On the failure of lower square function estimates in the non-homogeneous weighted setting

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Definition

We say that a weight w satisfies the martingale A_2 condition and write $w \in A_2^{\mathcal{D}}$ if

$$[w]_{2,\mathcal{D}} := \sup_{I \in \mathcal{D}} \langle w \rangle_I \langle w^{-1} \rangle_I < \infty.$$

Definition

We say that a weight w satisfies the classical A_2 condition and write $w \in A_2^{\mathrm{cl}}$ if

$$[w]_2^{\mathrm{cl}} = \sup_{I \subseteq [0,1]} \langle w \rangle_I \langle w^{-1} \rangle_I < \infty,$$

where the supremum runs over all intervals $I \subset [0,1]$.

Definition

For an interval I define the *localized* maximal function M_I,

$$M_I f(x) := \mathbf{1}_I(x) \sup_{J \subseteq I:\, x \in J} |\langle f \rangle_J|,$$

where the supremum runs over all intervals $J \subset I$ containing x.
For an interval $I \in \mathcal{D}$ define also the *martingale* localized maximal function $M_I^{\mathcal{D}}$,

$$M_I^{\mathcal{D}} f(x) = \mathbf{1}_I(x) \sup_{J \in \mathcal{D}(I):\, x \in J} \left|\langle f \rangle_J\right|$$

Definition

We say that a weight w satisfies the classical A_∞ condition and write $w \in A_\infty^{\text{cl}}$ if

$$[w]_{\infty,\text{cl}} = \sup_{I \subseteq [0,1]} \frac{\langle M_I w \rangle_I}{\langle w \rangle_I} < \infty.$$

where $M_I f$ is the localized classical maximal function defined above.

frame title

Definition

We say that a weight w satisfies the *semiclassical* A_∞ condition and write $w \in A_\infty^{\mathrm{scl}}$ if

$$[w]_{\infty,\mathrm{scl}} = \sup_{I\in\mathcal{D}} \frac{\langle M_I w\rangle_I}{\langle w\rangle_I} < \infty,$$

where again $M_I f$ is the classical maximal function localized to $I \in \mathcal{D}$.

Definition

We say that $w \in A_\infty^{\mathcal{D}}$ if

$$[w]_{\infty,\mathcal{D}} = \sup_{I\in\mathcal{D}} \frac{\langle M_I^{\mathcal{D}} w\rangle_I}{\langle w\rangle_I} < \infty,$$

where $M_I^{\mathcal{D}} f$ is the martingale maximal function localized to $I \in \mathcal{D}$.

Clearly from the previous definitions

$$[w]_{\infty,\text{cl}} \geqslant [w]_{\infty,\text{scl}} \geqslant [w]_{\infty,\mathcal{D}}$$

Proposition

For any atomic filtration

$$[w]_{\infty,\mathcal{D}} \leq 4[w]_{2,\mathcal{D}} \tag{1}$$

Theorem

$$\|f\|_{L^2(w)} \lesssim [w]_{2,\mathcal{D}}^{1/2}[w]_{\infty,\mathcal{D}}^{1/2}\|Sf\|_{L^2(w)} \le 2[w]_{2,\mathcal{D}}^{1}\|Sf\|_{L^2(w)}. \tag{2}$$

