Boundary Layers in Periodic Homogenization

Zhongwei Shen

University of Kentucky

Joint work with Jinping Zhuge

Supported in part by the NSF

Elliptic Operators with Rapidly Oscillating Coefficients

Consider a family of elliptic operators

$$\mathcal{L}_\varepsilon = -\mathrm{div}\big(A(x/\varepsilon)\nabla\big) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \qquad \varepsilon > 0$$

where

$$A = A(y) = \big(a_{ij}^{\alpha\beta}(y)\big)$$

with

$$1 \leq i, j \leq d \quad \text{and} \quad 1 \leq \alpha, \beta \leq m$$

Assume that

- A is real and uniformly elliptic
- A is 1-periodic
- A is smooth

Elliptic Operators with Rapidly Oscillating Coefficients

Consider a family of elliptic operators

$$\mathcal{L}_\varepsilon = -\operatorname{div}\big(A(x/\varepsilon)\nabla\big) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \qquad \varepsilon > 0$$

where

$$A = A(y) = \big(a_{ij}^{\alpha\beta}(y)\big)$$

with

$$1 \le i, j \le d \quad \text{and} \quad 1 \le \alpha, \beta \le m$$

Assume that

- A is real and uniformly elliptic
- A is 1-periodic
- A is smooth

Elliptic Operators with Rapidly Oscillating Coefficients

Consider a family of elliptic operators

$$\mathcal{L}_\varepsilon = -\mathrm{div}\big(A(x/\varepsilon)\nabla\big) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \qquad \varepsilon > 0$$

where

$$A = A(y) = \big(a_{ij}^{\alpha\beta}(y)\big)$$

with

$$1 \le i, j \le d \quad \text{and} \quad 1 \le \alpha, \beta \le m$$

Assume that

- A is real and uniformly elliptic
- A is 1-periodic
- A is smooth

Strongly Inhomogeneous Medium

- Media with rapidly oscillating and "self-similar" microstructure, such as composite materials,

$$A^{\varepsilon}(x) = A(x/\varepsilon),$$

$$\varepsilon > 0 \qquad \text{microscopic scale}$$

- $A(y)$ could be periodic, quasi-periodic, almost-periodic, or a realization of a stationary random field
- Direct computation of the characteristics of the medium may be costly
- Homogenization theory:
  Use asymptotic analysis to find effective (averaged, homogenized) characteristics

Strongly Inhomogeneous Medium

- Media with rapidly oscillating and "self-similar" microstructure, such as composite materials,

$$A^{\varepsilon}(x) = A(x/\varepsilon),$$

$$\varepsilon > 0 \qquad \text{microscopic scale}$$

- $A(y)$ could be periodic, quasi-periodic, almost-periodic, or a realization of a stationary random field
- Direct computation of the characteristics of the medium may be costly
- Homogenization theory:
  Use asymptotic analysis to find effective (averaged, homogenized) characteristics

Strongly Inhomogeneous Medium

- Media with rapidly oscillating and "self-similar" microstructure, such as composite materials,

$$A^\varepsilon(x) = A(x/\varepsilon),$$

$$\varepsilon > 0 \qquad \text{microscopic scale}$$

- $A(y)$ could be periodic, quasi-periodic, almost-periodic, or a realization of a stationary random field
- Direct computation of the characteristics of the medium may be costly
- Homogenization theory:
  Use asymptotic analysis to find effective (averaged, homogenized) characteristics

Strongly Inhomogeneous Medium

- Media with rapidly oscillating and "self-similar" microstructure, such as composite materials,

$$A^{\varepsilon}(x) = A(x/\varepsilon),$$

$$\varepsilon > 0 \qquad \text{microscopic scale}$$

- $A(y)$ could be periodic, quasi-periodic, almost-periodic, or a realization of a stationary random field
- Direct computation of the characteristics of the medium may be costly
- Homogenization theory:
  Use asymptotic analysis to find effective (averaged, homogenized) characteristics

Homogenization of Boundary Value Problems

Suppose

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega \\ u_\varepsilon = f \quad \text{or} \quad \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega \end{cases}$$

As $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega)$$

where

$$\begin{cases} \mathcal{L}_0(u_0) = F & \text{in } \Omega \\ u_0 = f \quad \text{or} \quad \dfrac{\partial u_0}{\partial \nu_0} = g & \text{on } \partial\Omega \end{cases}$$

and $\mathcal{L}_0$ is an elliptic operator with constant coefficients.

Theory of Homogenization

The strongly inhomogeneous medium with rapidly oscillating microstructure, such as composite material, may be approximately described via an effective homogeneous medium.

$$\mathcal{L}_0 = -\frac{\partial}{\partial x_i}\widehat{a}_{ij}^{\alpha\beta}\frac{\partial}{\partial x_j}$$

$-$ homogenized (effective) operator

$$\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta}) - \text{ homogenized (effective) coefficients}$$

Theorem (Convergence Rate)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\,\varepsilon\|u_0\|_{H^2(\Omega)}$$

Higher-order Convergence

Consider

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi_j(x/\varepsilon)\frac{\partial u_0}{\partial x_j} - \varepsilon^2\chi_{ij}(x/\varepsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_j}$$

