = 1$, weak-(1,1) (Seeger) $|\Lambda^j(b,h)| \lesssim 2^{-cj} \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_\infty}$

Interpolating (Riesz-Thorin or Marcinkiewicz) (2), (3) in h yields

$$|\Lambda^j(b,h)| \lesssim 2^{-\frac{c(p-1)}{p}j} \|\Omega\|_\infty \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_p}, \qquad p > 1$$

and we get $|\Lambda_Q(b,h)| \lesssim \dfrac{p}{p-1} \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_p}$ by summing up.

Localized estimates: maximal rough truncations

- Let K be a ω-smooth CZ kernel. Then for $2 < q \le \infty$

$$\Lambda_s^\sigma(f_1, f_2) = \int_{\mathbb{R}^d} \sup_{N, \sigma < t_1 < \cdots < t_N < s} \left\| \int_{t_j < |x-y| \le t_{j+1}} K(x,y) f_1(y)\, \mathrm{d}y \right\|_{\ell^q_j} f_2(x)\, \mathrm{d}x$$

satisfies $\Lambda_Q(b,h) + \Lambda_Q(h,b) \lesssim \|\omega\|_{\mathsf{Dini}} \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_1}$

- splitting (Duoandikoetxea-Rubio, Hytönen et. al.) for all $\Delta > 1$:

$$T_\Omega = \sum_{j \ge 1} T_j, \qquad \|T^j\|_{\mathsf{Dini}} \lesssim \Delta j, \qquad \|(T^j)^\star\|_{L^2} \lesssim 2^{-\Delta j}$$

leading to

$$\Lambda_Q[(T^j)^\star](b,h) + \Lambda_Q[(T^j)^\star](h,b) \lesssim \Delta j 2^{-cj} \|b\|_{\dot{\mathcal{X}}_{1+\frac{1}{\Delta}}} \|h\|_{\mathcal{Y}_{1+\frac{1}{\Delta}}}$$

and finally

$$\Lambda_Q[T_\Omega^\star](b,h) + \Lambda_Q[T_\Omega^\star](h,b) \lesssim \Delta \|b\|_{\dot{\mathcal{X}}_{1+\frac{1}{\Delta}}} \|h\|_{\mathcal{Y}_{1+\frac{1}{\Delta}}}$$

Sparse domination of $T_\Omega^\star$

Theorem (DP-K. Li, 2017, preprint)

For all $\Delta > 1$ there holds

$$\|T_\Omega^\star\|_{(1+\frac{1}{\Delta},1+\frac{1}{\Delta}),\mathrm{sparse}} \lesssim \Delta$$

Corollary:

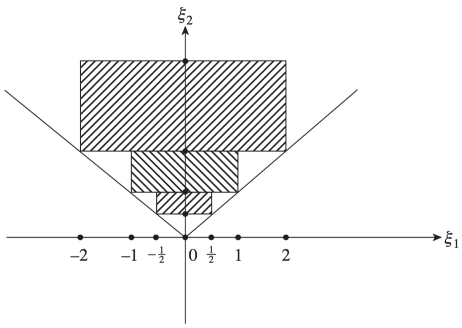
$$\|T_\Omega^\star\|_{L^t(w)} \le C_t \|\Omega\|_\infty [w]_{A_t}^{\frac{t}{t-1}} \qquad 1 < t < \infty$$

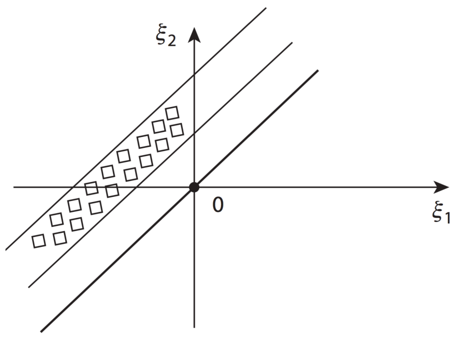
extending Hytönen-Roncal-Tapiola to maximal truncated case

Multilinear operators given by singular multipliers

We consider the multiplier forms

$$\Lambda_m(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} m(\xi)\Big(\prod_{j=1}^{3} \widehat{f_j}(\xi_j)\Big)\, \mathrm{d}\xi,$$