Theorem

Assumption $w \in A_\infty^{\mathrm{cl}}$ is not sufficient for an estimate

$$\|f\|_{L^2(w)} \le C([w]_{\infty,\mathrm{cl}})\|Sf\|_{L^2(w)}.$$

Theorem

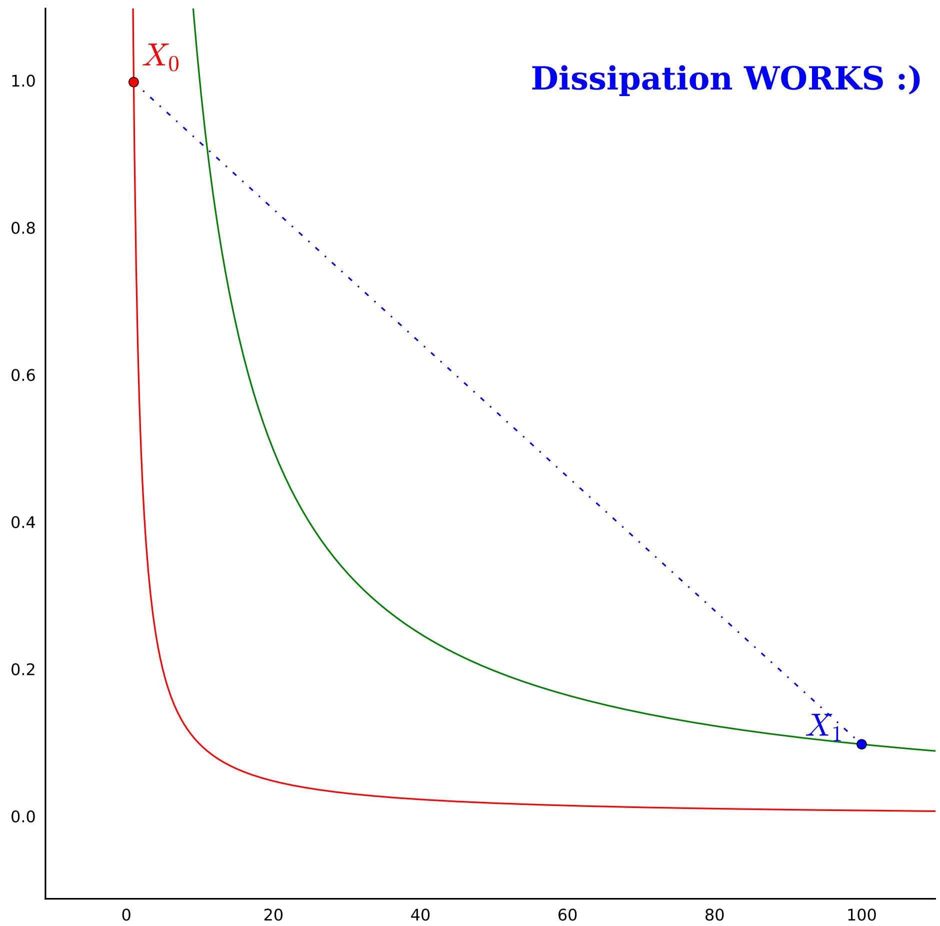
For the n-adic filtration

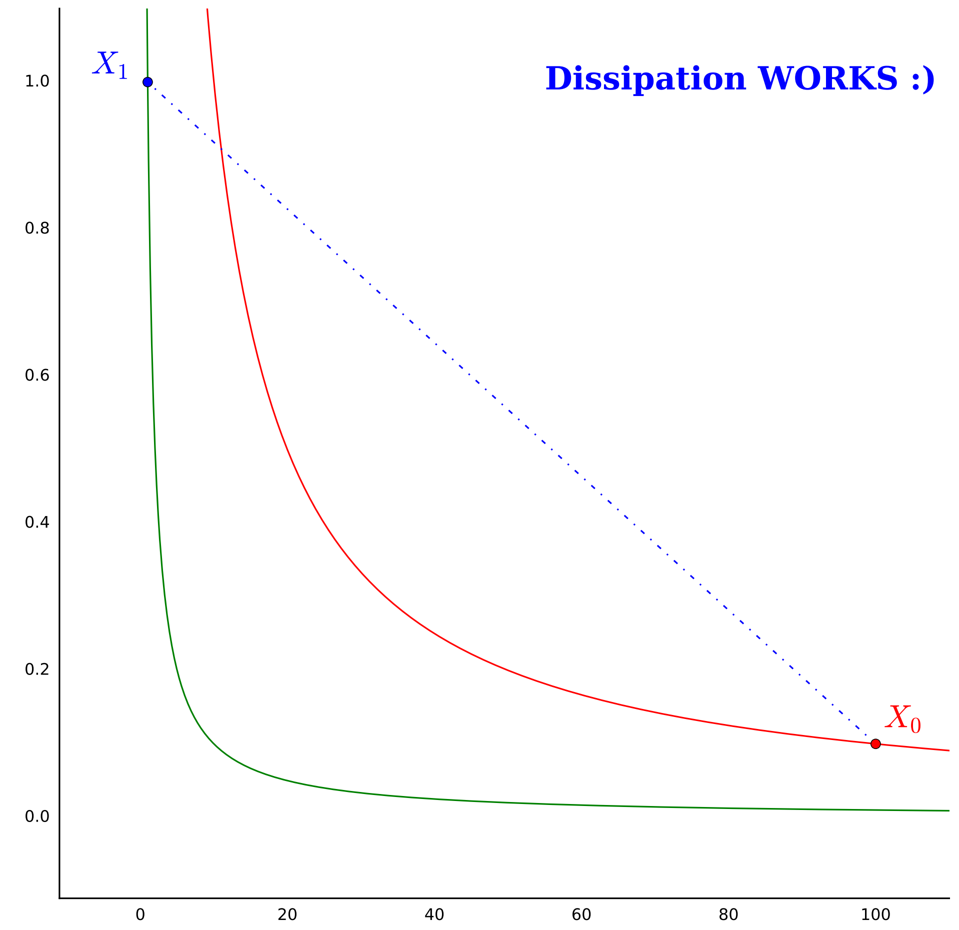
$$\|f\|_{L^2(w)} \lesssim n[w]_{\infty,\mathrm{scl}}^{1/2}\|Sf\|_{L^2(w)}.$$

