On approximations of functions in some critical Sobolev spaces

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An approximation theorem of Bourgain and Brezis

- We work on $\mathbb{R}^n$ where $n \geq 2$.
- Let $\dot{W}^{k,p}$ be the homogeneous Sobolev space on $\mathbb{R}^n$, that is the completion of the space of all C_c^∞ functions under the norm
$$\|f\|_{\dot{W}^{k,p}} := \|\nabla^k f\|_{L^p}$$
if $k \in \mathbb{N}$.
- It is well-known that $\dot{W}^{k,p}$ embeds into $L^{\frac{np}{n-kp}}$ if $1 \leq p < \frac{n}{k}$, and that the embedding fails if $p = \frac{n}{k}$
(e.g. $\dot{W}^{1,n}$ does not embed into L^∞ on $\mathbb{R}^n$).
- Nevertheless, Bourgain and Brezis proved the following remarkable theorem, that says a general $\dot{W}^{1,n}$ function can be 'well-approximated' by a bounded function on $\mathbb{R}^n$.

Theorem (Bourgain-Brezis)

Given any $\delta > 0$, there exists a constant C_δ such that for any function $f \in \dot{W}^{1,n}$, there exists a function $F \in L^\infty \cap \dot{W}^{1,n}$ satisfying

$$\sum_{i=1}^{n-1} \|\partial_i f - \partial_i F\|_{L^n} \leq \delta \|\nabla f\|_{L^n}$$

and

$$\|\nabla F\|_{L^n} + \|F\|_{L^\infty} \leq C_\delta \|\nabla f\|_{L^n}.$$

- The derivatives of F approximates the derivatives of f in all but one direction!
- This is the starting point of a long journey, and key to the proofs of many important results. We will list three shortly.

Outline of the talk

- Three consequences of this approximation theorem
- Some subsequent development
- A new approximation theorem for a whole range of critical Sobolev spaces $\dot{W}^{\alpha,p}(\mathbb{R}^n)$, where $\alpha p = n$; this is joint work with Pierre Bousquet, Emmanuel Russ and Yi Wang
- Indeed our theorem also works for a whole range of critical Triebel-Lizorkin spaces $\dot{F}^{\alpha,p}_q$ on $\mathbb{R}^n$, with $\alpha p = n$.

Three consequences of the approximation theorem for $\dot{W}^{1,n}$

- First comes the solution of the following PDE.

Proposition (Bourgain-Brezis)

For any function $f \in L^n$, there exists a vector field $Y \in \dot{W}^{1,n} \cap L^\infty$ such that

$$div\ Y = f$$

with $\|\nabla Y\|_{L^n} + \|Y\|_{L^\infty} \lesssim \|f\|_{L^n}$.

- By $\|\nabla Y\|_{L^n}$ we mean sum of the $\dot{W}^{1,n}$ norms of the components of Y; same for $\|Y\|_{L^\infty}$.
- Can always find $Y \in \dot{W}^{1,n}$ by Hodge decomposition, but $\dot{W}^{1,n}$ fails to embed into L^∞.
- But the equation is underdetermined: if Y is a solution, so is Y plus any divergence free vector field.
- The claim is one can find a solution that is not just in $\dot{W}^{1,n}$, but also bounded, by adding a divergence free vector field.

- More generally consider the Hodge-de Rham complex on $\mathbb{R}^n$.
- A differential ℓ-form on $\mathbb{R}^n$ is of the form

$$u = \sum_J u_J dx^J$$

where the sum is over all multiindices $J = (j_1, \ldots, j_\ell)$ of length ℓ, with $1 \leq j_1 < \cdots < j_\ell \leq n$,

$$dx^J := dx^{j_1} \wedge \cdots \wedge dx^{j_\ell},$$

and u_J is a function on $\mathbb{R}^n$ for each such J.
- The exterior derivative d maps ℓ-forms to $(\ell + 1)$ forms, via

$$du = \sum_{j=1}^{n} \frac{\partial u_J}{\partial x^j} dx^j \wedge dx^J$$

if u is as above.