RANK 0: Coifman-Meyer

$$\sup_{|\alpha|\le N} \sup_{\xi} |\xi|^{|\alpha|} \left|\partial_\xi^\alpha m(\xi)\right| \le C_N$$

RANK 1: bilinear Hilbert transform

$$\sup_{|\alpha|\le N} \sup_{\xi} |\mathrm{dist}(\xi,\Gamma)|^{|\alpha|} \left|\partial_\xi^\alpha m(\xi)\right| \le C_N$$

L^p bounds for multilinear multipliers

We expect Hölder type bounds

$$T_m : L^{q_1} \times L^{q_2} \to L^r, \qquad \tfrac{1}{q_1} + \tfrac{1}{q_2} = \tfrac{1}{r}, \qquad q_1, q_2 > 1$$

for the formal adjoints $\Lambda_m(f_1, f_2, f_3) = \langle T_m(f_1, f_2), f_3 \rangle$,

$$T_m(f_1, f_2)(x) = \int_{\xi_1+\xi_2+\xi_3=0} m(\xi) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \mathrm{e}^{ix(\xi_1+\xi_2)} \, \mathrm{d}\xi$$

- RANK 0: bilinear CZ translation invariant kernels, $r > \frac{1}{2}$
- RANK 1: most significant example $m(\xi) = \mathrm{sign}(\xi_1 - \xi_2)$ is bilinear HT

$$\mathrm{BHT}(f_1, f_2)(x) = \mathrm{p.v.} \int_{\mathbb{R}} f_1(x - t) f_2(x + t) \frac{\mathrm{d}t}{t}.$$

and $r \geq \frac{2}{3}$ necessary in such generality (Lacey-Thiele'97, Muscalu-Tao-Thiele'01)

Sparse domination of multilinear multipliers

Theorem (Culiuc-DP-Ou, arXiv:1603.05317)

There holds

$$\sup_{m \text{ RANK } 1} \|\Lambda_m\|_{\vec{p},\text{sparse}} \leq K(\vec{p}) C_{N(\vec{p})}$$

for all tuples $\vec{p} = (p_1, p_2, p_3)$ with $p_j > 1$, $\displaystyle\sum_{j=1}^{3} \frac{1}{\min\{p_j, 2\}} < 2$.

Example tuples: $(1^+, 2^-, 2^-)$, $(\frac{4}{3}^+, \frac{4}{3}^-, 2^-)$...

Rem. This result recovers L^t-theory and thus is sharp up to endpoints.

Corollaries: multilinear weighted estimates

Corollary

Let $\vec{q} = (q_1, q_2, q_3)$ and a weight vector $\vec{v} = (v_1, v_2, v_3)$ satisfying

$$1 < q_j < \infty, \qquad \sum_{j=1}^{3} \frac{1}{q_j} = 1, \qquad \prod_{j=1}^{3} v_j^{\frac{1}{q_j}} = 1$$

Then

$$\sup_{m \text{ RANK } 1} |\Lambda_m(f_1, f_2, f_3)| \leq \left(\inf_{\vec{p}} C(\vec{p}, \vec{q}) [\vec{v}]_{A_{\vec{q}}^{\vec{p}}}^{\max \frac{q_j}{q_j - p_j}} \right) \prod_{j=1}^{3} \|f_j\|_{L^{q_j}(v_j)}$$

where the infimum is taken over open admissible tuples $\vec{p}$ with $p_j < q_j$ and

$$[\vec{v}]_{A_{\vec{q}}^{\vec{p}}} := \sup_{Q} \prod_{j=1}^{3} \left\langle v_j^{\frac{p_j}{p_j - q_j}} \right\rangle_Q^{\frac{1}{p_j} - \frac{1}{q_j}}$$

Corollaries: weighted,vector-valued estimates

Corollary (sample) (unweighted: Silva'12 JLMS, Benea-Muscalu'15, APDE)

Let $\boldsymbol{m} = \{m_k\}$ be a sequence of RANK 1 multipliers and

$$T_{\boldsymbol{m}} : (\{f_{1k}\}, \{f_{2k}\}) \mapsto \{T_{m_k}(f_{1k}, f_{2k})\}$$