Stronger result holds even for $\|Mf\|_{L^p(w)}$

$$x^2 - y$$

$$x^2 w\left(2 - \frac{1}{vw} - \frac{log(vw)}{2Q}\right) - 40Qwy$$


X_0
Dissipation WORKS :)
1.0
0.8
0.6
0.4
0.2
0.0
X_1
0
20
40
60
80
100



Dissipation WORKS :)
X_1
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0.6
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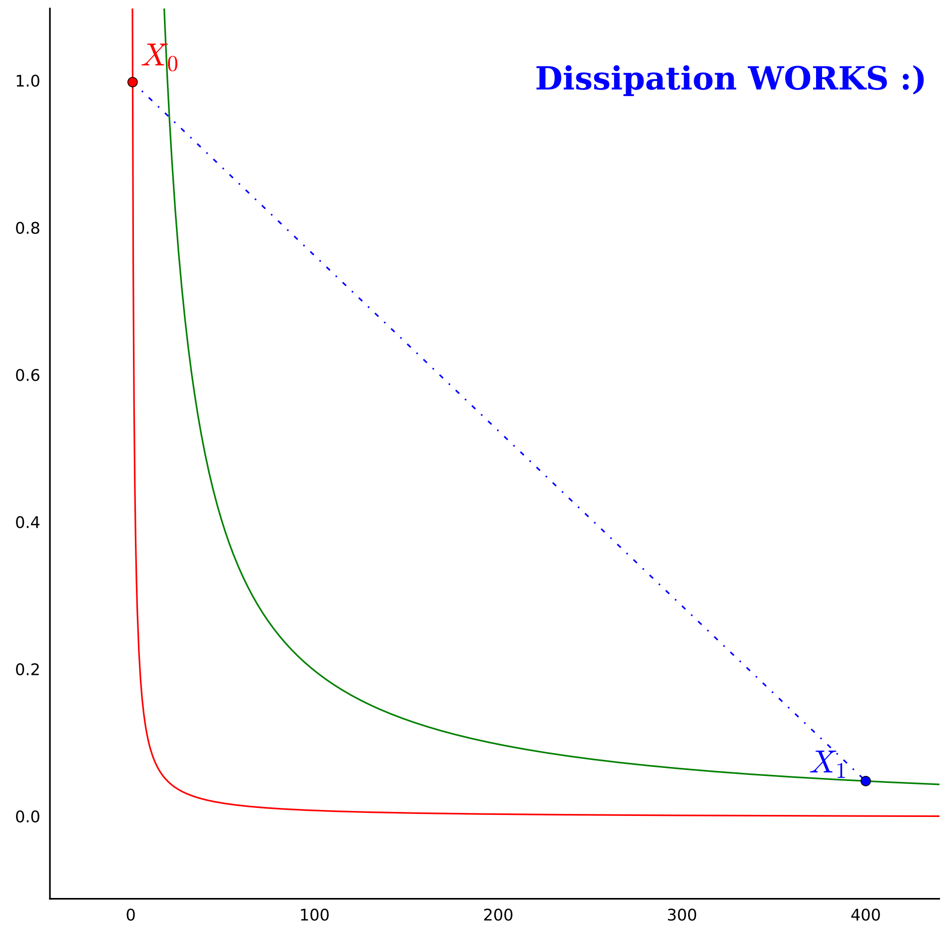
