Recent progress on decouplings

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Decouplings

Let $(f_j)_{j=1}^N$ be N elements of a Banach space X. The triangle inequality

$$\|\sum_{j=1}^N f_j\|_X \le \sum_{j=1}^N \|f_j\|_X$$

is universal, it does not incorporate any possible cancellations between the f_j. It leads to

$$\|\sum_{j=1}^N f_j\|_X \le N^{\frac{1}{2}}(\sum_{j=1}^N \|f_j\|_X^2)^{1/2}.$$

But if X is a Hilbert space (think $X = L^2(\mathbb{T}^n)$) and if $f_{\mathbf{j}} \in X, \mathbf{j} \in J$ are pairwise orthogonal (think $f_{\mathbf{j}}(\mathbf{x}) = e(\mathbf{x} \cdot \mathbf{j})$) then we have

l^2 **decoupling** $$\|\sum_{\mathbf{j}} f_{\mathbf{j}}\|_X \le (\sum_{\mathbf{j}} \|f_{\mathbf{j}}\|_X^2)^{1/2}$$

l^p **decoupling** $$\|\sum_{\mathbf{j}\in J} f_{\mathbf{j}}\|_X \le |J|^{\frac{1}{2}-\frac{1}{p}}(\sum_{\mathbf{j}\in J} \|f_{\mathbf{j}}\|_X^p)^{1/p}$$

Motivated in part by investigations by Thomas Wolff from late 1990s, Bourgain and I have developed a decoupling theory for L^p spaces. In a nutshell, our theorems go as follows:

Theorem (Abstract decoupling theorem)

Let $f : \mathcal{M} \to \mathbb{C}$ be a function on some compact manifold $\mathcal{M}$ in $\mathbb{R}^n$, with natural measure σ. Partition the manifold into caps τ of size δ (with some variations forced by curvature) and let $f_\tau = f 1_\tau$ be the restriction of f to τ. Then there is a critical index $p_c > 2$ and some $q \geq 2$ (both depending on the manifold) so that we have an l^2 (or sometimes just the analogous l^p) decoupling

$$\|\widehat{fd\sigma}\|_{L^p(B_{\delta^{-q}})} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{\tau:\delta-cap} \|\widehat{f_\tau d\sigma}\|^2_{L^p(B_{\delta^{-q}})}\Big)^{1/2}$$

for each ball $B_{\delta^{-q}}$ in $\mathbb{R}^n$ with radius δ^{-q} and each $2 \leq p \leq p_c$.

Most of the applications of our abstract decoupling theorem for estimating exponential sums rely on the very simple observation that for each $\xi \in \mathbb{R}^n$ the Fourier transform of the Dirac delta distribution

$$\delta_\xi(\eta) := \begin{cases} 1, & \eta = \xi \\ 0, & \eta \neq \xi \end{cases}$$

is an exponential

$$\widehat{\delta_\xi}(x) = e(x \cdot \xi)$$

Bourgain's observation (2011): To get from...

Theorem (Abstract decoupling theorem)

$$\|\widehat{fd\sigma}\|_{L^p(B_{\delta^{-q}})} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{\tau:\delta-cap}\|\widehat{f_\tau d\sigma}\|^2_{L^p(B_{\delta^{-q}})}\Big)^{1/2}$$

for each ball $B_{\delta^{-q}}$ in $\mathbb{R}^n$ with radius δ^{-q} and each $2 \le p \le p_c$.

...to the exponential sum estimate

Theorem (Discrete decoupling)

For each cap τ let $\xi_\tau \in \tau$ and $a_\tau \in \mathbb{C}$. Then

$$|B_{\delta^{-q}}|^{-1/p}\|\sum_\tau a_\tau e(\xi_\tau \cdot \mathbf{x})\|_{L^p(B_{\delta^{-q}})} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_\tau |a_\tau|^2\Big)^{1/2}$$

for each ball $B_{\delta^{-q}}$ in $\mathbb{R}^n$ with radius δ^{-q} and each $2 \le p \le p_c$,

apply the decoupling to (a smooth approximation of) $f(\xi) = \sum_\tau a_\tau \delta_{\xi_\tau}$