(two-scale expansion), where

$$\begin{aligned} \chi_j(y) &\quad - \text{first-order correctors} \\ \chi_{ij}(y) &\quad - \text{second-order correctors} \end{aligned}$$

Then

$$\mathcal{L}_\varepsilon(w_\varepsilon) = \varepsilon c_{ijk}(x/\varepsilon)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} + \varepsilon^2 \mathrm{div}(G(x/\varepsilon)\nabla^3 u_0)$$

Higher-order Convergence (Dirichlet Problem)

To correct the boundary discrepancy, introduce

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = c_{ijk}(x/\varepsilon)\dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} & \text{in } \Omega \\ v_\varepsilon = -\chi_j(x/\varepsilon)\dfrac{\partial u_0}{\partial x_j} & \text{on } \partial\Omega \end{cases}$$

Then

$$\|u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)\nabla u_0 - \varepsilon v_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon^2 \|u_0\|_{W^{3,\infty}(\Omega)}$$

what happens to v_ε, as $\varepsilon \to 0$?

Dirichlet Problem with Oscillating Data

Consider

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = f(x, x/\varepsilon) & \text{on } \partial\Omega \end{cases}$$

where

$$f(x, y) \text{ is 1-periodic in } y \in \mathbb{R}^d$$

Question:

Does u_ε have a limit u_0 in $L^2(\Omega)$, as $\varepsilon \to 0$?
What is the homogenized problem for the limit u_0?
Convergence rate?

The answers depend on the geometry of $\partial\Omega$, even for operators with constant coefficients.

Dirichlet Problem with Oscillating Data

- First-order correction for eigenvalues, F. Santosa - M. Vogelius (1993), S. Moskow - M. Vogelius (1997)
- Boundary layers for rectangular domains, G. Allaire - M. Amar (1999)
- Convex polygonal domains, strictly convex domains, homogenized data and their regularities, convergence rates, D. Gérard-Varet - N. Masmoudi (2011, 2012)
- Sharp rates for systems with constant coefficients, H. Aleksanyan - H. Shahgholian - P. Sjölin (2013 - 2015)
- Sharp convergence rates (for $d \geq 4$), S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)
- Fully nonlinear equations, W. Feldman (2014), W. Feldman - I. Kim - P. Souganidis (2015)

Dirichlet Problem with Oscillating Data

- First-order correction for eigenvalues, F. Santosa - M. Vogelius (1993), S. Moskow - M. Vogelius (1997)
- Boundary layers for rectangular domains,
  G. Allaire - M. Amar (1999)
- Convex polygonal domains, strictly convex domains, homogenized data and their regularities, convergence rates,
  D. Gérard-Varet - N. Masmoudi (2011, 2012)
- Sharp rates for systems with constant coefficients,
  H. Aleksanyan - H. Shahgholian - P. Sjölin (2013 - 2015)
- Sharp convergence rates (for $d \geq 4$),
  S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)
- Fully nonlinear equations, W. Feldman (2014),
  W. Feldman - I. Kim - P. Souganidis (2015)

Dirichlet Problem with Oscillating Data

- First-order correction for eigenvalues, F. Santosa - M. Vogelius (1993), S. Moskow - M. Vogelius (1997)
- Boundary layers for rectangular domains,
  G. Allaire - M. Amar (1999)
- Convex polygonal domains, strictly convex domains, homogenized data and their regularities, convergence rates,
  D. Gérard-Varet - N. Masmoudi (2011, 2012)
- Sharp rates for systems with constant coefficients,
  H. Aleksanyan - H. Shahgholian - P. Sjölin (2013 - 2015)
- Sharp convergence rates (for $d \geq 4$),
  S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)
- Fully nonlinear equations, W. Feldman (2014),
  W. Feldman - I. Kim - P. Souganidis (2015)

Dirichlet Problem with Oscillating Data

- First-order correction for eigenvalues, F. Santosa - M. Vogelius (1993), S. Moskow - M. Vogelius (1997)
- Boundary layers for rectangular domains,
  G. Allaire - M. Amar (1999)
- Convex polygonal domains, strictly convex domains, homogenized data and their regularities, convergence rates,
  D. Gérard-Varet - N. Masmoudi (2011, 2012)
- Sharp rates for systems with constant coefficients,
  H. Aleksanyan - H. Shahgholian - P. Sjölin (2013 - 2015)
- Sharp convergence rates (for $d \geq 4$),
  S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)
- Fully nonlinear equations, W. Feldman (2014),
  W. Feldman - I. Kim - P. Souganidis (2015)

Dirichlet Problem with Oscillating Data

- First-order correction for eigenvalues, F. Santosa - M. Vogelius (1993), S. Moskow - M. Vogelius (1997)
- Boundary layers for rectangular domains, G. Allaire - M. Amar (1999)
- Convex polygonal domains, strictly convex domains, homogenized data and their regularities, convergence rates, D. Gérard-Varet - N. Masmoudi (2011, 2012)
- Sharp rates for systems with constant coefficients, H. Aleksanyan - H. Shahgholian - P. Sjölin (2013 - 2015)
- Sharp convergence rates (for $d \geq 4$), S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)
- Fully nonlinear equations, W. Feldman (2014), W. Feldman - I. Kim - P. Souganidis (2015)