Dissipation WORKS :)
X_0
X_1
1.0
0.8
0.6
0.4
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0.0
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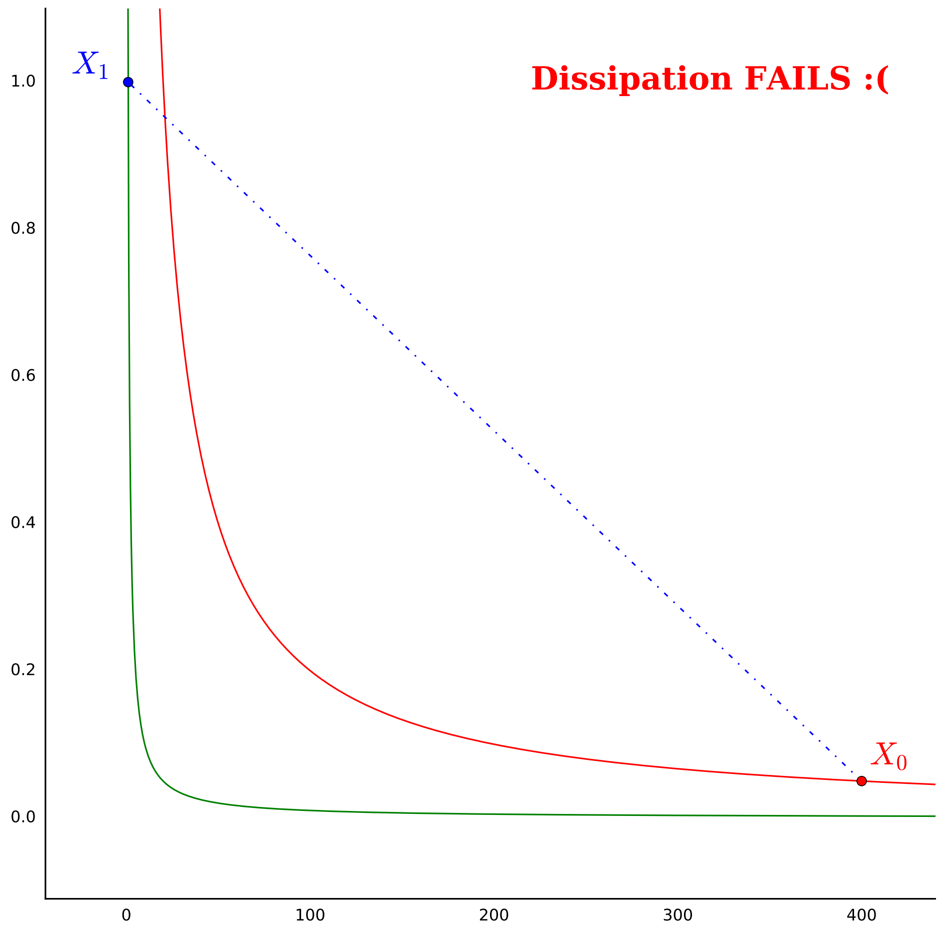
100
200
300
400


Dissipation FAILS :(
X_1
X_0
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400