Key ingredient: Multilinear Kakeya (a decoupling in disguise)

Theorem (Bennetr, Carbery, Tao, 2006)

Consider n families $\mathcal{T}_j$ consisting of $R \times R^{1/2} \times \ldots \times R^{1/2}$ tubes $T \subset B_{4R}$ in $\mathbb{R}^n$ having the following property

Transversality: *The direction of the long axis of $T \in \mathcal{T}_j$ is in a small neighborhood of $e_j = (0, \ldots, 1, \ldots, 0)$*

Then we have the following inequality ($⨏$ denotes the average)

$$⨏_{B_{4R}} |\prod_{j=1}^{n} F_j|^{\frac{1}{2n}\frac{2n}{n-1}} \lesssim_\epsilon R^\epsilon \left[\prod_{j=1}^{n} |⨏_{B_{4R}} F_j|^{\frac{1}{2n}}\right]^{\frac{2n}{n-1}} \quad (1)$$

for all functions F_j of the form

$$F_j = \sum_{T \in \mathcal{T}_j} c_T 1_T.$$

Some examples with sharp decoupling theory

• Hypersurfaces in $\mathbb{R}^n$ with nonzero Gaussian curvature ($p_c = \frac{2(n+1)}{n-1}$). **Many applications:** Optimal Strichartz estimates for Schrödinger equation on both rational and irrational tori in all dimensions, improved L^p estimates for the eigenfunctions of the Laplacian on the torus, etc

• The cone (zero Gaussian curvature) in $\mathbb{R}^n$ ($p_c = \frac{2n}{n-2}$). **Many applications:** progress on Sogge's "local smoothing conjecture for the wave equation", etc

• (Bourgain) Two dimensional surfaces in $\mathbb{R}^4$ ($p_c = 6$). **Application:** Bourgain used this to improve the estimate in the Lindelöf hypothesis for the growth of Riemann zeta

• (with Bourgain and Guth) Curves with torsion in $\mathbb{R}^n$ ($p_c = n(n+1)$). **Application:** Vinogradov's Mean Value Thm.

• (with Bourgain and Guo) Surfaces in $\mathbb{R}^9$ ($p_c = 20$). **Application:** Parsell-Vinogradov systems

Some recent developments by others

1. Fan, Staffilani, Wang and Wilson obtained a new proof of the decoupling theorem for the paraboloid in $\mathbb{R}^3$ that does not make use of trilinearity.

2. Du, Guth and Li combined the polynomial method with sharp decouplings for the parabola to prove

Theorem (Improved bilinear Strichartz estimate in $\mathbb{R}^2$)

Let I_1, I_2 be two separated intervals in $[-1,1]$ and let $f_i : I_i \to \mathbb{C}$. Let σ be the standard measure on the parabola. Let $S_1, \dots, S_N$ be (dyadic) squares with side length $\sqrt{R}$ inside $[-R,R]^2$ such that

$$\int_{S_j} |\widehat{f_i d\sigma}|^6 \sim C_i, \quad i = 1,2$$

for each S_j. Then

$$\| |\widehat{f_1 d\sigma} \widehat{f_2 d\sigma}|^{\frac12} \|_{L^6(S_1 \cup \dots S_N)} \lesssim N^{-\frac16} R^{\epsilon} (\|f_1\|_2 \|f_2\|_2)^{\frac12}$$

Note the $N^{-\frac16}$ gain over the classical (unrestricted) estimate.

Many questions are still left open, about decouplings on curves, on the cone, manifolds of intermediate co-dimension, with further potential applications to number theory.

Question

Can one make progress on the conjectured estimate

$$\|\sum_{n=1}^{N} a_n e(nx+n^3y)\|_{L^p_{dx,dy}([0,1]^2)} \lesssim N^{\epsilon}\|a_n\|_{l^2},\quad 2\le p\le 8$$

using decouplings?

