The hidden landscape of localization of eigenfunctions

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$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Dirichlet Laplacian eigenfunctions are uniformly distributed on boxes

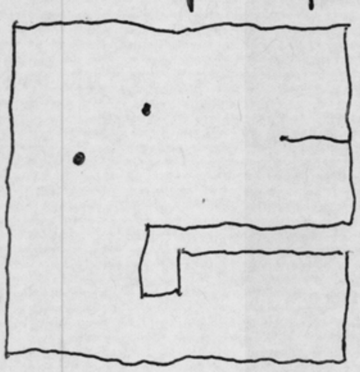
Questions: Given any domains will vibrations localize?
Which frequencies?
To what subregions? How severly?

For Bi-Laplacian - strong localization induced, clamping one point
random potential $-\Delta + V$

<u>Thm</u>: (Filoche, S. M. '12) Let L be a divergence form elliptic operator of order $2m$, $m \geq 1$, with complex bounded coefficients. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then, for any φ: $L\varphi = \lambda\varphi$, $\|\varphi\|_{L^\infty} = 1$, $\varphi \in \mathring{H}^m(\Omega)$,

$$|\varphi(x)| \leq \lambda u(x), \quad x \in \Omega$$

where $Lu = 1$, $u \in \mathring{H}^m(\Omega)$ (if the corresponding BVP satisfies the max principle) or more generally $u(x) = \int_\Omega |G(x,y)|\,dy$

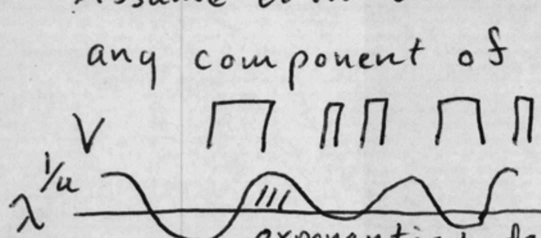


Laplacian / Bi-laplacian eigenfunctions localized
effective valley network
Anderson localization

$$u(x) = \sum_j \frac{1}{\lambda_j} \langle \varphi_j, 1 \rangle \varphi_j(x)$$

$1/u$ is the effective potential of quantum tunneling

<u>Thm</u>: (Arnold, David, Jerison, Mayboroda, Filoche, '15)
(Decay rate) $L = -\operatorname{div} A \nabla + V$ on a non-tangential (NTA) domain $L\Psi = \lambda\Psi$ and u solves $Lu = 1$ on Ω. Assume both u and Ψ have Neumann B.C. Let Ω_λ be any component of $\{x \in \Omega \mid 1/u(x) - \lambda > 0\}$ . . .



$1/u > \lambda$ penalized exponential region

exponential decay (governed by $1/u$)

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identity $\int_D |\nabla_A f|^2 + V f^2 dx = \int_D \frac{1}{u} f^2 + u^2 |\nabla_A (\frac{f}{u})|^2 dx + \int_{\partial D} \frac{\partial_\nu u}{u} f^2 d\sigma$

$$\int_D |\nabla \Psi|^2 + (V-\lambda)\Psi^2 dx = \int_D (\tfrac{1}{u} - \lambda)\Psi^2 + \text{positive and small}$$

to be compared with $\int |\nabla \Psi|^2 + (V-\lambda)\Psi^2 dx \geq \int (V-\lambda)_+ \Psi^2 dx$

Fefferman-Phong

$$m(x, V) = \inf_{r>0} \left\{ \frac{1}{r} \,\Big|\, \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}$$

Diagonalization theorem

Landscape function: knows both eigenfunctions and eigenvalues

Observation: $(\min 1/u)_j \approx c_n \lambda_j$ Density of states

Weyl Law

$N(\mu) \approx \mathrm{Vol}\{\xi^2 + V \leq \mu\}$ $N(\mu) \approx \mathrm{Vol}\{\xi^2 + 1/u \leq \mu\}$

V - Weyl law $\cancel{1/\mu}$ - Weyl law

Geometry of Ω_j's valleys as free boundary

David, Filoche, Jerison, S.M. '14 - 'A free boundary problem for the localization of eigenfunctions'

Valleys are free boundaries $\partial \Omega_i$ coming from minimizing

$$J(u, w) = \sum_{i=1}^{N} \int_{W_i} |\nabla u_i|^2 + u_i^2 f - u_i g \, dx + F(W_1, \ldots, W_N)$$

among all possible disjoint partitions $\Omega = \bigcup_{i=1}^{N} W_i$ and all u_i supported on W_i

- methods build from results of Alt, Cafarelli

Minimizers exist and are Lipschitz

Design GaN light emitting diode - Nobel prize '14

3D computation of QW absorption in InGaN with random alloy fluctuations