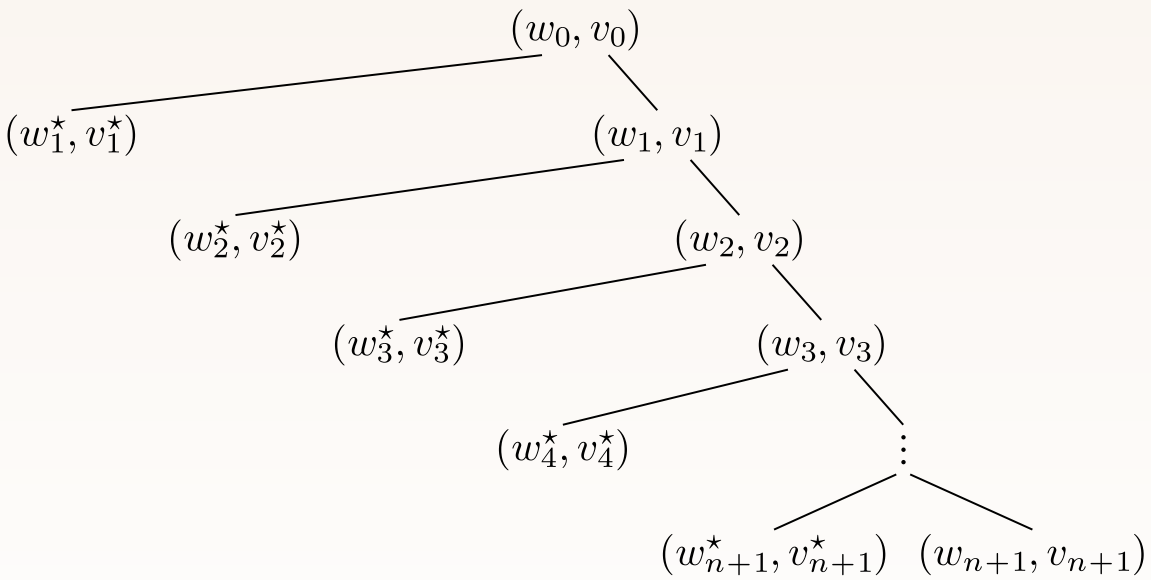
We want to make the expression

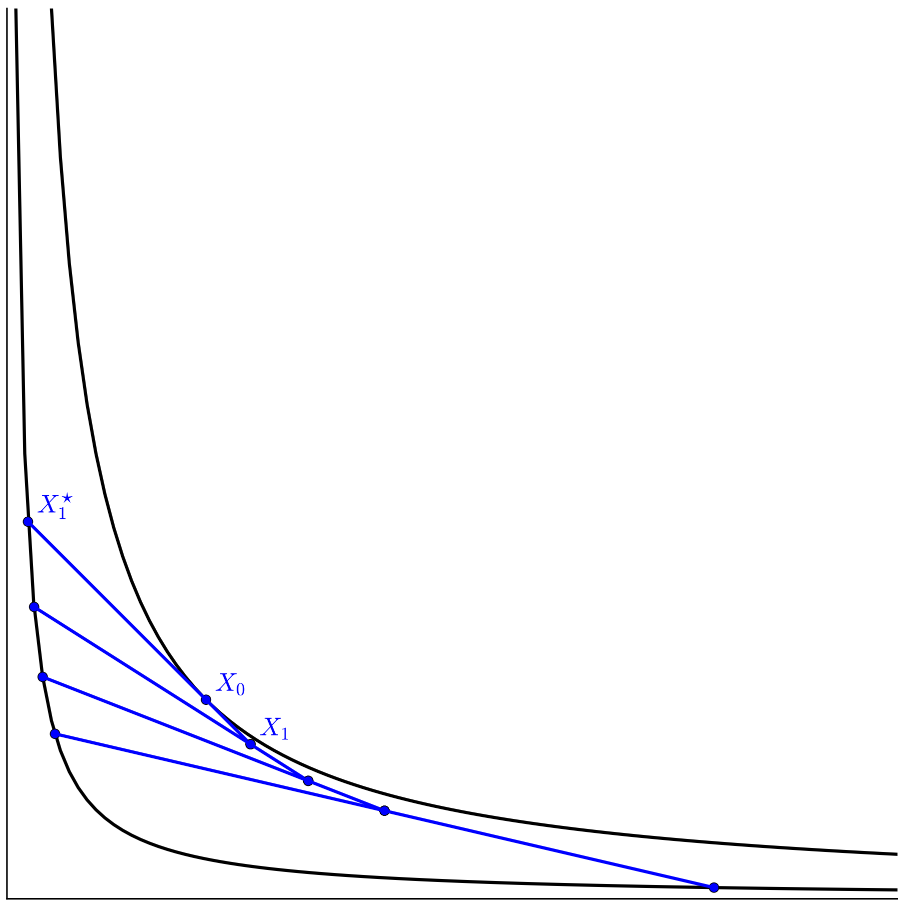
$$\sum_{I\in\mathcal{D}(I_0)} \frac{\alpha_-^I\alpha_+^I(\langle w\rangle_{I_+}-\langle w\rangle_{I_-})^2}{\alpha_-^I\langle w\rangle_{I_+}+\alpha_+^I\langle w\rangle_{I_-}}|I| = \sum_{I\in\mathcal{D}(I_0)} \frac{(\langle w\rangle_{I_+}-\langle w\rangle_{I_-})^2}{\left(\frac{\langle w\rangle_{I_+}}{\alpha_+^I}+\frac{\langle w\rangle_{I_-}}{\alpha_-^I}\right)}|I|$$

as large as possible.

