Tensor Valuations on Lattice Polytopes

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Valuations on Convex Bodies

- $\mathcal{F}$ family of subsets of $\mathbb{R}^n$
  - $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
  - $\mathcal{P}^n$ space of convex polytopes in $\mathbb{R}^n$
- $\langle \mathcal{A}, + \rangle$ Abelian semigroup
- A function $\mathrm{Z} : \mathcal{F} \to \langle \mathcal{A}, + \rangle$ is a *valuation* $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{F}$ such that $K \cup L, K \cap L \in \mathcal{F}$.

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Examples

- $K \mapsto V_n(K)$ n-dimensional volume of K
- $K \mapsto S(K)$ surface area of K
- $K \mapsto \operatorname{L}(K)$ number of points in $K \cap \mathbb{Z}^n$
- $K \mapsto \int_K x \, \mathrm{d}x$ moment vector of K

Classification of Valuations on Convex Bodies

- **Real valuations**:
  Blaschke 1937, Hadwiger 1949, McMullen 1977, Klain 1995, Alesker 1998, Ludwig 1999, Ludwig & Reitzner 1999, Bernig & Fu 2011, Haberl & Parapatits 2014, ...
- **Vector and tensor valuations**:
  Hadwiger & Schneider 1971, Schneider 1972, McMullen 1977, Alesker 1999, Ludwig 2002, Haberl & Parapatits 2016, Bernig & Hug 2017+, Ma & Zeng 2017+ ...
- **Convex body valued and star body valued valuations:**
  Schneider 1974 (Minkowski endomorphisms), Ludwig 2002 (Minkowski valuations), Kiderlen 2006, Haberl & Ludwig 2006, Ludwig 2006, Schneider & Schuster 2006, Schuster 2007, Haberl 2009, Abardia & Bernig 2011, Wannerer 2011, Schuster & Wannerer 2012, Abardia 2012, Parapatits 2014, Li & Yuan & Leng 2015, Li & Leng 2016, Villanueva 2016, Villanueva & Tradacete 2016, ...

The Hadwiger Classification Theorem 1952

Theorem

$Z : \mathcal{K}^n \to \mathbb{R}$ is a rigid motion invariant and continuous valuation

$$\Longleftrightarrow$$

$\exists\ c_0, c_1, \dots, c_n \in \mathbb{R}$:

$$Z(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

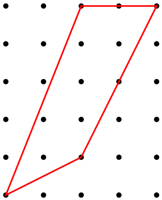
$V_0(K), \dots, V_n(K)$	intrinsic volumes of K
$V_n(K)$	n-dimensional volume of K
$2\, V_{n-1}(K) = S(K)$	surface area of K

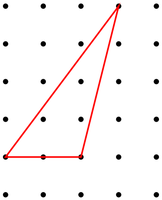
V_i is i-homogeneous: $V_i(s\,K) = s^i\, V_i(K)$

Steiner formula: $V_n(K + s\, B^n) = \sum_{i=0}^{n} s^{n-i} v_{n-i}\, V_i(K)$

Lattice Polytopes

- P lattice polytope in $\mathbb{R}^n$
  $\iff$ P is the convex hull of finitely many points from $\mathbb{Z}^n$





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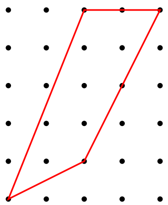

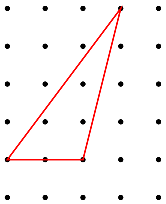

- Applications
  - Integer programming
  - Geometry of numbers
  - Algebraic geometry (Newton polytope)

Invariant Valuations on Lattice Polytopes

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$$\Longleftrightarrow$$

$\mathrm{Z}(\phi P) = \mathrm{Z}(P)$ for all $\phi \in \mathrm{SL_n}(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^n)$

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- $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{A}$ is translation invariant

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$$\mathrm{Z}(P + x) = \mathrm{Z}(P) \text{ for all } x \in \mathbb{Z}^n \text{ and } P \in \mathcal{P}(\mathbb{Z}^n)$$

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Question.

Classification of $\mathrm{SL_n}(\mathbb{Z})$ and translation invariant valuations on $\mathcal{P}(\mathbb{Z}^n)$.