For $1 < q_1, q_2, q_3 < \infty, \quad 2 \leq r_1, r_2, r_3 \leq \infty$

$$T_{\boldsymbol{m}} : L^{q_1}(v_1; \ell^{r_1}) \times L^{q_2}(v_2; \ell^{r_2}) \to L^{(q_3)'}(u; \ell^{(r_3)'}), \qquad u^{\frac{1}{(q_3)'}} = v_1^{\frac{1}{q_1}} v_2^{\frac{1}{q_2}}$$

whenever either of these hold (and more examples)

- $v_j \in A_{\frac{q_j+1}{2}} \cap RH_2, \quad j = 1, 2$
- $v_j \in A_{\frac{q_j}{2}}, \quad j = 1, 2$

Proof. Domination theorem + weighted Fefferman-Stein

$$\left\| \{\mathrm{M}_p f_k\} \right\|_{L^q(v;\ell^r)} \lesssim C\left([v]_{A_{\frac{q}{p}}} \right) \|\{f_k\}\|_{L^q(\mathbb{R};\ell^r)} \qquad 1 \leq p < \min\{q, r\}$$

Sparse domination of variational Carleson operators

Consider the r-variational Carleson operator

$$C_r f(x) = \sup_{N\in\mathbb{N}} \sup_{\xi_0<\cdots<\xi_N} \left(\sum_{j=1}^{N} \left| \int_{\xi_{j-1}}^{\xi_j} \widehat{f}(\xi)\, \mathrm{e}^{ix\xi}\, \mathrm{d}\xi \right|^r \right)^{1/r}$$

Thm (DP-Do-Uraltsev, arXiv:1612.03028, improving on Oberlin-Seeger-et. al. '09.

If $r > 2$ and $p > r'$ then $\|C_r\|_{(p,1),\text{sparse}} \leq C_p$.

Corollaries: improving on Do-Lacey'12

- $\|C_r\|_{L^t(w)} \leq C_\tau [w]_{A_\tau}^{\max\left\{1, \frac{\tau}{t(\tau-1)}\right\}}, \qquad 1 \leq \tau < \frac{t}{r'}$ (sharp)
- $\|C_r\|_{L^t(w)} \leq C(t, [w]_{A_{\frac{t}{r'}}})$ (sharp range)

Multitile maps

- tile $t = I_t \times \omega_t$ dyadic rectangle of area 1: $|I_t||\omega_t| = 1$
- rank 1 tritile $P = (P_1, P_2, P_3)$: triple of tiles P_j with

$$I_{P_1} = I_{P_2} = I_{P_3} =: I_P, \qquad \omega_P := \omega_{P_1} \times \omega_{P_2} \times \omega_{P_3}, \quad \operatorname{dist}(\omega_P, \mathbb{R}(1,1,1)) \sim |\omega_{P_j}|$$

- $\mathbb{P}_0$ rank 0 collection if above holds and $(\xi, \xi, \xi) \in C\omega_P$ for all $P \in \mathbb{P}_0$.
- Multitile map

$$\Lambda_{\mathbb{P}}(f_1, f_2, f_3) = \sum_{P \in \mathbb{P}} |I_P| \prod_{j=1}^{3} F_j(f_j)(P), \quad F_j(f)(P) := \sup |\langle f, \phi_{P_j} \rangle|$$

supremum taken over P_j-adapted functions, L^1-normalized:

$$\operatorname{supp} \widehat{\phi_t} \subset \omega_t, \quad |\phi_t^{(N)}(x)| \leq C_N |I_t|^{-N-1} \left(1 + \frac{\operatorname{dist}(I_t, x)}{|I_t|}\right)^{-N}$$

$$m \text{ RANK k} \implies |\Lambda_m(f_1, f_2, f_3)| \leq \sum_{j \leq C(\Gamma)} \Lambda_{\mathbb{P}_j}(f_1, f_2, f_3) \text{ with } \mathbb{P}_j \text{ of rank } k.$$

