Applications of decoupling-type estimates to the cubic NLSE

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Background

We first consider a complex-valued function, $f : \mathbb{T} \to \mathbb{C}$, defined on the one-dimensional torus, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$,

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e(nx)$$

where $a_n \in \mathbb{C}$, $e(y) := e^{2\pi i y}$.

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where $a_n \in \mathbb{C}$, $e(y) := e^{2\pi i y}$.
Our first question is, for which $p \in [1, \infty]$ and in what sense does the following inequality hold

$$\|f\|_{L^p(\mathbb{T})} \leq C_p \left(\sum_{n\in\mathbb{Z}} |a_n|^2 \right)^{1/2}$$

where $\|f\|_{L^p(\mathbb{T})} = \|f\|_p := \left(\int_{\mathbb{T}} |f|^p \, dm \right)^{1/p}$

Background

Similarly, let $\{I_k\}$ be a disjoint decomposition of $\mathbb{Z}$. Then we consider the inequality

$$\|f\|_p \leq C_p \left(\sum_k \|f_{I_k}\|_p^2 \right)^{1/2}$$

where $f_{I_k}(x) = \sum_{n \in I_k} a_n e(nx)$. We will refer to inequalities of this form as *decoupling inequalities*.

Littlewood-Paley Theorem

For any decomposition, $\{I_k\}$, of $\mathbb{Z}$, define the square function

$$Sf(x) := \left(\sum_{k \in \mathbb{Z}_+} |f_{I_k}(x)|^2 \right)^{1/2}$$

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Let $\beta \geq \alpha > 1$ and define $\{m_n\}_{n \geq 1} \subset \mathbb{Z}_+$ satisfying $\alpha m_n \leq m_{n+1} \leq \beta m_n$. Let $m_0 = 0$.

Theorem (Littlewood-Paley (1937))

Let $1 < p < \infty$. Consider $I_k = (-m_k, -m_{k-1}] \cup [m_{k-1}, m_k)$, for $k = 1, 2, 3, \dots$.
Then

$$C_{p,\alpha,\beta}^{-1} \|Sf\|_p \leq \|f\|_p \leq C_{p,\alpha,\beta} \|Sf\|_p$$

Example: $m_n := 2^n$.

Higher Dimensions

Consider $f : \mathbb{T}^d \to \mathbb{C}$ defined as

$$f(x) = \sum_{n \in \mathbb{Z}^d} a_n e(n \cdot x)$$

Theorem (L-P ('37))

Let $1 < p < \infty$, $I_k := \{n \in \mathbb{Z}^d \mid 2^{k-1} \leq \|n\| < 2^k\}$. *Then*

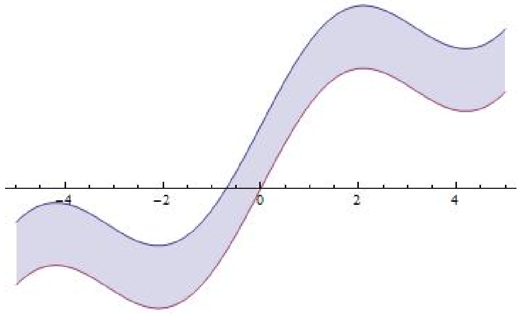
$$C_p^{-1} \|Sf\|_p \leq \|f\|_p \leq C_p \|Sf\|_p$$

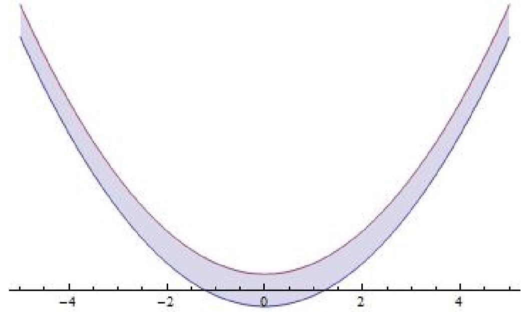
Note: Rubio de Francia (1985) and Journe (1985) showed that for an arbitrary decomposition of $\mathbb{Z}^d$, $\{I_k\}$, $1 < p \leq 2$,

$$\|f\|_p \leq C_p \|Sf\|_p$$

Higher Dimensions

For $d > 1$, the geometry of the support of (a_n) becomes important. We will focus on (a_n) supported near hyper-surfaces:





