Quantitative differentiation

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Recent developments in harmonic analysis
MSRI, Berkeley, 15–19 May 2017

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$$\forall f \text{ with } \|f\|_{\mathrm{Lip(B_X,Y)}} := \sup_{x \neq y} \frac{\|f(x) - f(y)\|_Y}{\|x - y\|_X} \leq 1$$

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Also fails: $(\bigoplus_{n=1}^{\infty} \ell_n^q)_{\ell^2}$, $q \in \{1,\infty\}$ (reflexive but not super-)

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We prove: $r^{X \to Y}(\varepsilon) \geq \exp(-C_Y e^{cn+c_Y}/\varepsilon^{cn+c_Y})$ (e^{cn} vs. $n^{cn} = e^{cn \log n}$)

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Point: If an affine f is bi-Lipschitz on a fine enough net, then it is bi-Lipschitz on all of X. So need $f \approx$ affine on a large enough ball.

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$$\Rightarrow \exists \delta \geq r, \ x \in X, \ \text{ affine } P : X \to Y : \|P\|_{\text{Lip}} \leq 3\|f\|_{\text{Lip}},$$

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Below: case $\| \ \|_X = \| \ \|_2$.

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Idea: $K \ge \int_0^1 \phi(t)\frac{\mathrm{d}t}{t} \ge \int_r^1 \frac{\mathrm{d}t}{t}\inf_{[r,1]}\phi = \log\frac{1}{r}\inf_{[r,1]}\phi.$

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$$\Rightarrow \inf_{[r,1]}\phi \le \frac{K}{\log\frac{1}{r}} = \varepsilon \quad \text{if} \quad r = \exp\Big(-\frac{K}{\varepsilon}\Big).$$

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We need: extension to $\dim Y=\infty$, good control on $\dim\mathbb{R}^n=n$.

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D's proof: fractional spaces first; integer order by interpolation!

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Dorronsoro follows if $J : f \mapsto f(y) - P_{x,t}f(y)$ maps

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Each "$\eqsim$" and "$\hookrightarrow$" comes with an implicit constant to control!

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$\Rightarrow$ Dorronsoro with $n^c \Rightarrow r^{X\to Y}(\varepsilon) \geq \exp\Big(-\frac{e^{cn+c_Y}}{\varepsilon^{cn+c_Y}}\Big)$, $Y \in$ UMD

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(Higher order extension by T.H. & J. Merikoski (arXiv 2017):
$\dot{W}^{k,q} \hookrightarrow \dot{B}^k_{qq} \hookrightarrow k$th order local approx space.)

Proof sketch

$$\begin{aligned} f - P &= f - \mathrm{Taylor}_x^1\, e^{t^2\Delta} f \\ &= \left(f - e^{t^2\Delta} f\right) + \left(e^{t^2\Delta} f - \mathrm{Taylor}_x^1\, e^{t^2\Delta} f\right) = I + II \end{aligned}$$

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Both boil down to "Littlewood–Paley inequality"

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Proof sketch

$$f - P = f - \mathsf{Taylor}_x^1\, e^{t^2\Delta} f$$
$$= (f - e^{t^2\Delta} f) + (e^{t^2\Delta} f - \mathsf{Taylor}_x^1\, e^{t^2\Delta} f) = I + II$$

$$I = -t^2 \int_0^1 \Delta e^{t^2 s\Delta} f \,\mathrm{d}s = -t^2 \int_0^1 \mathrm{div}\, e^{t^2 s\Delta} \nabla f \,\mathrm{d}s$$

$$\frac{\|f - P\|_{L^q}}{t} \le \int_0^1 \|t \,\mathrm{div}\, e^{t^2 s\Delta}(\nabla f)\|_{L^q} \,\mathrm{d}s$$

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$$\sqrt{t}\nabla e^{t\Delta}f = \sum_{k=-1}^{\infty} 2^{-k/2}\sqrt{2^k t}\nabla e^{2^k t\Delta}(e^{2^k t\Delta} - e^{3\cdot 2^k t\Delta})f$$

Similar idea $\Rightarrow$ removal of subordination assumption from abstract Littlewood–Paley theory by Q. Xu (personal communication, 2016)!

The end

Thanks for your attention!