The Betke & Kneser Theorem 1985

Theorem

$\operatorname{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\operatorname{SL_n}(\mathbb{Z})$ *and translation invariant valuation*

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Betke & Kneser: unimodular valuations

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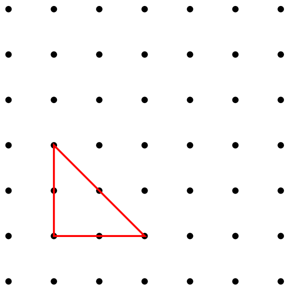

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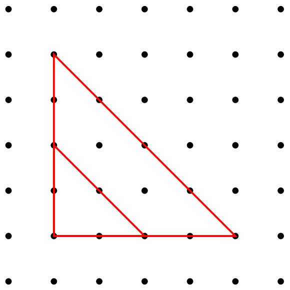

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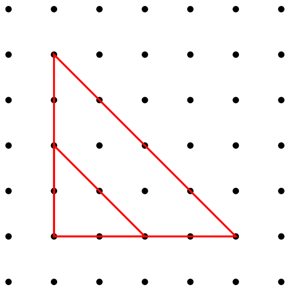

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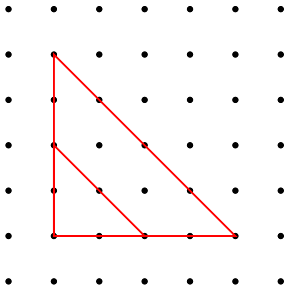

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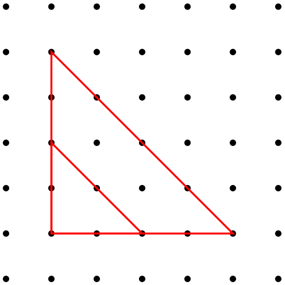

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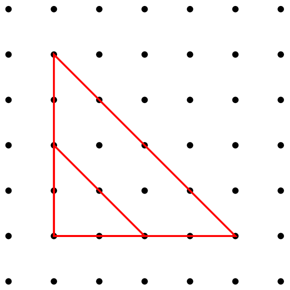

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- Computational aspects

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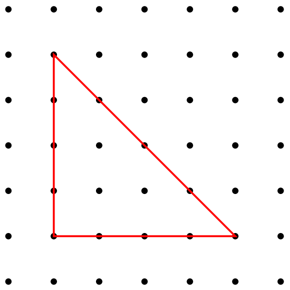

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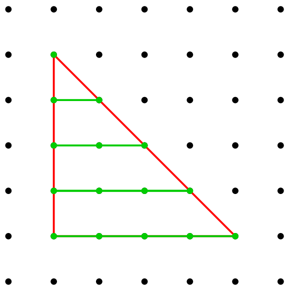

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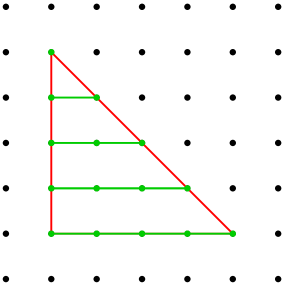

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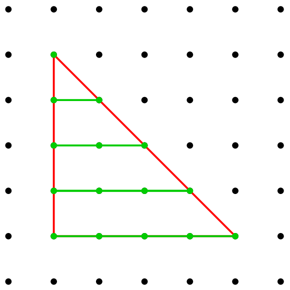

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- S basic n-dimensional simplex $\Rightarrow$

  $S = \phi T_n$ with $\phi \in \mathrm{SL_n}(\mathbb{Z})$

  $\mathrm{L}(k\, S) = \sum_{i=0}^{n} \mathrm{L}_i(S)\, k^i$

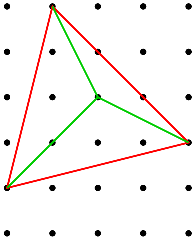
Inclusion-Exclusion Principle

Theorem (Betke 1979; McMullen: AiM 2009)

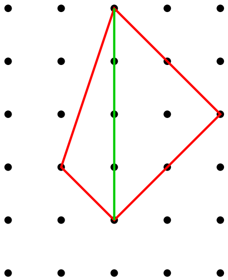
Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *be a valuation. If* $\mathcal{F}$ *is a cell decomposition of the n-dimensional lattice polytope* P*, then*

$$\mathrm{Z}(P) = \sum_F (-1)^{n-\dim F}\, \mathrm{Z}(F),$$

where we sum over all faces of $\mathcal{F}$ *that intersect the interior of* P*.*



Dissections of Lattice Polytopes



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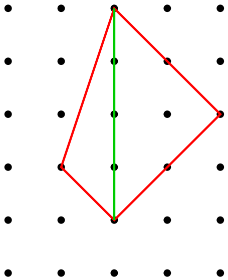