If yes, it would have to involve a very subtle/novel argument, for the following reason.

$$C_1 = \{(t, t^2)\}$$

$$C_2 = \{(t, \phi(t) := t^3) : t \sim 1\} : \text{ Note that } \phi'', \phi''' \sim 1$$

We know that for both C_1, C_2 the following l^2 decoupling holds

$$\|\widehat{fd\sigma_{C_i}}\|_{L^p(B_{\delta^{-2}})} \lesssim_\epsilon \delta^{-\epsilon}(\sum_{\tau\subset C_i:\delta-arc} \|\widehat{f_\tau d\sigma_{C_i}}\|^2_{L^p(B_{\delta^{-2}})})^{1/2},$$

within the range $2 \le p \le 6$. This range is sharp both C_i:

$$\|\widehat{fd\sigma_{C_i}}\|_{L^p(B_1)} \lesssim_\epsilon \delta^{-\epsilon}(\sum_{\tau\subset C_i:\delta-arc} \|\widehat{f_\tau d\sigma_{C_i}}\|^2_{L^p(\mathbb{R}^2)})^{1/2}$$

is false for $p > 6$, even when $\mathbb{R}^2$ is placed on the right. (test with $f \equiv 1$.)

Question

Is it true that for each ~ 1-separated set Γ with $\sim N$ points on

$$\{(x, x^3) : 1 \le x \le N\}$$

we have

$$|\{(\lambda_1, \dots, \lambda_8) \in \Gamma^8 : \lambda_1 + \dots + \lambda_4 = \lambda_5 + \dots + \lambda_8\}| \lesssim_\epsilon N^{4+\epsilon}?$$

Let

$$E_I g(x_1,\ldots,x_n)=\widehat{g_I d\sigma}(x_1,\ldots,x_n)=\int_I g(t)e(tx_1+\ldots+t^n x_n)dt$$

The next question is about $l^p(L^p)$ decoupling at spatial scale N^2.

Question

What is the largest $p=p_n$ for which we have $l^p(L^p)$ decoupling

$$\|E_{[0,1]}g\|_{L^p(B_{N^2})}\lesssim_\epsilon N^{\frac12-\frac1p+\epsilon}\Big(\sum_{|I|=N^{-1}}\|E_I g\|^p_{L^p(B_{N^2})}\Big)^{1/p}.$$

If one integrates over larger balls B_{N^n} then Bourgain-D-Guth proved the above holds for p as large as $n(n+1)$.
This is connected with open questions about the size of

$$\int_{\mathbb{T}^{n-1}}\int_0^{\frac{1}{N^\alpha}}|\sum_{k\sim N}e(kx_1+\ldots+k^n x_n)|^p dx_1\ldots dx_n,\quad 0<\alpha<1$$

Question

What is the largest $p = p_n$ for which we have

$$\|E_{[0,1]}g\|_{L^p(B_{N^2})} \lesssim_\epsilon N^{\frac{1}{2}-\frac{1}{p}+\epsilon}\Big(\sum_{|I|=N^{-1}} \|E_I g\|^p_{L^p(B_{N^2})}\Big)^{1/p}.$$

We know that p_n is nondecreasing, but also bounded from above ($p_n \le 22$).

We also know $p_2 = 6$ (standard **linear** decoupling for the parabola)

Using **bilinear** methods one can prove $p_3 \ge 8$ and $p_4 \ge 12$.

I will next show how to use **trilinear** arguments to prove that $p_5 \ge 14$.
None of these lower bounds is known to be sharp.

For two symmetric matrices $A_1, A_2 \in M_3(\mathbb{R})$ consider the quadratic forms

$$Q_i(r,s,t) = [r,s,t]A_i[r,s,t]^T$$

and the associated three dimensional quadratic surface in $\mathbb{R}^5$

$$S_{Q_1,Q_2} := \{(r,s,t,Q_1(r,s,t),Q_2(r,s,t)) : (r,s,t) \in [0,1]^3\}.$$

Theorem (D,Guo,Shi, to appear in Revista)