Dirichlet Problem with Oscillating Data

- First-order correction for eigenvalues, F. Santosa - M. Vogelius (1993), S. Moskow - M. Vogelius (1997)
- Boundary layers for rectangular domains,
  G. Allaire - M. Amar (1999)
- Convex polygonal domains, strictly convex domains, homogenized data and their regularities, convergence rates,
  D. Gérard-Varet - N. Masmoudi (2011, 2012)
- Sharp rates for systems with constant coefficients,
  H. Aleksanyan - H. Shahgholian - P. Sjölin (2013 - 2015)
- Sharp convergence rates (for $d \geq 4$),
  S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)
- Fully nonlinear equations, W. Feldman (2014),
  W. Feldman - I. Kim - P. Souganidis (2015)

Higher-order Convergence for Neumann Problems

The two-scale expansion leads to the Neumann problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \varepsilon^{-1} n_i(x) b_{ij}(x/\varepsilon) \dfrac{\partial u_0}{\partial x_j} & \text{on } \partial\Omega \end{cases}$$

where $b_{ij}(y)$ is 1-periodic,

$$\int_{\mathbb{T}^d} b_{ij} = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i}(b_{ij}) = 0$$

Write $b_{ij} = \frac{\partial}{\partial y_k} \phi_{kij}$, where ϕ is 1-periodic, and

$$\phi_{kij} = -\phi_{ikj}$$

Neumann Problems with First-order Oscillating Boundary Data

Consider

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = T_{ij} \cdot \nabla_x \{g_{ij}(x, x/\varepsilon)\} + g_0(x, x/\varepsilon) - \gamma_\varepsilon & \text{on } \partial\Omega \end{cases} \tag{1}$$

where

$T_{ij} = n_i e_j - n_j e_i$ — a tangential vector field on $\partial\Omega$

$\dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = n \cdot A(x/\varepsilon)\nabla u_\varepsilon$

- the conormal derivative associated with $\mathcal{L}_\varepsilon$

$g_{ij}(x, y), g_0(x, y)$ are 1-periodic in y

Energy Estimates for Dirichlet and Neumann Problems

We have

$$\|u_\varepsilon\|_{H^1(\Omega)} \sim \varepsilon^{-\frac{1}{2}}$$

Also,

$$\|u_\varepsilon\|_{H^1(K)} \leq C, \qquad \|u_\varepsilon\|_{H^{1/2}(\Omega)} \leq C$$

In fact,

$$\int_\Omega |\nabla u_\varepsilon(x)|^2 \, \mathrm{dist}(x, \partial\Omega)\, dx \leq C$$

(C. Kenig - S., Kenig - F. Lin - S.)

Homogenization for Neumann Problems

Assume that

$$\Omega \text{ is smooth and strictly convex}$$

We show that

$$u_\varepsilon \to u_0 \quad \text{strongly in } L^2(\Omega)$$

where

$$\begin{cases} \mathcal{L}_0(u_0) = 0 & \text{in } \Omega \\ \dfrac{\partial u_0}{\partial \nu_0} = T_{ij} \cdot \nabla_x \overline{g}_{ij} + \langle g_0 \rangle - \gamma_0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

with

$$\langle g_0 \rangle(x) = \fint_{\mathbb{T}^d} g_0(x, y)\, dy$$

The formula for $\{\overline{g}_{ij}\}$ on $\partial\Omega$ is given explicitly. Its value at $x \in \partial\Omega$ depends only on A, $\{g_{ij}(x, \cdot)\}$, and the unit normal n to $\partial\Omega$ at x.

Sharp Convergence Rates

Theorem (S. - J. Zhuge, CPAM)

Let Ω be a bounded smooth, strictly convex domain in $\mathbb{R}^d$, $d \geq 3$. Let u_ε and u_0 be solutions of (1) and (2), respectively, with

$$\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0.$$

Then for any $\sigma \in (0, 1/2)$ and $\varepsilon \in (0, 1)$,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C_\sigma \, \varepsilon^{\frac{1}{2} - \sigma},$$

where C_σ depends only on d, m, σ, A, Ω, and $g = \{g_0, g_{ij}\}$.

Regularity for Homogenized Data

Theorem (S. - Zhuge)

Under the same assumptions on A and Ω, the homogenized data $\overline{g} = \{\overline{g}_{ij}\}$ in (2) satisfy

$$\|\overline{g}\|_{W^{1,q}(\partial\Omega)} \le C_q \sup_{y\in\mathbb{T}^d} \|g(\cdot, y)\|_{C^1(\partial\Omega)} \qquad \text{for any } q < d-1,$$

where C_q depends only on d, m, μ, q and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d) \ge 1$.

Boundary Layers for Neumann Problems

There exists $\Omega_\varepsilon = \Omega_{\varepsilon,\sigma}$ such that

$$\left\{x \in \Omega : \ \delta(x) \leq c_0 \varepsilon\right\} \subset \Omega_\varepsilon \subset \left\{x \in \Omega : \ \delta(x) \leq c_1 \sqrt{\varepsilon}\right\}$$

with

$$|\Omega_\varepsilon| \leq C\, \varepsilon^{1-\sigma}$$

and for $x \in \Omega \setminus \Omega_\varepsilon$,

$$|u_\varepsilon(x) - u_0(x)| \leq C\, \varepsilon^{\frac{1}{2}-\sigma} \int_{\partial\Omega} \frac{\left[M_{\partial\Omega}(\kappa^{-q})(y)\right]^{\frac{1-\rho}{q}}}{|x-y|^{d-1}}\, dy$$

where $1 < q < d-1$ and $\rho \in (0,1)$.