Fourier Analysis on $\mathbb{R}^d$

Let
$P^{d-1} = \{(\xi_1, \xi_2, ..., \xi_{d-1}, \xi_1^2 + \xi_2^2 + \cdots + \xi_{d-1}^2) \mid |\xi_i| \leq 1\} \subset \mathbb{R}^d$.
For $a : P^{d-1} \to \mathbb{C}$, and for

$$Ea(x) := \widehat{a d\sigma} = \int_{P^{d-1}} a(\xi) e(x \cdot \xi)\, d\sigma(\xi)$$

we ask, for which p,

$$\|Ea\|_{L^p(\mathbb{R}^d)} \leq C_p \|a\|_{L^2(d\sigma)}$$

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Stein's Restriction conjecture: This holds for $p \geq 2\frac{d+1}{d-1}$. (As well as analogous inequalities for different norms on a)

Background on Restriction

Replacing $\|a\|_{L^2(d\sigma)}$ with $\|a\|_{L^q(d\sigma)}$ and considering more general surfaces we have

- Tomás-Stein Theorem: Conjecture true for $q = 2$, $p \geq 2\frac{d+1}{d-1}$.

Background on Restriction

Replacing $\|a\|_{L^2(d\sigma)}$ with $\|a\|_{L^q(d\sigma)}$ and considering more general surfaces we have

- Tomás-Stein Theorem: Conjecture true for $q = 2$, $p \geq 2\frac{d+1}{d-1}$.
- Wolff, Tao, Bennett, Carbery, Bourgain, Guth, Demeter among many others: various results in linear, bilinear, k-linear, multilinear from 1990s-Now.

Discrete Case

Let $0 \leq \delta \leq 1$. Consider a set of points $\Lambda \subset P^{d-1}$ such that for each pair of points $x, y \in \Lambda$, $\|x - y\| > \delta^{1/2}$. Then

$$\|Ea\|_{L^p(\mathbb{R}^d)} \leq C_p \|a\|_{L^2(d\sigma)}$$

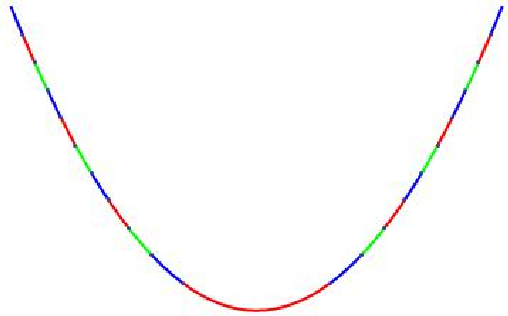
is equivalent to

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x) \right|^p \right)^{1/p} \leq C_p \delta^{\frac{d}{2p} - \frac{d-1}{4}} \|a_\xi\|_{\ell^2(\Lambda)}$$

where $B_R \subset \mathbb{R}^d$ is a ball of radius $R \sim \delta^{-1/2}$.

- Let $\mathcal{N}_\delta = \{x \in \mathbb{R}^d \mid \text{dist}(x, P^{d-1}) < \delta\}$.
- Let $\mathcal{P}_\delta$ be a finitely overlapping cover of $\mathcal{N}_\delta$ with curved regions of the form

$$\theta = \{(\xi_1, ..., \xi_{d-1}, \eta + \xi_1^2 + \cdots + \xi_{d-1}^2 \mid (\xi_1, ..., \xi_{d-1}) \in C_\theta, |\eta| \leq 2\delta\},$$



where $\{C_\theta\} = \{c + [-\frac{\delta^{1/2}}{2}, \frac{\delta^{1/2}}{2}]^{d-1}\}_c$ with $c \in \frac{\delta^{1/2}}{2}\mathbb{Z}^{d-1} \cap [-1,1]^{d-1}$.