- d maps ℓ-forms with $\dot{W}^{1,n}$ coefficients to $(\ell+1)$-forms with L^n coefficients.
- A $(\ell+1)$-form $f \in L^n$ is said to be in the image of d, if

$$f = dX$$

for some ℓ-form $X \in \dot{W}^{1,n}$.

Theorem (Bourgain-Brezis)

If $\ell \neq 0$, then for any $(\ell+1)$-form $f \in L^n$ that is in the image of d, there exists a ℓ-form $Y \in \dot{W}^{1,n} \cap L^\infty$ such that

$$dY = f$$

with $\|\nabla Y\|_{L^n} + \|Y\|_{L^\infty} \lesssim \|f\|_{L^n}$.

- The case $\ell = n-1$ is the earlier proposition about the equation $\operatorname{div} Y = f$.

- Next comes the following compensation phenomenon.
- For two Banach spaces A and B, their sum is a Banach space $A+B=\{a+b\colon a\in A, b\in B\}$, with norm

$$\|u\|_{A+B}=\inf\{\|a\|_A+\|b\|_B\colon u=a+b, a\in A, b\in B\}.$$

- For any locally integrable function v on $\mathbb{R}^n$, we define

$$\|v\|_{\dot{W}^{-1,\frac{n}{n-1}}}=\sup\left\{\int_{\mathbb{R}^n} v\,\varphi\,dx\colon \varphi\in C_c^\infty, \|\nabla\varphi\|_{L^n}=1\right\}.$$

- We can thus discuss the $L^1+\dot{W}^{-1,\frac{n}{n-1}}$ norm of a C^∞ function, or C^∞ vector field.

Theorem (Bourgain-Brezis)

If u is a C^∞ vector field and div $u = 0$, then for any vector field $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^n} u \cdot \Phi \, dx \right| \lesssim \|u\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}} \|\nabla \Phi\|_{L^n}.$$

- In particular, since $\|u\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}} \leq \|u\|_{L^1}$, this says

  $$\left| \int_{\mathbb{R}^n} u \cdot \Phi \, dx \right| \lesssim \|u\|_{L^1} \|\nabla \Phi\|_{L^n},$$

  if u is a divergence-free vector field on $\mathbb{R}^n$.
- The latter inequality would be trivial if $\dot{W}^{1,n}$ embeds into L^∞. So this is some remedy of failure of this critical Sobolev embedding when one test a $\dot{W}^{1,n}$ vector field against something divergence free (inequality fails otherwise).

Theorem (Bourgain-Brezis)

If u is a C^∞ vector field and div $u = 0$, then for any vector field $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^n} u \cdot \Phi \, dx \right| \lesssim \|u\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}} \|\nabla \Phi\|_{L^n}.$$

- In particular, since $\|u\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}} \leq \|u\|_{L^1}$, this says

  $$\left| \int_{\mathbb{R}^n} u \cdot \Phi \, dx \right| \lesssim \|u\|_{L^1} \|\nabla \Phi\|_{L^n},$$

  if u is a divergence-free vector field on $\mathbb{R}^n$.
- Van Schaftingen gave a simple and elegant proof of the latter inequality. This would also give a simple proof of a special case of the earlier proposition, namely a solution to the equation div $Y = f$ with $Y \in L^\infty$, if $f \in L^n$.
- But the proof of the full theorem remains quite involved.

- Finally comes a Gagliardo-Nirenberg inequality for differential forms.
- Recall Gagliardo-Nirenberg: If a function $u \in C_c^\infty(\mathbb{R}^n)$, then

$$\|u\|_{L^{n/(n-1)}} \lesssim \|\nabla u\|_{L^1}.$$

- Let d^* be the adjoint of d under the standard L^2 inner product of differential forms on $\mathbb{R}^n$.