Outer L^q and Hölder inequalities

For $F : \mathbb{P} \to \mathbb{C}$, and a *tree* $\mathbf{T}$ (rank 0 collection with $I_P \subset I_{\mathbf{T}}$ for all $P \in \mathbf{T}$)

$$\mathrm{s}(F)(\mathbf{T}) = \left(\frac{1}{|I_{\mathbf{T}}|} \sum_{P \in \mathbf{T}^*} |I_P||F(P)|^2 \right)^{\frac{1}{2}} + \sup_{P \in \mathbf{T}} |F(P)|$$

we consider (out of Do-Thiele'15, Bull. AMS) outer L^q norms

$$\|F\|_{L^q(\mathrm{s})} := \left(\int_0^\infty p\lambda^{p-1} \mu(\mathrm{s}(F) > \lambda)\, \mathrm{d}\lambda \right)^{\frac{1}{p}}$$

where, if μ outer measure generated by trees with premeasure $\sigma(\mathbf{T}) = |I_{\mathbf{T}}|$,

$$\mu(\mathrm{s}(F) > \lambda) := \inf \left\{ \mu(E_\lambda) : S(F\mathbf{1}_{(E_\lambda)^c}) \le \lambda \right\}.$$

Outer Hölder inequality

$$|\Lambda_{\overline{\mathbb{P}}}(f_1, f_2, f_3)| \le C \prod_{j=1}^{3} \|F_j(f_j)\mathbf{1}_{\overline{\mathbb{P}}}\|_{L^{q_j}(\mathrm{s})} \qquad \forall\, \overline{\mathbb{P}} \subset \mathbb{P}$$

Carleson embedding theorems with stopping times

Define

$$\mathbb{G}_{f,p,Q} := \left\{ P \in \mathbb{P} : \inf_{x \in 3I_P} \mathrm{M}_p(f \mathbf{1}_{3Q})(x) > C\langle f \rangle_{p,3Q} \right\}$$

Localized Carleson embeddings (DP-Y.Ou'15, arXiv:1510.06433)

$$\left\| F_j(f\mathbf{1}_{3Q})\mathbf{1}_{\mathbb{G}_{f,p,Q}} \right\|_{L^q(\mathsf{s})} \le C_{p,q}|Q|^{\frac{1}{q}}\langle f \rangle_{3Q,p} \qquad \begin{cases} p = 1, q > 1 & \mathbb{P} \text{ RANK } 0 \\ 1 < p < 2, q > p' & \mathbb{P} \text{ RANK } 1 \end{cases}$$

In the RANK 0 case, this amounts to interpolating the two inequalities

$$\sup_{R \subset \mathbb{R}} \frac{1}{|R|} \sum_{\substack{P \in \mathbb{G}_{f,p,Q} \\ I_P \subset R}} |I_P||\langle f, \phi_{P_j} \rangle|^2 \lesssim \langle f \rangle_{3Q,1}^2, \qquad \mu\left(\mathbb{P} \backslash \mathbb{G}_{f,p,Q}\right) \le \frac{1}{2}|Q|$$

the first of which is the Calderón-Zygmund decomposition.

Sparse domination in UMD function lattices

Theorem (a sparse version of Corollary 4 DP-Ou, arXiv:1506.05827)

Let $\mathcal{X}_j = L^{q_j}(v_j), \quad 1 < q_j < \infty, \quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1.$

Let m be a $B(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$-valued multiplier with

$$\mathcal{M} = \left\{ |\xi|^\alpha \partial_\xi^\alpha m(\xi)(\cdot,\cdot,\cdot) : \xi, |\alpha| \leq N \right\}$$

Assume that uniformly over n, $\{\mathrm{m}_j : j = 1, \dots n\} \subset \mathcal{M}$, permutations σ.

$$\begin{aligned}
&\sum_{j=1}^{n} \mathrm{m}_j[g_1^j, g_2^j, g_3^j] \\
&\leq C_{\mathcal{M}} \left\| g^j_{\sigma(1)} \right\|_{L^{q_1}(v_3;\ell^2_j)} \left\| g^j_{\sigma(2)} \right\|_{L^{q_2}(v_2;\ell^2_j)} \left\| \sup_j |g^j_{\sigma(3)}| \right\|_{L^{q_3}(v_3)}
\end{aligned}$$

Then for all $L^{q_j}(v_j)$-valued functions f_j there exists a sparse collection $\mathcal{S}$

$$|\Lambda_m(f_1, f_2, f_3)| \lesssim \sum_{Q \in \mathcal{S}} |Q| \prod_{j=1}^{3} \langle \|f_j\|_{L^{q_j}(v_j)} \rangle_{1,Q}$$