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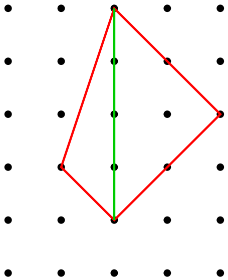

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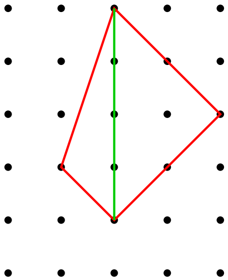

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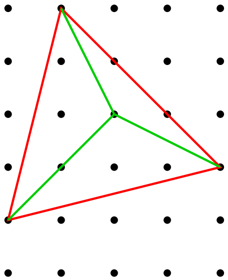
  Reeve tetrahedron [(0,0,0), (1,0,0), (0,1,0), (1,1,k)] with $k \in \mathbb{N}$

Dissections and Complementations



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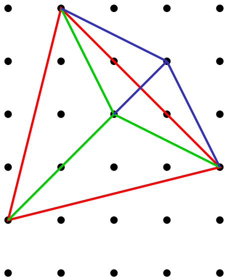

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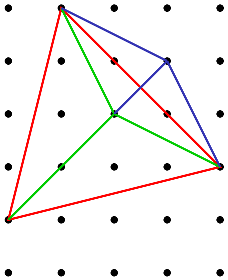

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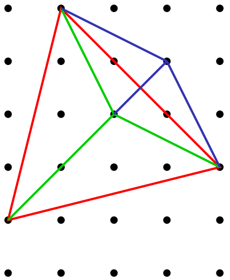

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- Similar arguments for dissecting and complementing by basic simplices

  $\Rightarrow$ existence of Ehrhart polynomial

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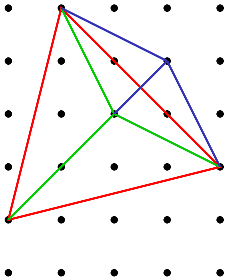

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  $\Rightarrow$ Betke & Kneser theorem

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- Ehrhart Theory

$$\mathrm{L}(kP) = \sum_{j=0}^{n} h_j^*(P) \binom{k+n-j}{n}$$

$\dim P = n \Rightarrow h_j^*(P)$ non-negative, monotone
(Stanley 1980)

Classification Theorems

Theorem (Betke & Kneser)

$\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation*

$$\iff$$

$\exists\ c_0, \dots, c_n \in \mathbb{R}$:

$$\mathrm{Z} = c_0\, \mathrm{L}_0 + \cdots + c_n\, \mathrm{L}_n$$

Theorem (Hadwiger)

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  - $\mathcal{P}(\mathbb{Z}^n)$ space of lattice polytopes in $\mathbb{R}^n$
- $\langle \mathcal{K}^n, + \rangle$ convex bodies with Minkowski addition

  $K + L = \{x + y : x \in K, y \in L\}$ Minkowski sum of K and L

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- $\mathcal{F}$ family of subsets of $\mathbb{R}^n$
- - $\mathcal{K}^n$ space of convex bodies in $\mathbb{R}^n$
  - $\mathcal{P}^n$ space of convex polytopes in $\mathbb{R}^n$
  - $\mathcal{P}(\mathbb{Z}^n)$ space of lattice polytopes in $\mathbb{R}^n$
- $\langle \mathcal{K}^n, + \rangle$ convex bodies with Minkowski addition

  $K + L = \{x + y : x \in K, y \in L\}$ Minkowski sum of K and L
- A function $\operatorname{Z} : \mathcal{F} \to \langle \mathcal{K}^n, + \rangle$ is a **Minkowski valuation** $\iff$

$$\operatorname{Z} K + \operatorname{Z} L = \operatorname{Z}(K \cup L) + \operatorname{Z}(K \cap L)$$

for all $K, L \in \mathcal{F}$ such that $K \cup L, K \cap L \in \mathcal{F}$.

Classification of Minkowski Valuations

- $\operatorname{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is translation invariant $\iff$

$$\operatorname{Z}(K + x) = \operatorname{Z} K$$

for $x \in \mathbb{R}^n$, $K \in \mathcal{K}^n$

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- $\operatorname{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is $\mathrm{SL_n}(\mathbb{R})$ equivariant $\iff$

$$\operatorname{Z}(\phi K) = \phi \operatorname{Z} K$$

  for $\phi \in \mathrm{SL_n}(\mathbb{R})$, $K \in \mathcal{K}^n$

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- Corresponding definitions for $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}^n$

Classification of Minkowski Valuations

Theorem (L.: TAMS 2005)

$\mathrm{Z} : \mathcal{P}^n \to \langle \mathcal{K}^n, + \rangle$ *is an* $\mathrm{SL_n}(\mathbb{R})$ *equivariant and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists\;\; c \geqslant 0:$

$$\mathrm{Z}\, P = c(P + (-P))$$

for every $P \in \mathcal{P}^n$.