Instead of choosing a filtration F and a weight $w(x)$ we choose a filtration F and successively find w by starting with its roughest approximation $w_0 = E(w \mid F_0)$ and that of $v_0 = E(w^{-1} \mid F_0)$.

The filtration F we choose has to be very unbalanced.




$X_1^\star$
X_0
X_1

We want a big jump on the left and a small jump on the right, with a balanced denominator in the contribution

$$\frac{(w_1 - w_1^\star)^2}{\frac{w_1}{\alpha_0} + \frac{w_1^\star}{\alpha_0^\star}} |I_0|$$

with $\frac{w_1}{\alpha_0} \sim \frac{w_1^\star}{\alpha_0^\star}$.

From the picture we see (for large w_0) that $w_1^\star \sim \frac{1}{w_0}$ and $w_1 - w_1^\star \sim w_0$ and with $\alpha_0^\star \sim \frac{1}{w_0^2}$.

Finally for one step

$$\frac{(w_1 - w_1^\star)^2}{\frac{w_1}{\alpha_1} + \frac{w_1^\star}{\alpha_1^\star}} |I_0| \gtrsim w_0 |I_0|$$

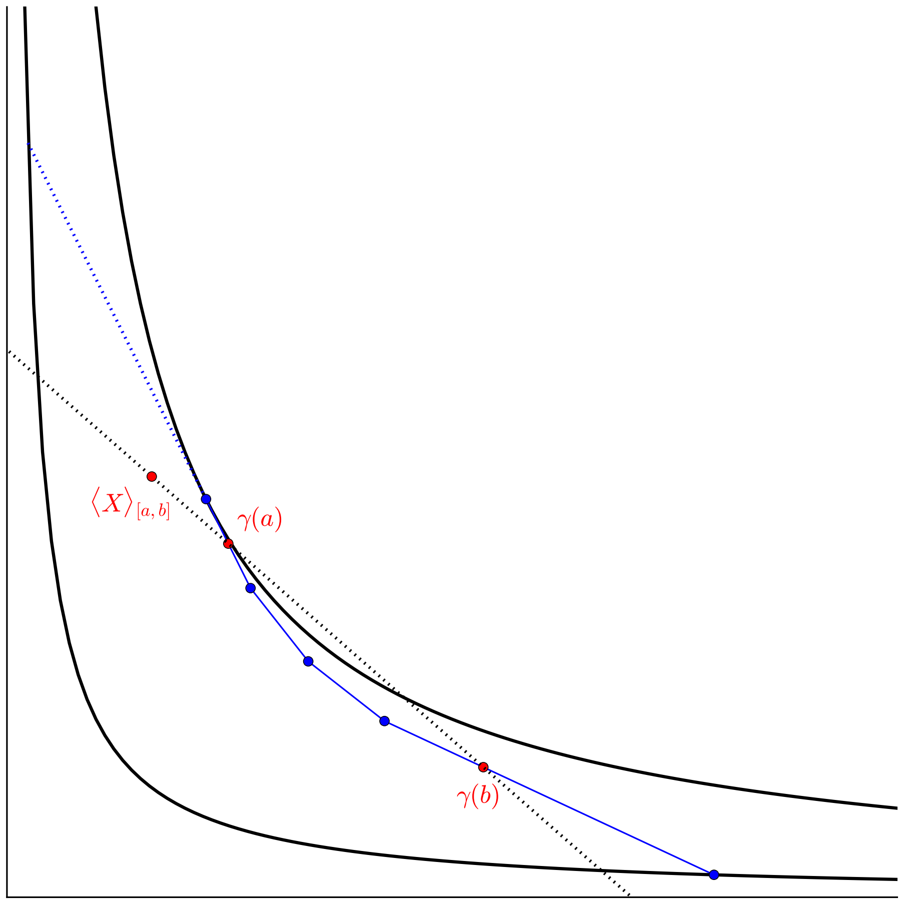
Lead by this, take $Q_0 \gg 1$. Set on the diagonal
$w_0 = \langle w \rangle_{I_0} = v_0 = \langle w^{-1} \rangle_{I_0} = \sqrt{Q_0}$ we choose $\alpha_0^\star = 1/Q_0$.
This allows us to easily calculate that $w_1 \sim (1 + 1/Q_0) w_0$ and $v_1 \sim (1 - 1/Q_0) v_0$ so that we know how far we have gone down already after this first step: $Q_1 \sim (1 - 1/Q_0^2) Q_0$
and after k steps, as long as we $Q_k \geq Q_0/2$ we have $Q_k \sim (1 - c/Q_0^2)^k Q_0$.