ℓ^2 Decoupling
A Littlewood-Paley Formulation

Let $f : \mathbb{R}^d \to \mathbb{C}$, $f(\xi) = \int a(\xi) e(\xi \cdot x)\, dx$ and supp $a \subset \mathcal{N}_\delta$. If we define the square function as

$$Sf(x) := \left(\sum_\theta |f_\theta(x)|^2 \right)^{1/2}.$$

we wonder whether

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C_{p,\delta} \|Sf\|_{L^p(\mathbb{R}^d)}$$

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holds. Particularly

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C_p \delta^{-\varepsilon} \|Sf\|_{L^p(\mathbb{R}^d)} \qquad p \in [2, 2\tfrac{d+1}{d-1})$$
$$\|f\|_{L^p(\mathbb{R}^d)} \leq C_p \delta^{-\frac{d-1}{4} + \frac{d+1}{2p} - \varepsilon} \|Sf\|_{L^p(\mathbb{R}^d)} \quad p \in [2\tfrac{d+1}{d-1}, \infty)$$

This is unresolved.

ℓ^2 Decoupling

However, we have the decoupling form of this result

Theorem (Bourgain-Demeter (2014))

Let S be a compact C^2 hypersurface in $\mathbb{R}^d$ with positive definite second fundamental form. If supp $a \subset \mathcal{N}_\delta$ then for $p \geq 2$ and $\varepsilon > 0$

$$\|f\|_p \leq C_p \delta^{-\varepsilon} \left(1 + \delta^{-\frac{d-1}{4} + \frac{d+1}{2p}}\right) \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2\right)^{1/2}$$

Consider the Cauchy problem

$$\begin{cases} iu_t + \Delta u - |u|^2 u = 0, & x \in \mathbb{T}^d, t \geq 0 \\ u(x,0) = u_0(x) \in H^2(\mathbb{T}^d), & \end{cases} \tag{1}$$

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Definition

(1) is said to be locally well-posed in $H^s(\mathbb{T}^d)$ if for any initial data $u_0 \in H^s(\mathbb{T}^d)$ there exists a time $T = T(\|u_0\|_{H^s})$ such that a unique solution to the initial value problem exists on the time interval $[0, T]$. We also require that $u(t,x) \in C_t^0 H_x^s([0, T] \times \mathbb{T}^d)$. If $T = \infty$ we say that a Cauchy problem is globally well-posed.

Let $S(t)$ be the solution operator for the linear Schrödinger equation: $iu_t + \Delta u = 0$.

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Note that for

$$u_0(x) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ \|\xi\| \leq N}} a(\xi) e(\xi \cdot x)$$

$$S(t)u_0(x) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ \|\xi\| \leq N}} a(\xi) e(\xi \cdot x - 2\pi \|\xi\|^2 t)$$

For $d = 2$, global well-posedness for $s \geq 1$ (Bourgain 1993) can be reduced to showing

$$\|\eta(t)S(t)u_0\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T}^2)} \leq CN^{\varepsilon}\|\hat{u}_0\|_{\ell^2} \tag{2}$$

for supp $\hat{u}_0 \subset [-N, N]^2$, $\eta \in C^\infty(\mathbb{R})$ compactly supported.