Boundary Layers

$$\Omega_\varepsilon = \cup_j B(x_j, r_j) \cap \Omega$$

with $x_j \in \partial\Omega$ and $c_0\varepsilon \le r_j \le c_1\sqrt{\varepsilon}$,

$$r_j \sim \varepsilon^{1-\sigma} \Big/ \left(\fint_{B(x_j, r_j)\cap\partial\Omega} \kappa^p \right)^{1/p}, \qquad p > d-1$$

where $\kappa(x)$ is defined by

$$|(I - n(x) \otimes n(x))\xi| \ge \kappa |\xi|^{-2} \quad \text{for any } \xi \in \mathbb{Z}^d \setminus \{0\}$$

It is known

$$\frac{1}{\kappa} \in L^{d-1,\infty}(\partial\Omega) \qquad \text{if } \Omega \text{ is convex}$$

Dirichlet Problem with Oscillating Data

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = f(x, x/\varepsilon) & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $f(x,y)$ is smooth in $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and 1-periodic in y.

- D. Gérard-Varet - N. Masmoudi (JEMS, 2011) (Acta Math, 2012). Homogenization and convergence rates,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\,\varepsilon^{\frac{d-1}{3d+5}-},$$

where Ω is smooth and strictly convex,

$$\mathcal{L}_0(u_0) = 0 \quad \text{in } \Omega \quad \text{and} \quad u_0 = \bar{f}(x) \quad \text{on } \partial\Omega,$$

and the homogenized data $\bar{f}$ is identified.

Optimal Rates for Dirichlet Problem

- S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq \begin{cases} C\,\varepsilon^{\frac{1}{2}-} & \text{for } d \geq 4 \\ C\,\varepsilon^{\frac{1}{3}-} & \text{for } d = 3 \\ C\,\varepsilon^{\frac{1}{6}-} & \text{for } d = 2 \end{cases}$$

- S. - Zhuge (2016)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq \begin{cases} C\,\varepsilon^{\frac{1}{2}-} & \text{for } d = 3 \\ C\,\varepsilon^{\frac{1}{4}-} & \text{for } d = 2 \end{cases}$$

- Zhuge (2016), general domains of finite type

Optimal Rates for Dirichlet Problem

- S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le \begin{cases} C\,\varepsilon^{\frac{1}{2}-} & \text{for } d \ge 4 \\ C\,\varepsilon^{\frac{1}{3}-} & \text{for } d = 3 \\ C\,\varepsilon^{\frac{1}{6}-} & \text{for } d = 2 \end{cases}$$

- S. - Zhuge (2016)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le \begin{cases} C\,\varepsilon^{\frac{1}{2}-} & \text{for } d = 3 \\ C\,\varepsilon^{\frac{1}{4}-} & \text{for } d = 2 \end{cases}$$

- Zhuge (2016), general domains of finite type

Optimal Rates for Dirichlet Problem

- S.N. Armstrong - T. Kuusi - J.C. Mourrat - C. Prange (2016)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le \begin{cases} C\,\varepsilon^{\frac{1}{2}-} & \text{for } d \ge 4 \\ C\,\varepsilon^{\frac{1}{3}-} & \text{for } d = 3 \\ C\,\varepsilon^{\frac{1}{6}-} & \text{for } d = 2 \end{cases}$$

- S. - Zhuge (2016)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le \begin{cases} C\,\varepsilon^{\frac{1}{2}-} & \text{for } d = 3 \\ C\,\varepsilon^{\frac{1}{4}-} & \text{for } d = 2 \end{cases}$$

- Zhuge (2016), general domains of finite type

Regularity for Homogenized Data $\overline{f}$

$$\overline{f} \in W^{1,q}(\partial\Omega)$$

- Gérard-Varet - Masmoudi, $q < \frac{d-1}{2}$
- Armstrong - Kuusi - Mourrat - Prange, $q < \frac{2(d-1)}{3}$
- S. - Zhuge, $q < d - 1$
  (for both Dirichlet and Neumann problems)

The $O(\sqrt{\varepsilon})$ convergence rate is sharp even for operators with constant coefficients
(H. Aleksanyan - H. Shahgholian - P. Sjölin, 2013-2015)

Regularity for Homogenized Data $\overline{f}$

$$\overline{f} \in W^{1,q}(\partial\Omega)$$

- Gérard-Varet - Masmoudi, $q < \frac{d-1}{2}$
- Armstrong - Kuusi - Mourrat - Prange, $q < \frac{2(d-1)}{3}$
- S. - Zhuge, $q < d - 1$
  (for both Dirichlet and Neumann problems)

The $O(\sqrt{\varepsilon})$ convergence rate is sharp even for operators with constant coefficients
(H. Aleksanyan - H. Shahgholian - P. Sjölin, 2013-2015)

Regularity for Homogenized Data $\bar{f}$

$$\bar{f} \in W^{1,q}(\partial\Omega)$$

- Gérard-Varet - Masmoudi, $q < \frac{d-1}{2}$
- Armstrong - Kuusi - Mourrat - Prange, $q < \frac{2(d-1)}{3}$
- S. - Zhuge, $q < d - 1$
  (for both Dirichlet and Neumann problems)