As long as we stay high enough the contributions do not change very much

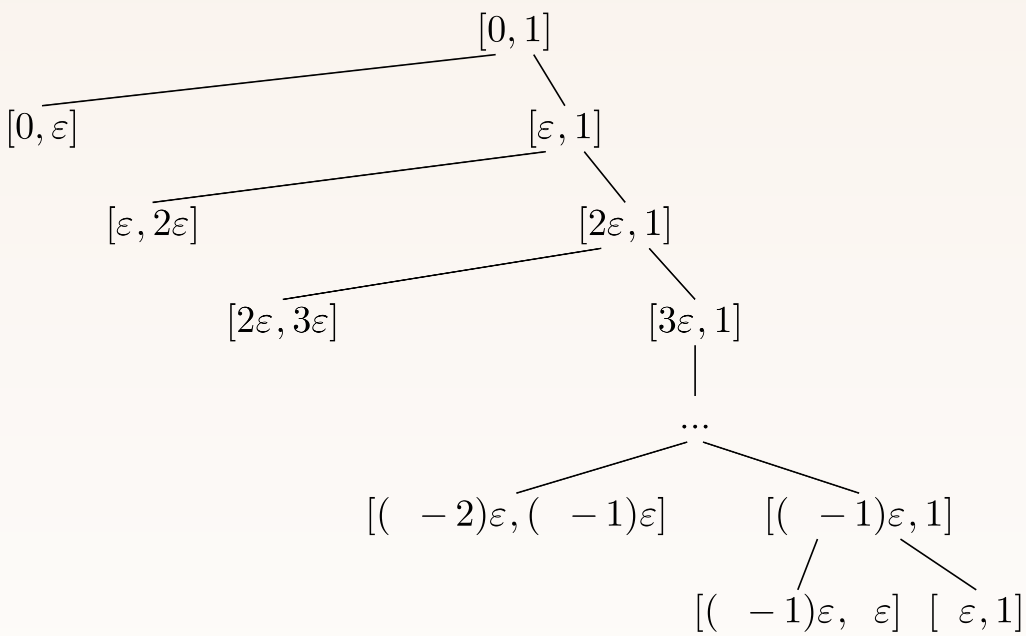
$$u_k |I_k| \geq (1 - c/Q_0^2)^k u_0 |I_0|$$

both happen on the order of Q_0^2 times.

Why is this weight in A_2^{cl}. Set $X = (w, v)$ and $\gamma(t) = \langle X \rangle_{[t,1]}$



Take $w(x) = x$ on $\Omega = [0,1]$ and the filtration



Take $\varepsilon = 1/N$, compute $\langle w \rangle_{I_k^\pm}$ and $\alpha_{I_k^\pm}$. It follows for any N

$$\sum_{I \in \mathcal{D}(I_0)} \frac{\alpha_-^I \alpha_+^I (\langle w \rangle_{I_+} - \langle w \rangle_{I_-})^2}{\alpha_-^I \langle w \rangle_{I_+} + \alpha_+^I \langle w \rangle_{I_-}} |I| \gtrsim \sum_{k=1}^{N} \frac{(1 - k/N)^3}{k} \gtrsim \sum_{k=1}^{N} \frac{1 - 3k/N}{k} \gtrsim \ln N$$