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Observation: The space-time Fourier support of $S(t)u_0$ lies in

$$P_N^2 = \{(\xi_1, \xi_2, \xi_1^2 + \xi_2^2) \in \mathbb{Z}^3 \mid |\xi_i| \leq N\}$$

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Thus, our problem reduces to showing

$$\left\| \sum_{\xi \in P_N^2} a_{\xi} e(\xi \cdot x) \right\|_4 \leq CN^{\varepsilon}\|a_{\xi}\|_{\ell^2}$$

For $s < 1$ (in which case the Energy/Hamiltonian may not exist) there have been many results including

- Bourgain's High-Low Method (1993)
- Colliander, Keel, Staffilani, Takaoka and Tao, I-Method (2001-2002)

Applications
Nonlinear Schrödinger Equation

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- Bourgain's High-Low Method (1993)
- Colliander, Keel, Staffilani, Takaoka and Tao, I-Method (2001-2002)
- De Silva, Pavlovic, Staffilani, Tzirakis (2006) showed GWP for $s > \frac{2}{3}$ using a bilinear version of (2):

$$\|\eta(t)S(t)u_0 S(t)v_0\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}^2)} \leq CN_2^{\varepsilon}\|\hat{u}_0\|_{\ell^2}\|\hat{v}_0\|_{\ell^2}$$

for supp $\hat{u}_0 \subset \{\frac{1}{2}N_1 \leq |\xi| \leq \frac{3}{2}N_1\}$ and supp $\hat{v}_0 \subset \{\frac{1}{2}N_2 \leq |\xi| \leq \frac{3}{2}N_2\}$. Where $N_2 \ll N_1$.

In 1998, Bourgain considered the following Cauchy problem

$$\begin{cases} iu_t + \Delta u - |u|^2 u = 0, & x \in \mathbb{R}^2, t \geq 0 \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^2), & s > 2/3 \end{cases} \tag{3}$$

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Well-posedness here can be reduced to the following refined Strichartz estimate:

$$\|S(t)u_0 S(t)v_0\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \leq C\left(\frac{N_2}{N_1}\right)^{\frac{1}{2}} \|u_0\|_{L^2(\mathbb{R}^2)}\|v_0\|_{L^2(\mathbb{R}^2)}$$

for supp $\hat{u}_0 \subset \{|\xi| \sim N_1\}$ and supp $\hat{v}_0 \subset \{|\xi| \sim N_2\}$. Where $N_2 < N_1$.

The proof follows simply enough in the continuous case.

By Parseval and Cauchy-Schwarz

$$\|S(t)u_0 S(t)v_0\|^2_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \leq C\|u_0\|_2^2\|v_0\|_2^2 \left[\sup_{\tau,\,|\xi|\sim N_1} \mathcal{L}^1(A_{\tau,\xi})\right]$$

where

$$A_{\tau,\xi} := \left\{\xi_1 \in \mathbb{R}^2 \mid |\xi_1| \sim N_2 \text{ and } |\xi_1|^2 + |\xi - \xi_1|^2 = \tau\right\}$$

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Then $\mathcal{L}^1(A_{\tau,\xi}) \lesssim N_2/N_1$.

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$$\|\eta(t)S_\lambda(t)u_0 S_\lambda(t)v_0\|_{L^2_{t,x}(\mathbb{R}\times(\lambda\mathbb{T})^2)} \leq C(\lambda N_2)^{\varepsilon}\|\hat{u}_0\|_{\ell^2}\|\hat{v}_0\|_{\ell^2}$$

Applications
Nonlinear Schrödinger Equation, Approximation of Continuous Case

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For $\lambda \gg 1$

$$\|\eta(t)S_\lambda(t)u_0 S_\lambda(t)v_0\|_{L^2_{t,x}(\mathbb{R}\times(\lambda\mathbb{T})^2)} \leq C\left(\frac{1}{\lambda}+\frac{N_2}{N_1}\right)^{\frac{1}{2}}\|\hat{u}_0\|_{\ell^2}\|\hat{v}_0\|_{\ell^2}$$

Lemma

Let $\mathcal{C}$ be a circle of radius R. If γ is an arc on $\mathcal{C}$ of length $|\gamma| < (\frac{3}{4}R)^{1/3}$, then γ contains at most two lattice points.