The $O(\sqrt{\varepsilon})$ convergence rate is sharp even for operators with constant coefficients
(H. Aleksanyan - H. Shahgholian - P. Sjölin, 2013-2015)

An Approach to Dirichlet Problem (Armstrong - Kuusi - Mourrat - Prange)

$$u_\varepsilon(x) = \int_{\partial\Omega} P_\varepsilon(x,y) f(y, y/\varepsilon)\, dy$$

Homogenization of Poisson Kernels

M. Avellaneda - Lin (1989), Kenig - Lin - S. (2014)

$$= \int_{\partial\Omega} P_0(x,y) f(y, y/\varepsilon) \omega_\varepsilon(y)\, dy + \text{ error}$$

Calderón-Zygmund decomposition on $\partial\Omega$ adapted to $\kappa(x)$

$$= \sum_j \int_{\Delta_j} \eta_j(y) P_0(x,y) f(y_j, y/\varepsilon) \omega_\varepsilon(y)\, dy + \text{error}$$

Approximation by solutions in half-spaces

$$= \sum_j \int_{\partial\mathbb{H}_j} \eta_j(y) P_0(x,y) f(y_j, y/\varepsilon) \widetilde{\omega}(y_j, y/\varepsilon)\, dy + \text{error}$$

Continued

$$u_\varepsilon(x) = \sum_j \int_{\partial \mathbb{H}_j} \eta_j(y) P_0(x,y) f(y_j, y/\varepsilon) \widetilde{\omega}(y_j, y/\varepsilon)\, dy + \text{error}$$

mean values of quasi-periodic functions

$$= \sum_j \int_{\partial \mathbb{H}_j} \eta_j(y) P_0(x,y) \overline{f(y_j, \cdot)\widetilde{\omega}(y_j, \cdot)} dy + \text{ error}$$

regularity of the homogenized data

$$= \sum_j \int_{\Delta_j} \eta_j(y) P_0(x,y) \overline{f(y, \cdot)\widetilde{\omega}(y, \cdot)} dy + \text{ error}$$

$$= \int_{\partial \Omega} P_0(x,y) \overline{f(y, \cdot)\widetilde{\omega}(y, \cdot)} dy + \text{ error}$$

A Similar and Improved Approach to Neumann Problems

- Homogenization of first-order derivatives of Neumann functions (Kenig - Lin - S., 2014).
- Construction and optimal estimates of solutions to Neumann problems in half-spaces with periodic data.
- Approximation of oscillating factors by solutions in half-spaces.

A Similar and Improved Approach to Neumann Problems

- Homogenization of first-order derivatives of Neumann functions (Kenig - Lin - S., 2014).
- Construction and optimal estimates of solutions to Neumann problems in half-spaces with periodic data.
- Approximation of oscillating factors by solutions in half-spaces.

A Similar and Improved Approach to Neumann Problems

- Homogenization of first-order derivatives of Neumann functions (Kenig - Lin - S., 2014).
- Construction and optimal estimates of solutions to Neumann problems in half-spaces with periodic data.
- Approximation of oscillating factors by solutions in half-spaces.

Homogenization of Neumann Functions
Kenig - Lin - S. (CPAM, 2014)

Let $N_\varepsilon(x, y)$ be the matrix of Neumann functions for $\mathcal{L}_\varepsilon$ in Ω. Then

$$|N_\varepsilon(x,y) - N_0(x,y)| \le \frac{C\,\varepsilon \ln\left[\varepsilon^{-1}|x-y|+2\right]}{|x-y|^{d-1}},$$

$$|\nabla_y \big\{N(x,y)\big\}^T - \nabla_y \Psi_\varepsilon^*(y) \nabla_y \big\{N_\varepsilon(x,y)\big\}^T| \le \frac{C_\sigma\,\varepsilon^{1-\sigma}}{|x-y|^{d-\sigma}},$$

for any $x, y \in \Omega$ and $\sigma \in (0,1)$,
where Ψ_ε^* denotes the Neumann corrector for $\mathcal{L}_\varepsilon^*$ in Ω,

$$\mathcal{L}_\varepsilon^*(\Psi_\varepsilon^*) = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial}{\partial \nu_\varepsilon^*}(\Psi_\varepsilon^*) = \frac{\partial}{\partial \nu_0^*}(x) \quad \text{on } \partial\Omega.$$

Neumann Problems in a Half-Space

For $n \in \mathbb{S}^{d-1}$ and $a \in \mathbb{R}$, let

$$\mathbb{H}_n(a) = \{x \in \mathbb{R}^d : x \cdot n < -a\}.$$

For $g \in C^\infty(\mathbb{T}^d)$, consider the Neumann problem

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = 0 & \text{in } \mathbb{H}_n(a) \\ n \cdot A\nabla u = T \cdot \nabla g & \text{on } \partial\mathbb{H}_n(a) \end{cases}$$

where $T \in \mathbb{R}^d$, $|T| \leq 1$, and $T \cdot n = 0$.
Assume that n satisfies the Diophantine condition

$$|(I - n \otimes n)\xi| \geq \kappa|\xi|^{-2} \quad \text{for any } \xi \in \mathbb{Z}^d \setminus \{0\},$$

for some $\kappa > 0$.