Theorem (Bourgain-Brezis)

Suppose $0 \leq \ell \leq n-2$. *Then for any* ℓ*-form* $u \in C_c^\infty$ *with* $d^* u = 0$, *we have*

$$\|u\|_{L^{\frac{n}{n-1}}} \lesssim \|du\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}}.$$

Theorem (Bourgain-Brezis)

Suppose $0 \le \ell \le n-2$. *Then for any* ℓ*-form* $u \in C_c^\infty$ *with* $d^* u = 0$*, we have*

$$\|u\|_{L^{\frac{n}{n-1}}} \lesssim \|du\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}}.$$

- In particular, since $\|du\|_{L^1 + \dot{W}^{-1,\frac{n}{n-1}}} \le \|du\|_{L^1}$, we have

  $$\|u\|_{L^{\frac{n}{n-1}}} \lesssim \|du\|_{L^1}$$

  whenever u is a C_c^∞ ℓ-form on $\mathbb{R}^n$, $0 \le \ell \le n-2$, with $d^* u = 0$.
- Since d^* of a function is always zero, when $\ell = 0$ this is just Gagliardo-Nirenberg.
- Lanzani and Stein gave a proof of the latter inequality in a similar spirit of Van Schaftingen.

Some subsequent development (partial list)

- There has been lots of work around the L^1 inequalities we discussed.
- For example, the inequality

$$\left| \int_{\mathbb{R}^n} u \cdot \Phi \, dx \right| \lesssim \|u\|_{L^1} \|\nabla \Phi\|_{L^n} \quad \text{whenever } \operatorname{div} u = 0$$

  has been generalized in various ways:
- The norm $\|\nabla \Phi\|_{L^n}$ can be replaced by a critical Besov or Triebel-Lizorkin norm of Φ, such as $\|\Phi\|_{\dot{W}^{\alpha,p}}$ with $\alpha p = n$ (Van Schaftingen)
- $\mathbb{R}^n$ can be replaced by homogeneous groups, such as the Heisenberg group (Chanillo-Van Schaftingen)
- $\mathbb{R}^n$ can be replaced by any globally Riemannian symmetric spaces of non-compact type, such as the hyperbolic spaces or $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ (Chanillo-Van Schaftingen-Y.)
- But very few results along the lines of the full theorem, where one considers $L^1 + \dot{W}^{-1,\frac{n}{n-1}}$ in place of L^1.

New results

- Joint work with Pierre Bousquet, Emmanuel Russ, Yi Wang
- We prove an approximation theorem not just for $\dot{W}^{1,n}$, but for a range of Sobolev or Triebel-Lizorkin spaces on $\mathbb{R}^n$ that barely fail to embed into L^∞.
- Let $\mathcal{S}'$ be the space of tempered distributions on $\mathbb{R}^n$, and $\mathcal{P}$ be the subspace of all polynomials on $\mathbb{R}^n$.
- We write $\mathcal{Z}'$ for the quotient space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$.
- Pick a Schwartz function Δ on $\mathbb{R}^n$, so that $\widehat{\Delta}$ is supported on the annulus $1/2 \leq |\xi| \leq 2$, and $\sum_{j\in\mathbb{Z}} \widehat{\Delta}(2^{-j}\xi) = 1$.
- For $\alpha \in \mathbb{R}$ and $p, q \in (1, \infty)$, we say that f is in the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha,p}_q$, if $f \in \mathcal{Z}'$ and

$$\|f\|_{\dot{F}^{\alpha,p}_q} := \left\| \left\| 2^{\alpha j} \Delta_j f(x) \right\|_{\ell^q} \right\|_{L^p} < \infty,$$

  where $\Delta_j f(x) := f * \Delta_j(x)$ and $\Delta_j(x) := 2^{jn} \Delta(2^j x)$.

$$\|f\|_{\dot{F}_q^{\alpha,p}} := \left\| \left\| 2^{\alpha j} \Delta_j f(x) \right\|_{\ell^q} \right\|_{L^p}$$