Lemma

Let K be a convex domain in $\mathbb{R}^2$. If

$$N(\lambda) = \#\{\mathbb{Z}^2 \cap \lambda K\}$$

then, for $\lambda \gg 1$

$$N(\lambda) = \lambda^2 |K| + O(\lambda).$$

The methods of De Silva, Pavlovic, Staffilani, Tzirakis rely on the simple detail that on $\mathbb{T}^2$,

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Definition

Let $\alpha_1, .., \alpha_{d-1} \in [1/2, 1]$, we define a d–dimensional torus $\mathbb{T}^d$ as $\mathbb{T}^d = \mathbb{T} \times \alpha_1\mathbb{T} \times \cdots \times \alpha_{d-1}\mathbb{T}$. We say that the torus is irrational if the vector $(1, \alpha_1, ..., \alpha_{d-1})$ is irrational, i.e. $m \cdot (1, \alpha_1, ..., \alpha_{d-1}) = 0$ admits no solutions for $m \in \mathbb{Z}^d$.

Theorem (Fan, Staffilani, Wang, W. (2016))

Let $\phi_1, \phi_2 \in L^2(\mathbb{T}^d_\lambda)$ be two initial data, $\eta(t)$ be a time cut-off function, $\operatorname{supp}\eta \subset [0,1]$, assume $\operatorname{supp}\phi_i \subset \{k : k \sim N_i\}$, $i = 1, 2$, for some large $N_1 \geq N_2$, then

$$\|\eta(t)S_\lambda(t)\phi_1 \cdot \eta(t)S_\lambda(t)\phi_2\|_{L^2_{x,t}} \lesssim N_2^\varepsilon \left(\frac{1}{\lambda} + \frac{N_2^{d-1}}{N_1}\right)^{\frac{1}{2}} \|\phi_1\|_{L^2}\|\phi_2\|_{L^2} \tag{4}$$

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For $\lambda \geq N_1$, the same estimate follows (without the N_2^ε loss) for general compact manifolds due to Hani (2012).

Theorem (Fan, Staffilani, Wang, W. (2016))

Given $\lambda \geq 1$, $N_1 \geq N_2 \geq 1$. Let f_1 be supported on P where $|\xi| \sim 1$, and let f_2 be supported on P where $|\xi| \sim \frac{N_2}{N_1}$. Let $\Omega = \{(t,x) \in [0, N_1^2] \times [0, (\lambda N_1)^2]^d\}$. For a finitely overlapping covering of the ball B of caps $\{\theta\}$, $|\theta| = \frac{1}{\lambda N_1}$, we have the following estimate: for any small $\varepsilon > 0$,

$$\|Ef_1 Ef_2\|_{L^2_{avg}(w_\Omega)}$$

$$\lesssim_\varepsilon (N_2)^\varepsilon \lambda^{d/2} \left(\frac{1}{\lambda} + \frac{N_2^{d-1}}{N_1}\right)^{1/2} \prod_{j=1}^{2} \left(\sum_{|\theta| = \frac{1}{\lambda N_1}} \|Ef_{j,\theta}\|^2_{L^4_{avg}(w_\Omega)} \right)^{1/2},$$

where w_Ω is a weight adapted to Ω.

Applications: Key Tools

Let $v < 1$. For $i = 1, 2$, define f_i such that supp $f_i \subset B_i \cap P$, where B_1 is a ball of radius v centered at $(0, 1, 1)$ and B_2 is a ball of radius v centered at the origin.