Assume that for each nonzero vector $(u,v,w) \in \mathbb{R}^3$, the following **curvature** *condition holds: the determinant*

$$P(r,s,t) := \det \begin{bmatrix} \frac{\partial Q_1}{\partial r} & \frac{\partial Q_1}{\partial s} & \frac{\partial Q_1}{\partial t} \\ \frac{\partial Q_2}{\partial r} & \frac{\partial Q_2}{\partial s} & \frac{\partial Q_2}{\partial t} \\ u & v & w \end{bmatrix}$$

is not the zero polynomial, when regarded as a function of r,s,t. Then there is an $l^p(L^p)$ decoupling for $2 \le p \le \frac{14}{3}$. (sharp range)

• This is a decoupling on (spatial) balls B_{N^2} into (frequency) cubes with side length N^{-1}.

• Restriction theorems for 3 dimensional manifolds in $\mathbb{R}^5$ have been established under various assumptions by Christ (PhD thesis), De Carli and Iosevich (1998)...

• Our curvature condition is very general, in fact it may be also (at least very close to being) necessary in order to have the range $2 \le p \le \frac{14}{3}$.

• The very symmetric manifold

$$\{(r,s,t,r^2+s^2+t^2,rs+rt+st) : (r,s,t) \in [0,1]^3\}$$

fails to satisfy our curvature condition. We could prove that there is no l^p decoupling for $p > 4$.

The main new difficulties associated with our theorem (compared to previous decoupling results) are

• Identifying the "correct" notion of curvature

• Linear algebra associated with checking Brascamp-Lieb-type transversality conditions.

• Subtle analytic issues associated with cylindrical decoupling needed to address the complexity of the geometry of our manifolds.

• Lower dimensional contribution in the Bourgain-Guth iteration may come from a **non planar** 2-variety in $\mathbb{R}^3$ (the zero set of $P(r,s,t)$). We use an approximation argument/induction on scales à la Seeger-Pramanik, Bourgain-D. which most closely resembles a very recent argument of Oh. The point is to make quantitative use of the fact that 2-varieties are locally close to planes.

Back to the original problem, i.e proving that we have an $l^p(L^p)$ decoupling at spatial scale N^2

$$\|E_{[0,1]}g\|_{L^p(B_{N^2})} \lesssim_\epsilon N^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})+\epsilon}(\sum_{|I|=N^{-1}} \|E_I g\|^p_{L^p(B_{N^2})})^{1/p}$$

for $2 \le p \le 14$ for the moment curve (t, t^2, t^3, t^4, t^5).

The idea is to first get the L^{14} estimate for the trilinear term. Note that with

$$\gamma_{5,I} = \{(t, t^2, t^3, t^4, t^5) : t \in I\}$$

we have (I_i separated intervals in $[0,1]$) that

$$\gamma_{5,I_1} + \gamma_{5,I_2} + \gamma_{5,I_3}$$

is a subset of the manifold

$$S := (r, s, t, \frac{r^4}{6} - r^2 s + \frac{4rt}{3} + \frac{s^2}{2}, \frac{r^5}{6} - \frac{5r^3 s}{6} + \frac{5r^2 t}{6} + \frac{5st}{6})$$

via Newton's identities.

$$S := (r, s, t, Q_1 := \frac{r^4}{6} - r^2 s + \frac{4rt}{3} + \frac{s^2}{2}, Q_2 := \frac{r^5}{6} - \frac{5r^3 s}{6} + \frac{5r^2 t}{6} + \frac{5st}{6})$$

While the entries Q_1 and Q_2 are not quadratic (as in our theorem), we can use Taylor's formula and induction on scales à la Seeger-Pramanik to reduce matters to our theorem.
Our curvature condition turns out to be good enough.

This argument gives a trilinear decoupling for our moment curve via the key formula

$$14 = \frac{14}{3} \times 3$$

The passage to a linear decoupling is done via a (slightly less standard) Bourgain-Guth type iteration.

Other questions that I would love to see solved (hopefully making use of, or drawing inpiration from decouplings) until we meet again at MSRI are

• The restriction conjecture (at least in $\mathbb{R}^3$, at least modulo Kakeya-type estimates)

• The L^4 square function estimate for the cone in $\mathbb{R}^3$