Multi-parameter mixed norm weighted estimates

Theorem (extends DP-Ou, arXiv:1506.05827, see also Benea-Muscalu, 1511.04948)

Let $m = m(\xi, \eta)$ be a bi-parameter Rank 0-Rank 1 multiplier:

$$\sup_{\xi,\eta} |\xi|^{|\alpha|} \mathrm{dist}(\eta, \mathbb{R}(1,1,1))^{|\beta|} \left| \partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta) \right| \leq K, \qquad |\alpha|, |\beta| \leq N$$

Then, adjoints T to

$$\Lambda_m(f_1, f_2, f_3) = \int_{\substack{\xi_1+\xi_2+\xi_3=0 \\ \eta_1+\eta_2+\eta_3=0}} m(\xi,\eta) \left(\prod_{j=1}^{3} \widehat{f_j}(\xi_j, \eta_j) \right) \mathrm{d}\xi \mathrm{d}\eta$$

extend to bounded linear operators

$$T : L^1(w_1; L^{q_1}(v_1)) \times L^1(w_2; L^{q_2}(v_2)) \to L^{\frac{1}{2},\infty}(w; L^{q_3'}(u))$$

for $1 < q_1, q_2, q_3 < \infty$, $\quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$ whenever $w_j \in A_1, \ v_j \in A_{\frac{q_j+1}{2}} \cap RH_2$.

Thank you for your attention!

Work being reported

- FDP and Y. Ou *Banach-valued multilinear singular integrals*, arXiv:1506.05827, IUMJ
- FDP and Y. Ou *A modulation invariant Carleson embedding theorem outside local L^2*, arXiv:1510.06433, J. d'Analyse
- A. Culiuc, FDP and Y. Ou *Domination of multilinear singular integrals by positive sparse forms*, arXiv:1603.05317, submitted
- A. Culiuc, FDP and Y. Ou, *Uniform sparse domination of singular integrals via dyadic shifts*, arXiv:1610.01958, Math. Res. Lett
- Yen Q. Do, FDP and Gennady N. Uraltsev, *Positive sparse domination of variational Carleson operators*, arXiv:1612.03028, Ann. Sci. Scuola Norm. Sup.
- J. M. Conde-Alonso, A. Culiuc, FDP and Y. Ou *A sparse domination principle for rough singular integrals*, arXiv:1612.09201, Analysis and PDE

Sparse domination of singular integral operators

Francesco Di Plinio 16 May 2017

ϕ Schwartz function, $\operatorname{supp}\phi \subset [0,1]$

$\phi_n(x) = e^{2\pi i x n}\phi(x)$

$$(f_1, f_2, f_3) \longrightarrow \sum_{n=0}^{N} \langle f_1, \phi_n\rangle\langle f_2, \phi_{n+1}\rangle\langle f_3, \phi_{n+2}\rangle$$

Bound in best way possible. Try Cauchy-Schwartz

$$\left|\sum_{n=0}^{N} \langle f_1, \phi_n\rangle\langle f_2, \phi_{n+1}\rangle\langle f_3, \phi_{n+2}\rangle\right| \le \sup_n \langle f_1, \phi_n\rangle \prod_{j=1}^{2}\left(\sum |\langle f_j, \phi_{n+j+1}\rangle|^2\right)^{1/2}$$

$$\le \|f_1\|_1 \|f_2\|_2 \|f_3\|_2$$

interpolation between 1 and 2: $\lesssim \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}$

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \le 2, \quad 1 \le p_1 \le 2$$

Sparse domination and weak type

Case $m=2$, $p_1 = p_2 = 1$

$\|T\|_{1,-n_1,\infty} \lesssim \|\langle T\cdot, \cdot\rangle\|_{1,1}$ $\quad E = \{|Tf| > \lambda\}$, $|E'| \ge \frac{1}{2}|E|$

$$\lambda|E| \lesssim \int_{E'} |Tf|\,dx \lesssim \|T\|_{(1,1)} \int_{E''} M_1 f M_1 1_E\,dx$$

E'' special: $\|Mf\|_{L^\infty(E')} \lesssim |E|^{-1}$