Lemma

For a covering, $\{\tau_i\}$, of supp f_i with (v, v^2)- "plates". If $R > v^{-2}$, then

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim \sum_{\tau_1, \tau_2} \int |Ef_{1,\tau_1} Ef_{2,\tau_2}|^2 w_{B_R}.$$

Lemma

For a covering, $\{\theta_i\}$, of supp f_i with finitely overlapping balls of radius v^{-2}. If $R > v^{-2}$, then

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim v^{-1} \sum_{|\theta_i| = v^2} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_R}.$$

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ℓ^2 Decoupling and conservation of mass imply

$$\|S(t)f\|_{L^p_{t,x}([0,T]\times\mathbb{T}^d)} \lesssim N^{\varepsilon}(1 + N^{\frac{d}{2}-\frac{d+2}{p}})T^{1/p}\|f\|_{L^2}$$

if supp $\hat{f} \subset [-N, N]^d$.

Conjecture (Deng, Germain, Guth (2017))

Let $d \geq 2$. For generic $(\alpha_1, \ldots, \alpha_{d-1})$, for $\varepsilon > 0$, one has

$$\|S(t)f\|_{L^p([0,T]\times\mathbb{T}^d)} \lesssim N^{\varepsilon}(1 + N^{\frac{d}{2}-\frac{d+2}{p}}) \left[1 + \left(\frac{T}{N^{\theta(p)}}\right)^{\frac{1}{p}}\right] \|f\|_{L^2}$$

for $N \geq 1$, $T \geq 1$, where

$$\theta(p) = \begin{cases} 0, & p \in [2, \frac{2(d+2)}{d}), \\ \frac{d}{2}\left(p - \frac{2(d+2)}{d}\right), & p \in [\frac{2(d+2)}{d}, 6), \\ 2d-2, & p \in [6, \infty). \end{cases}$$

$$\|S(t)f\|_{L^4([0,T]\times\mathbb{T}^d)}^4 = \int_{\mathbb{T}^d}\int_0^T |S(t)f(x)|^4\,dt\,dx$$

$$\|S(t)f\|^4_{L^4([0,T]\times\mathbb{T}^d)} = \int_{\mathbb{T}^d}\int_0^T |S(t)f(x)|^4\,dt\,dx$$

$$= \int_{\mathbb{T}^d}\int_0^T \left(\sum_{\xi} a(\xi)e(\xi\cdot x + t\|\xi\|^2)\right)^2 \left(\sum_{\xi} \bar{a}(\xi)e(-\xi\cdot x - t\|\xi\|^2)\right)^2$$

$$= \int_{\mathbb{T}^d}\sum \Phi_{\xi_1,\xi_2,\xi_3,\xi_4}(x)\int_0^T e(t[\|\xi_1\|^2 + \|\xi_2\|^2 - \|\xi_3\|^2 - \|\xi_4\|^2])\,dt\,dx$$

$$\|S(t)f\|_{L^4([0,T]\times\mathbb{T}^d)}^4 = \int_{\mathbb{T}^d}\int_0^T |S(t)f(x)|^4 \, dt \, dx$$

$$= \int_{\mathbb{T}^d}\int_0^T \left(\sum_{\xi} a(\xi)e(\xi\cdot x + t\|\xi\|^2)\right)^2 \left(\sum_{\xi} \bar{a}(\xi)e(-\xi\cdot x - t\|\xi\|^2)\right)^2$$

$$= \int_{\mathbb{T}^d}\sum \Phi_{\xi_1,\xi_2,\xi_3,\xi_4}(x)\int_0^T e(t[\|\xi_1\|^2+\|\xi_2\|^2-\|\xi_3\|^2-\|\xi_4\|^2])\, dt\, dx$$

$$= \int_{\mathbb{T}^d}\sum \Phi_{\xi_1,\xi_2,\xi_3,\xi_4}(x)\frac{E_{\xi_1,\xi_2,\xi_3,\xi_4}(T)}{\|\xi_1\|^2+\|\xi_2\|^2-\|\xi_3\|^2-\|\xi_4\|^2}\, dx$$

We must control

$$\frac{1}{\|\xi_1\|^2 + \|\xi_2\|^2 - \|\xi_3\|^2 - \|\xi_4\|^2} = [k \cdot (1, \alpha_1^2, ..., \alpha_{d-1}^2)]^{-1}$$