Neumann Problems in a Half-Space

Let

$$u(x) = V(x - (x \cdot n)n, -x \cdot n)$$

where $V(\theta, t)$ is a function of $(\theta, t) \in \mathbb{T}^d \times [a, \infty)$. Then

$$\begin{cases} \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} V = 0 & \text{in } \mathbb{T}^d \times (a, \infty) \\ -e_{d+1} \cdot B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} V = T \cdot \nabla_\theta \widetilde{g} & \text{on } \mathbb{T}^d \times \{a\} \end{cases}$$

where $Me_d = -n$,

$$\begin{aligned} &B = B(\theta, t) = M^T A(\theta - tn) M, \\ &\widetilde{g}(\theta, t) = g(\theta - tn), \\ &NN^T + n \otimes n = I. \end{aligned}$$

Existence and Preliminary Estimates

Theorem (S. - Zhuge, 2016)

The Neumann problem has a smooth solution u satisfying

$$|u(x)| \leq \frac{C_{\alpha,\ell}}{\kappa(1+\kappa|x\cdot n + a|)^{\ell}},$$
$$|\partial_x^\alpha u(x)| \leq \frac{C_{\alpha,\ell}}{(1+\kappa|x\cdot n + a|)^{\ell}},$$

for any $|\alpha| \geq 1$ *and* $\ell \geq 1$.

Note that

$$\text{dist}(x, \partial\mathbb{H}_n(a)) = |x\cdot n + a|$$

A Refined Estimate

Theorem (S. - Zhuge, 2016)

The solution given by the last theorem satisfies

$$|\nabla u(x)| \le \frac{C\,\|g\|_\infty}{|x \cdot n + a|} \qquad \text{for any } x \in \mathbb{H}_n(a),$$

where C depends only on d, m, the ellipticity constant μ, and some Hölder norm of A.

- The proof uses the representation by Neumann functions and integration by parts on the boundary $\partial\Omega$.
- This theorem plays the same role as the maximum principle in the Dirichlet problem.

A Refined Estimate

Theorem (S. - Zhuge, 2016)

The solution given by the last theorem satisfies

$$|\nabla u(x)| \leq \frac{C \, \|g\|_\infty}{|x \cdot n + a|} \qquad \textit{for any } x \in \mathbb{H}_n(a),$$

where C depends only on d, m, the ellipticity constant μ, and some Hölder norm of A.

- The proof uses the representation by Neumann functions and integration by parts on the boundary $\partial\Omega$.
- This theorem plays the same role as the maximum principle in the Dirichlet problem.

A Refined Estimate

Theorem (S. - Zhuge, 2016)

The solution given by the last theorem satisfies

$$|\nabla u(x)| \leq \frac{C\,\|g\|_\infty}{|x \cdot n + a|} \qquad \text{for any } x \in \mathbb{H}_n(a),$$

where C depends only on d, m, the ellipticity constant μ, and some Hölder norm of A.

- The proof uses the representation by Neumann functions and integration by parts on the boundary $\partial\Omega$.
- This theorem plays the same role as the maximum principle in the Dirichlet problem.

Approximation of Neumann Correctors

Fix $x_0 \in \partial\Omega$. For $g \in C^\infty(\mathbb{T}^d)$, approximate the solution to the Neumann problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = T(x) \cdot \nabla g(x/\varepsilon) & \text{on } \partial\Omega \end{cases}$$

where $T(x) = n_i(x) e_j - n_j(x) e_i$, by a solution to

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = 0 & \text{in } \mathbb{H}_n(a) \\ \dfrac{\partial v_\varepsilon}{\partial \nu_\varepsilon} = T(x_0) \cdot \nabla g(x/\varepsilon) & \text{on } \partial\mathbb{H}_n(a) \end{cases}$$

where $a = -x_0 \cdot n$ and $\partial\mathbb{H}_n(a)$ is the tangent plane of $\partial\Omega$ at x_0.

Continued

Theorem (S. - Zhuge, 2016)

Let

$$\varepsilon \le r \le \sqrt{\varepsilon} \quad \textit{and} \quad \sigma \in (0,1).$$

Then

$$\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B(x_0,r)\cap\Omega)} \le C\sqrt{\varepsilon}\big\{1 + |\ln \varepsilon|\big\} + C\varepsilon^{-1-\sigma} r^{2+\sigma}.$$

- Use representation by Neumann functions, boundary Lipschitz estimates, and lots of integration by parts on the boundary.
- Similar estimates were obtained for Dirichlet problem by Armstrong - Kuusi - Mourrat - Prange.

Continued

Theorem (S. - Zhuge, 2016)

Let

$$\varepsilon \le r \le \sqrt{\varepsilon} \quad \textit{and} \quad \sigma \in (0,1).$$

Then

$$\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B(x_0,r)\cap\Omega)} \le C\sqrt{\varepsilon}\{1 + |\ln \varepsilon|\} + C\varepsilon^{-1-\sigma} r^{2+\sigma}.$$

- Use representation by Neumann functions, boundary Lipschitz estimates, and lots of integration by parts on the boundary.
- Similar estimates were obtained for Dirichlet problem by Armstrong - Kuusi - Mourrat - Prange.