- When $\alpha = k \in \mathbb{N}$ and $q = 2$, the space $\dot{F}_q^{\alpha,p}$ is isomorphic to the homogeneous Sobolev space $\dot{W}^{k,p}$, with

$$\|f\|_{\dot{F}_2^{k,p}} \simeq \|\nabla^k f\|_{L^p}.$$

- When $\alpha \in (0,1)$ and $q = p$, the space $\dot{F}_q^{\alpha,p}$ is isomorphic to a fractional Sobolev space $\dot{W}^{\alpha,p}$, with

$$\|f\|_{\dot{F}_p^{\alpha,p}} \simeq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha p}} dxdy \right)^{1/p}.$$

- When $\alpha > 0$, $1 < q < \infty$ and $1 < p < n/\alpha$, the space $\dot{F}_q^{\alpha,p}$ embeds continuously into $L^{np/(n-\alpha p)}$.
- But when $\alpha p = n$, $\dot{F}_q^{\alpha,p}$ fails to embed into L^∞.

Theorem (Bousquet-Russ-Wang-Y.)

Suppose $\alpha > 0$, $p, q \in (1, \infty)$ *and* $\alpha p = n$. *Let* κ *be the largest positive integer that satisfies*

$$\kappa < \min\{p, n\}.$$

Then for every $\delta > 0$, *there exists a constant* C_δ, *such that for every* $f \in \dot{F}^{\alpha,p}_q(\mathbb{R}^n)$, *there exists* $F \in \dot{F}^{\alpha,p}_q \cap L^\infty(\mathbb{R}^n)$ *satisfying*

$$\sum_{i=1}^{\kappa} \|\partial_i f - \partial_i F\|_{\dot{F}^{\alpha-1,p}_q(\mathbb{R}^n)} \leq \delta \|f\|_{\dot{F}^{\alpha,p}_q(\mathbb{R}^n)}$$

and

$$\|F\|_{\dot{F}^{\alpha,p}_q(\mathbb{R}^n)} + \|F\|_{L^\infty(\mathbb{R}^n)} \leq C_\delta \|f\|_{\dot{F}^{\alpha,p}_q(\mathbb{R}^n)}.$$

- This reduces to the result of Bourgain and Brezis if $\alpha = 1$, $p = n$ and $q = 2$.

Corollary (Bousquet-Russ-Wang-Y.)

Suppose $\alpha > 0$, $p, q \in (1, \infty)$ and $\alpha p = n$. Let κ be the largest positive integer that satisfies

$$\kappa < \min\{p, n\}.$$

Let $\ell \in \mathbb{N}$ satisfy $\ell \in [n-\kappa, n-1]$. Then for any ℓ-form $\varphi \in \dot{F}^{\alpha,p}_q$ on $\mathbb{R}^n$, there exists an ℓ-form $\psi \in \dot{F}^{\alpha,p}_q \cap L^\infty$ on $\mathbb{R}^n$, such that

$$d\psi = d\varphi,$$

and

$$\|\psi\|_{\dot{F}^{\alpha,p}_q} + \|\psi\|_{L^\infty} \lesssim \|d\varphi\|_{\dot{F}^{\alpha-1,p}_q}.$$

- The special case of this corollary when $\ell = n-1$, $p \geq 2$, $q \in [2, p]$ and $\alpha > 1/2$ is an earlier result of Bousquet, Mironescu and Russ.

Corollary (Bousquet-Russ-Wang-Y.)

Suppose $\alpha > 0$, $p, q \in (1, \infty)$ and $\alpha p = n$. Let κ be the largest positive integer that satisfies

$$\kappa < \min\{p, n\}.$$

If $u = (u_1, \ldots, u_{\kappa+1})$ has components in $C_c^\infty(\mathbb{R}^n)$ with

$$\sum_{i=1}^{\kappa+1} \partial_i u_i = 0,$$

then for any $\varphi = (\varphi_1, \ldots, \varphi_{\kappa+1})$ with components in $\dot{F}_q^{\alpha,p}(\mathbb{R}^n)$, we have

$$\left| \int_{\mathbb{R}^n} \langle u, \varphi \rangle dx \right| \lesssim \|u\|_{L^1 + \dot{F}_{q'}^{-\alpha,p'}} \|\varphi\|_{\dot{F}_q^{\alpha,p}}.$$

Corollary (Bousquet-Russ-Wang-Y.)