We must control

$$\frac{1}{\|\xi_1\|^2 + \|\xi_2\|^2 - \|\xi_3\|^2 - \|\xi_4\|^2} = [k \cdot (1, \alpha_1^2, ..., \alpha_{d-1}^2)]^{-1}$$

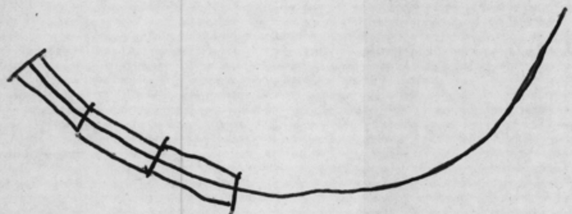
which can be done using a Diophantine condition such as

$$|k_1 + \alpha_1^2 k_2 + \cdots + \alpha_{d-1}^2 k_d| \\ \gtrsim \frac{1}{(|k_1| + \cdots + |k_d|)^{d-1} \log(|k_1| + \cdots + |k_d|)^{2d}}$$

Thank you for listening

Applications of decoupling type estimates to the cubic NLSE

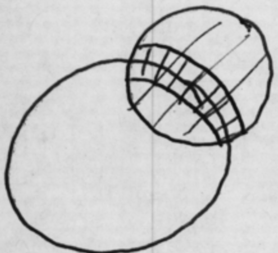
Bobby Wilson 18 May 2017



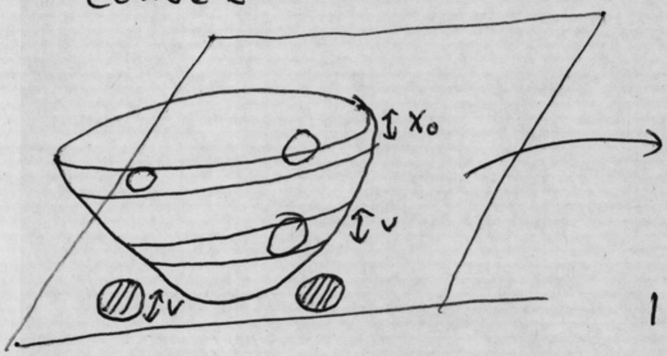
parabola

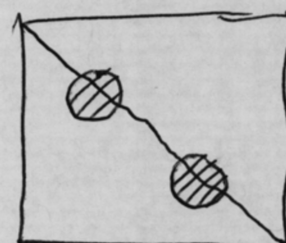
$\Theta = \delta \times \delta^{1/2}$

$$H(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int |u|^4$$

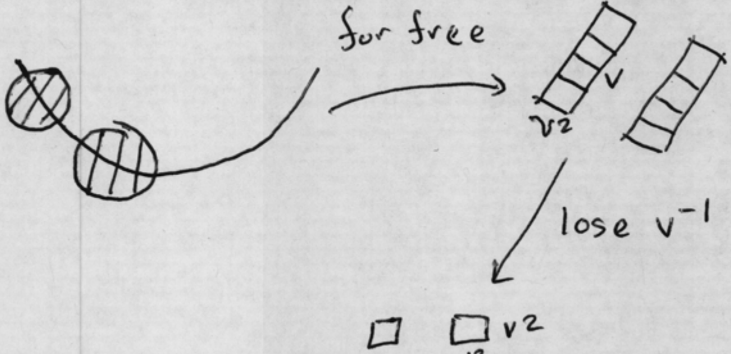


count lattice points in body which is not necessarily convex





1D Strichartz estimate



$$\|\xi_1\|^2 + \|\xi_2\|^2 - \|\xi_3\|^2 = \sum \pm K_j^2$$

$$|\xi_1'|^2 + |\xi_2'|^2 - |\xi_3'|^2$$