Continued

Theorem (S. - Zhuge, 2016)

Let

$$\varepsilon \le r \le \sqrt{\varepsilon} \quad \textit{and} \quad \sigma \in (0,1).$$

Then

$$\begin{aligned} &\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B(x_0,r)\cap\Omega)} \\ &\quad \le C\sqrt{\varepsilon}\big\{1 + |\ln\varepsilon|\big\} + C\varepsilon^{-1-\sigma} r^{2+\sigma}. \end{aligned}$$

- Use representation by Neumann functions, boundary Lipschitz estimates, and lots of integration by parts on the boundary.
- Similar estimates were obtained for Dirichlet problem by Armstrong - Kuusi - Mourrat - Prange.

Approximation of Neumann Correctors

Let

$$\phi_\varepsilon^*(x) = \Psi_\varepsilon^*(x) - x - \varepsilon\chi^*(x/\varepsilon).$$

Then, for any $x \in B(x_0, r) \cap \Omega$,

$$|\nabla\Big(\phi_\varepsilon^*(x) - \varepsilon V^* \left(\frac{x - (x \cdot n + a)n}{\varepsilon}, -\frac{x \cdot n + a}{\varepsilon}\right)|$$
$$\leq C\sqrt{\varepsilon}\big\{1 + |\ln \varepsilon|\big\} + C\,\varepsilon^{-1-\sigma} r^{2+\sigma},$$

where

$$\varepsilon \leq r \leq \sqrt{\varepsilon} \quad \text{and} \quad \sigma \in (0, 1/2)$$

Sharp Estimates for the Homogenized Data

Let $x, y \in \partial\Omega$ and $|x - y| \leq c_0$. Suppose $n(x)$ and $n(y)$ satisfy the Diophantine condition with constants $\kappa(x)$ and $\kappa(y)$, respectively. Let $\overline{g} = (\overline{g}_k^\beta)$ be the homogenized data. Then

$$|\overline{g}(x) - \overline{g}(y)| \leq \frac{C_\sigma |x-y|}{\kappa^{1+\sigma}} \left(\frac{|x-y|}{\kappa} + 1 \right) \sup_{z \in \mathbb{T}^d} \|g(\cdot, z)\|_{C^1(\partial\Omega)}$$

where

$$\kappa = \max(\kappa(x), \kappa(y)).$$

- The estimate also holds for homogenized data in the case of Dirichlet problem. This improved a result of Armstrong - Kuusi - Mourrat - Prange (by a power of $1/2$), which in turn improved an early work of Gérard-Varet - Masmoudi.
- The improvement for Dirichlet Problem leads to the sharp convergence rates for $d = 2$ or 3.

Sharp Estimates for the Homogenized Data

Let $x, y \in \partial\Omega$ and $|x - y| \leq c_0$. Suppose $n(x)$ and $n(y)$ satisfy the Diophantine condition with constants $\kappa(x)$ and $\kappa(y)$, respectively. Let $\overline{g} = (\overline{g}_k^\beta)$ be the homogenized data. Then

$$|\overline{g}(x) - \overline{g}(y)| \leq \frac{C_\sigma |x - y|}{\kappa^{1+\sigma}} \left(\frac{|x - y|}{\kappa} + 1 \right) \sup_{z \in \mathbb{T}^d} \|g(\cdot, z)\|_{C^1(\partial\Omega)}$$

where

$$\kappa = \max(\kappa(x), \kappa(y)).$$

- The estimate also holds for homogenized data in the case of Dirichlet problem. This improved a result of Armstrong - Kuusi - Mourrat - Prange (by a power of $1/2$), which in turn improved an early work of Gérard-Varet - Masmoudi.
- The improvement for Dirichlet Problem leads to the sharp convergence rates for $d = 2$ or 3.

Let $n, \widetilde{n} \in \mathbb{S}^{d-1}$. Show that

$$\int_{\mathbb{T}^d} |N_n^T \nabla_\theta \big(V_n^*(\theta, 0) - V_{\widetilde{n}}^*(\theta, 0) \big)| \, d\theta \leq \frac{C_\sigma |n - \widetilde{n}|}{\kappa^{1+\sigma}} \left(\frac{|n - \widetilde{n}|}{\kappa} + 1 \right),$$

where $\kappa > 0$ is the constant in the Diophantine condition for $\widetilde{n}$.
Let

$$W(\theta, t) = V_n^*(\theta, t) - V_{\widetilde{n}}^*(\theta, t).$$

By Sobolev imbedding it suffices to show that

$$\int_0^1 \int_{\mathbb{T}^d} \left\{ |N_n^T \nabla_\theta W|^2 + |\nabla_\theta \partial_t W|^2 \right\} d\theta dt \leq C_\sigma \left\{ \frac{|n - \widetilde{n}|^2}{\kappa^{2+\sigma}} + \frac{|n - \widetilde{n}|^4}{\kappa^{4+\sigma}} \right\},$$

for any $\sigma \in (0, 1)$.