Suppose $\alpha > 0$, $p, q \in (1, \infty)$ and $\alpha p = n$. Let κ be the largest positive integer that satisfies

$$\kappa < \min\{p, n\}.$$

Let ℓ be an integer with $0 \leq \ell \leq \kappa - 1$. Then for any smooth and compactly supported ℓ-form u on $\mathbb{R}^n$ with $d^* u = 0$, we have

$$\|u\|_{\dot{F}^{1-\alpha,p'}_{q'}} \lesssim \|du\|_{L^1 + \dot{F}^{-\alpha,p'}_{q'}}.$$

Corollary (Bousquet-Russ-Wang-Y.)

Suppose $p, q \in (1, \infty)$. Then for any smooth function u with compact support on $\mathbb{R}^n$, we have

$$\|u\|_{\dot{F}_q^{1-\frac{n}{p'},p}} \lesssim \|\nabla u\|_{L^1 + \dot{F}_q^{-\frac{n}{p'},p}}.$$

The proof of Bourgain and Brezis

- Below we review the strategy of the original proof of Bourgain and Brezis for their approximation theorem, and explain the new difficulties we had to overcome to prove our approximation theorem.
- Let $f \in \dot{W}^{1,n}$ on $\mathbb{R}^n$. While f may not be in L^∞, Bernstein's inequality shows that each Littlewood-Paley piece of f is in L^∞:
$$\|\Delta_j f\|_{L^\infty} \leq C\|\nabla f\|_{L^n}.$$
- Thus if $f = \Delta_j f$ for some j, i.e. if f is localized in a frequency band, then one can prove the approximation theorem by simply setting $F = f$.
- Since in general $f = \sum_j \Delta_j f$, the difficulty in the general case is to sum up the different frequencies.

- To sum up the different frequencies, the following 'partition of unity' identity is useful:
- Given any N numbers $a_1, \ldots, a_N$, we have

$$1 = \sum_{j=1}^{N} a_j \prod_{j'>j} (1 - a_{j'}) + \prod_{j=1}^{N} (1 - a_j)$$

- This is nothing but

$$\begin{aligned} 1 &= a_N + (1 - a_N) \\ &= a_N + a_{N-1}(1 - a_N) + (1 - a_{N-1})(1 - a_N) \\ &= a_N + a_{N-1}(1 - a_N) + a_{N-2}(1 - a_{N-1})(1 - a_N) \\ &\qquad + (1 - a_{N-2})(1 - a_{N-1})(1 - a_N) \\ &= \ldots \end{aligned}$$

$$1 = \sum_{j=1}^{N} a_j \prod_{j'>j} (1 - a_{j'}) + \prod_{j=1}^{N} (1 - a_j)$$

- In particular, if $\{a_j\}_{j\in\mathbb{Z}}$ is a sequence with $0 \leq a_j \leq 1$ for all j, then
$$\sum_j a_j \prod_{j'>j} (1 - a_{j'}) \leq 1.$$
- Suppose from now on $f \in \dot{W}^{1,n}$ and $\|\nabla f\|_{L^n}$ is small (so that $\|\Delta_j f\|_{L^\infty} \leq 1$ for all j, as possible by Bernstein's inequality).
- One is then tempted to set
$$F = \sum_j \Delta_j f \prod_{j'>j} (1 - |\Delta_{j'} f|)$$
as an approximation of $f = \sum_j \Delta_j f = \sum_j \Delta_j f \cdot 1$, for at least F is automatically in L^∞.

- This approach is doomed to fail, for the construction of F does not distinguish between the good directions $\partial_1, \dots, \partial_{n-1}$ from the bad direction ∂_n, while the goal was to construct F so that $\|\partial_i(f - F)\|_{L^n}$ is particularly small when $1 \leq i \leq n-1$.
- The way out: Bourgain and Brezis introduced a family of controlling functions ω_j's, so that for any $j \in \mathbb{Z}$, we have

$$|\Delta_j f(x)| \leq \omega_j(x) \leq \|\Delta_j f\|_{L^\infty}$$

$$|\partial_i \omega_j(x)| \lesssim 2^{j-\sigma} \omega_j(x) \quad \text{for } i = 1, \dots, n-1,$$

and

$$|\partial_n \omega_j(x)| \lesssim 2^j \omega_j(x).$$

Here σ is a large parameter depending on the small number δ.

$$|\Delta_j f(x)| \leq \omega_j(x) \leq \|\Delta_j f\|_{L^\infty}$$
$$|\partial_i \omega_j(x)| \lesssim 2^{j-\sigma}\omega_j(x) \quad \text{for } i = 1, \ldots, n-1,$$
$$|\partial_n \omega_j(x)| \lesssim 2^j \omega_j(x).$$

- One is then tempted to construct the approximating function F by setting
$$F = \sum_j \Delta_j f \prod_{j'>j} (1 - \omega_{j'}).$$
In that case, F would be automatically in L^∞, but this still would not work: indeed, we have
$$f - F = \sum_j \omega_j \mu_j, \quad \text{where} \quad \mu_j := \sum_{j'<j} \Delta_{j'} f \prod_{j'<j''<j} (1 - \omega_{j''}).$$
Note that $\|\mu_j\|_{L^\infty} \leq 1$ for all j.

$$f - F = \sum_j \omega_j \mu_j, \quad \text{with} \quad \|\mu_j\|_{L^\infty} \le 1 \quad \text{for all } j.$$

- So to estimate $\|\partial_i (f - F)\|_{L^n}$ for $i = 1, \dots, n-1$, we may need to bound a term like $\left\| \sum_j |\partial_i \omega_j| |\mu_j| \right\|_{L^n} \le \left\| \sum_j |\partial_i \omega_j| \right\|_{L^n}$.
- Recall $|\partial_i \omega_j| \lesssim 2^{j-\sigma} \omega_j$ for $i = 1, \dots, n-1$, with σ large. Thus we are led to estimate

$$2^{-\sigma} \left\| \sum_j 2^j \omega_j \right\|_{L^n}.$$

- But there is no hope estimating the above L^n norm: the L^n norm is even bigger than $\left\| \sum_j 2^j |\Delta_j f| \right\|_{L^n}$, while we want a bound like $\| \| 2^j |\Delta_j f| \|_{\ell^2} \|_{L^n} \simeq \|\nabla f\|_{L^n}$ by Littlewood-Paley inequality.

- There is another clever way out: if instead of $\left\|\sum_j 2^j \omega_j\right\|_{L^n}$ we need only estimate $\left\|\sum_j 2^j \omega_j \chi_{E_j}\right\|_{L^n}$, where

$$E_j := \left\{ x \in \mathbb{R}^n \colon 2^j \omega_j(x) > \sum_{k<j} 2^k \omega_k(x) \right\},$$

then since pointwisely we have

$$\sum_j 2^j \omega_j \chi_{E_j} \leq 2 \sup_j 2^j \omega_j,$$

which for comparison is smaller than $2\|2^j \omega_j\|_{\ell^2}$, we have some hope of estimating its L^n norm.

- So Bourgain and Brezis wrote

$$f = \sum_j \Delta_j f = \sum_j \Delta_j f \chi_{E_j} + \sum_j \Delta_j f \chi_{E_j^c} := h + g,$$

  and approximated each of h and g by functions in $\dot{W}^{1,n} \cap L^\infty$ using ideas we discussed above;
- Indeed, the above heuristics suggests that one can construct $\tilde{h} \in \dot{W}^{1,n} \cap L^\infty$ such that $\|\partial_i (h - \tilde{h})\|_{L^n}$ is under control.
- It turns out that one can also construct $\tilde{g} \in \dot{W}^{1,n} \cap L^\infty$ such that $\|\partial_i (g - \tilde{g})\|_{L^n}$ is under control.
- They concluded the proof by setting $F := \tilde{g} + \tilde{h}$.