Note that W is a solution of the Neumann problem,

$$-\begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} W = \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} G + H$$

in $\mathbb{T}^d \times \mathbb{R}_+$,

with

$$-e_{d+1} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} W = h + e_{d+1} \cdot G \quad \text{on } \mathbb{T}^d \times \{0\}$$

Weighted Estimates - Neumann Problem

Suppose that $n \in \mathbb{S}^{n-1}$ satisfies the Diophantine condition. Let U be a smooth solution of

$$\begin{cases} -\begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} U = \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} F & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} U = e_{d+1} \cdot F & \text{on } \mathbb{T}^d \times \{0\}. \end{cases}$$

Assume that

$$(1+t)\|\nabla_{\theta,t} U(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} + (1+t)\|F(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} < \infty.$$

Then, for any $-1 < \alpha < 0$,

$$\int_0^\infty \int_{\mathbb{T}^d} \left\{ |N_n^T \nabla_\theta U|^2 + |\partial_t U|^2 \right\} t^\alpha \, d\theta dt \le C_\alpha \int_0^\infty \int_{\mathbb{T}^d} |F|^2 \, t^\alpha \, d\theta dt,$$

where C_α depends only on d, m, μ, α as well as some Hölder norm of A.

Reduction to Weighted Estimates for Half-Spaces

Let

$$\Omega = \mathbb{H}_n(a) \quad \text{and} \quad \mathcal{L} = -\mathrm{div}(A(x)\nabla).$$

Consider the Dirichlet problem,

$$\begin{cases} \mathcal{L}(u) = \mathrm{div}(f) + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and the Neumann problem,

$$\begin{cases} \mathcal{L}(u) = \mathrm{div}(f) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = -n \cdot f & \text{on } \partial\Omega \end{cases}$$

We are interested in the weighted L^2 estimate,

$$\int_\Omega |\nabla u(x)|^2 \left[\delta(x)\right]^\alpha dx$$
$$\leq C \int_\Omega |f(x)|^2 \left[\delta(x)\right]^\alpha dx + C \int_\Omega |h(x)|^2 \left[\delta(x)\right]^{\alpha+2} dx,$$

where $-1 < \alpha < 0$ and

$$\delta(x) = \text{dist}(x, \partial\Omega) = |a + (x \cdot n)|.$$

- Use a weighted (and duel) version of the Calderón-Zygmund theory (Caffarelli - Peral (1998), Shen (2005), ...)
- Reduce the problem to a weak reverse Hölder inequality.
- If $-1 < \alpha < 0$, then $\omega_\alpha(x) = [\delta(x)]^\alpha$ is an A_1 weight.

We are interested in the weighted L^2 estimate,

$$\int_\Omega |\nabla u(x)|^2 \left[\delta(x)\right]^\alpha dx$$
$$\leq C \int_\Omega |f(x)|^2 \left[\delta(x)\right]^\alpha dx + C \int_\Omega |h(x)|^2 \left[\delta(x)\right]^{\alpha+2} dx,$$

where $-1 < \alpha < 0$ and

$$\delta(x) = \text{dist}(x, \partial\Omega) = |a + (x \cdot n)|.$$

- Use a weighted (and duel) version of the Calderón-Zygmund theory (Caffarelli - Peral (1998), Shen (2005), ...)
- Reduce the problem to a weak reverse Hölder inequality.
- If $-1 < \alpha < 0$, then $\omega_\alpha(x) = [\delta(x)]^\alpha$ is an A_1 weight.

We are interested in the weighted L^2 estimate,

$$\int_\Omega |\nabla u(x)|^2 \big[\delta(x)\big]^\alpha dx$$
$$\leq C \int_\Omega |f(x)|^2 \big[\delta(x)\big]^\alpha dx + C \int_\Omega |h(x)|^2 \big[\delta(x)\big]^{\alpha+2} dx,$$

where $-1 < \alpha < 0$ and

$$\delta(x) = \text{dist}(x, \partial\Omega) = |a + (x \cdot n)|.$$

- Use a weighted (and duel) version of the Calderón-Zygmund theory (Caffarelli - Peral (1998), Shen (2005), ...)
- Reduce the problem to a weak reverse Hölder inequality.
- If $-1 < \alpha < 0$, then $\omega_\alpha(x) = [\delta(x)]^\alpha$ is an A_1 weight.

Higher-order Convergence for Neumann Problems

Let u_ε be the solution to

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega$$

with $\int_\Omega u_\varepsilon = 0$.

Let u_0 be the solution of the homogenized problem.

Then there exists a function v^{bl}, independent of ε, such that

$$\|u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)\nabla u_0 - \varepsilon v^{bl}\|_{L^2(\Omega)} \le C_\sigma\, \varepsilon^{\frac{3}{2}-\sigma}\|u_0\|_{W^{3,\infty}(\Omega)},$$

for any $\sigma \in (0, 1/2)$, where C_σ depends only on d, m, σ, A and Ω.

Continued

The function v^{bl} is a solution to the Neumann problem

$$\mathcal{L}_0(v^{bl}) = F_* \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v^{bl}}{\partial \nu_0} = g_* \quad \text{on } \partial\Omega,$$

where

$$F_* = \overline{c}_{ki\ell} \frac{\partial^3 u_0}{\partial x_k \partial x_i \partial x_\ell}$$

for some constants $\overline{c}_{ki\ell}$, and g_* satisfies

$$\|g_*\|_{L^q(\partial\Omega)} \leq C_q \|u_0\|_{W^{2,\infty}(\Omega)},$$

for any $1 < q < d - 1$.

Thank You!