Difficulty #1: $\alpha > 1$

- First, Bourgain and Brezis used the following definition of ω_j: for $x = (x', x_n) \in \mathbb{R}^n$, they defined

$$\omega_j(x) := \sup_{y \in \mathbb{R}^n} |\Delta_j f(y)| e^{-2^{j-\sigma}|x'-y'|} e^{-2^j |x_n - y_n|}.$$

- As such ω_j is supremum of smooth functions in x, which is in general at best Lipschitz.
- When we prove an approximation theorem for $\dot{F}^{\alpha,p}_q$ with $\alpha > 1$ (e.g. $\dot{W}^{k,p}$ for $k > 1$), we naturally needed to differentiate ω_j more than once.
- So we used a different construction: morally speaking, we defined ω_j using a discrete ℓ^p convolution:

$$\omega_j(x) := \left(\sum_{r \in 2^{-j}\mathbb{Z}^n} \left(|\Delta_j f(r)| e^{-2^{j-\sigma}|x'-r'|} e^{-2^j |x_n - r_n|} \right)^p \right)^{1/p}.$$

- With the definition of ω_j in place, we define the sets E_j by

$$E_j := \left\{ x \in \mathbb{R}^n \colon 2^{j\alpha}\omega_j(x) > \sum_{k<j} 2^{k\alpha}\omega_k(x) \right\},$$

  and split

$$f = \sum_j \Delta_j f = \sum_j \Delta_j f \chi_{E_j} + \sum_j \Delta_j f \chi_{E_j^c} := h + g;$$

- We'd try to construct $\tilde{g}$ and $\tilde{h}$ in $\dot{F}_q^{\alpha,p} \cap L^\infty$ such that

$$\|\partial_i(g - \tilde{g})\|_{\dot{F}_q^{\alpha-1,p}} \quad \text{and} \quad \|\partial_i(h - \tilde{h})\|_{\dot{F}_q^{\alpha-1,p}}$$

  are both small, if $i = 1, \ldots, \kappa$.

Difficulty #2: $q > p$

- The success of such approach depends on being able to bound

$$\left\| \sup_j 2^{\alpha j} \omega_j \right\|_{L^p}$$

by a reasonably small multiple of $\|f\|_{\dot{F}^{\alpha,p}_q}$.

- There is an easy argument when $q \leq p$, since then we have an embedding $\ell^q \hookrightarrow \ell^p$.
- But this is not so easy when $q > p$ (which arises, for instance, when we want to prove an approximation theorem for $\dot{W}^{k,p}$ on $\mathbb{R}^n$ with $n/2 < k < n$, since then $q = 2 > n/k = p$).
- It turns out that we needed to use a vector-valued bound for a 'shifted' maximal function when $q > p$.

- It is well-known that the shifted maximal function satisfies a scalar L^p bound:

Lemma

Let φ be the characteristic function of the unit ball centered at the origin in $\mathbb{R}^n$, and $r \in \mathbb{R}^n$. Then $\varphi(r + \cdot)$ is the characteristic function of a unit ball centered at r. Define

$$k_j(x) = 2^{jn}\varphi(r + 2^j x), \quad \text{and} \quad \mathfrak{M}f(x) = \sup_{j \in \mathbb{Z}} |f| * k_j(x).$$

Then for $1 < p < \infty$, we have the following inequality:

$$\|\mathfrak{M}f\|_{L^p} \lesssim [\log(2 + |r|)]^{1/p} \|f\|_{L^p} .$$

Lemma

More generally, let φ be any non-negative integrable function on $\mathbb{R}^n$, satisfying

$$\int_{\mathbb{R}^n} \varphi(y)dy \lesssim 1,$$

$$\int_{|y|\geq R} \varphi(y)dy \lesssim R^{-1} \quad \textit{for all } R \geq 1,$$

and

$$\int_{\mathbb{R}^n} |\varphi(y-x) - \varphi(y)|dy \lesssim |x| \quad \textit{for all } x \in \mathbb{R}^n.$$

For $r \in \mathbb{R}^n$, let $k_j(x) = 2^{jn}\varphi(r + 2^j x)$, $\mathfrak{M}f(x) = \sup_{j\in\mathbb{Z}} |f| * k_j(x)$. Then for $1 < p, q < \infty$, we have the following vector-valued inequality:

$$\left\| \|\mathfrak{M}f_i\|_{\ell^q} \right\|_{L^p} \lesssim [\log(2 + |r|)]^{1/p} \left\| \|f_i\|_{\ell^q} \right\|_{L^p}.$$

- The proof of this lemma proceeds via a lemma of Zó; indeed one proves that

$$\int_{|y|\geq 4|x|} \sup_{j\in\mathbb{Z}} |k_j(y-x) - k_j(y)| dy \lesssim \log(2+|r|).$$

- This shows

$$\|\mathfrak{M}f\|_{L^p} \lesssim [\log(2+|r|)]^{1/p} \|f\|_{L^p},$$

and more generally the vector-valued bound

$$\left\| \|\mathfrak{M}f_i\|_{\ell^q} \right\|_{L^p} \lesssim [\log(2+|r|)]^{1/p} \left\| \|f_i\|_{\ell^q} \right\|_{L^p}.$$

- With this lemma about shifted maximal function in hand, one can control $\left\|\sup_j 2^{\alpha j} \omega_j\right\|_{L^p}$, and finish the proof of the theorem when α is a positive integer.

Difficulty #3: fractional values of α

- Recall we had split the problem of approximating f by setting

$$E_j := \left\{ x \in \mathbb{R}^n \colon 2^{\alpha j}\omega_j(x) > \sum_{k<j} 2^{\alpha k}\omega_k(x) \right\}$$

and writing

$$f = \sum_j \Delta_j f \chi_{E_j} + \sum_j \Delta_j f \chi_{E_j^c} := h + g;$$

we'd construct an approximating function $\tilde{h}$ and $\tilde{g}$ for h and g respectively.

- The way Bourgain and Brezis estimated $\|\partial_i(h-\tilde{h})\|_{L^n}$ was to write $h-\tilde{h}$ as a sum of products, and then apply Leibniz rule to evaluate $\partial_i(h-\tilde{h})$, before computing its L^n norm.
- When $\alpha > 0$ is not an integer, we would need to estimate

  $$\|\partial_i(h-\tilde{h})\|_{\dot{F}^{\alpha-1,p}_q}$$

  and the above argument needs to be replaced by a fractional version of Leibniz rule.
- It is a bit more complicated than that, since we need to differentiate a sum of products, and we want to keep the sum inside the $\dot{F}^{\alpha-1,p}_q$ norm.
- In addition, we needed to be careful in extracting some additional cancellations from certain Littlewood-Paley projections when $\alpha \in (0,1)$.

Difficulty #4: $\alpha \in (0, 1/2]$

- Recall

$$E_j := \left\{ x \in \mathbb{R}^n \colon 2^{\alpha j}\omega_j(x) > \sum_{k<j} 2^{\alpha k}\omega_k(x) \right\}$$

and we split

$$f = \sum_j \Delta_j f \chi_{E_j} + \sum_j \Delta_j f \chi_{E_j^c} := h + g.$$

- The sets E_j depends on α, and one can check that the smaller the α is, the smaller the sets E_j become.
- Thus when α is small, the function g is big, and it is relatively harder to approximate g by a function $\tilde{g} \in \dot{F}^{\alpha,p}_q \cap L^\infty$.
- It turns out that we do need a new estimate for

$$\|\partial_i(g - \tilde{g})\|_{\dot{F}^{\alpha-1,p}_q}$$

when $\alpha \in (0, 1/2]$.