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Theorem (Böröczky & L.: JEMS 2016+)

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$$\iff$$

$\exists \;\; a, b \geqslant 0 :$

$$\mathrm{Z}\, P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

Discrete Moment Vectors

$$\ell(P) = \sum_{x \in P \cap \mathbb{Z}^n} x \qquad \text{discrete moment vector of } P$$

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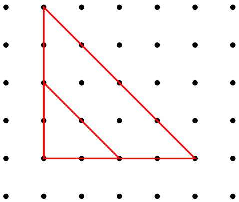

$$\ell(kP) = \sum_{i=1}^{n+1} \ell_i(P)\, k^i$$

(Ehrhart vector polynomial, McMullen 1977)

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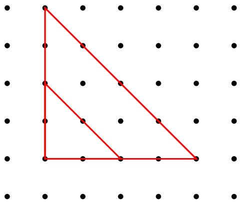

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(Ehrhart vector polynomial, McMullen 1977)

- $\ell_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ is an $\mathrm{SL_n}(\mathbb{Z})$ equivariant valuation, which is homogeneous of degree i
- $\ell_{n+1}(P) = \int_P x \, \mathrm{d}x = m_{n+1}(P)$ moment vector of $P \in \mathcal{P}(\mathbb{Z}^n)$

Discrete Steiner Point

Theorem (Böröczky & L.: JEMS 2016+)

$\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation equivariant valuation*

$$\Longleftrightarrow$$

$$\mathrm{Z} = \ell_1$$

- $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is translation equivariant $\Longleftrightarrow$
  $\mathrm{Z}(P + x) = \mathrm{Z}(P) + x \quad \forall\, x \in \mathbb{Z}^n,\ P \in \mathcal{P}(\mathbb{Z}^n)$

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$$\iff$$

$$\mathrm{Z} = \ell_1$$

Theorem (Schneider 1972)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is a rigid motion equivariant and continuous valuation*

$$\iff$$

$$\mathrm{Z} = m_1$$

- $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is translation equivariant $\iff$
  $\mathrm{Z}(P + x) = \mathrm{Z}(P) + x \quad \forall\, x \in \mathbb{Z}^n,\ P \in \mathcal{P}(\mathbb{Z}^n)$
- $m_1(K) = \dfrac{1}{v_n} \displaystyle\int_{\mathbb{S}^{n-1}} u\, h(K, u)\, \mathrm{d}u$ Steiner point of K

Classification of Vector Valuations

Theorem (L. & Silverstein: AiM 2017+)

$\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant, translation covariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_1, \dots, c_{n+1} \in \mathbb{R}: \quad \mathrm{Z} = c_1 \ell_1 + \cdots + c_{n+1}\, \ell_{n+1}$

- $\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is translation covariant $\iff \exists\, \mathrm{Z}^0 : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}:$ $\mathrm{Z}(P + x) = \mathrm{Z}(P) + \mathrm{Z}^0(P)x \quad \forall\ x \in \mathbb{Z}^n,\ P \in \mathcal{P}(\mathbb{Z}^n)$

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Theorem (Hadwiger & Schneider 1971)

$\operatorname{Z}:\mathcal{K}^n\to\mathbb{R}^n$ *is a rotation equivariant, translation covariant, continuous valuation*

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- Steiner formula: $m_{n+1}(K+s\,B^n)=\sum_{j=1}^{n+1}s^{n+1-j}v_{n+1-j}\,m_j(K)$

Classification of Vector Valuations

Theorem (L. & Silverstein: AiM 2017+)

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Theorem (Ludwig 2002; Haberl & Parapatits 2016)

$\mathrm{Z}: \mathcal{P}^n_{(0)} \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *equivariant and measurable valuation*

$$\iff$$

$\exists\ c \in \mathbb{R}: \quad \mathrm{Z} = c\, m_{n+1}$

Tensor Valuations

- $\mathbb{T}^r$ symmetric tensors of rank r in $\mathbb{R}^n$
- Valuations $\mathrm{Z} : \mathcal{K}^n \to \mathbb{T}^r$
  - McMullen 1997
  - Polynomial valuations: Khovanskiĭ & Pukhlikov 1992

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- $\mathrm{M}^r(K) = \frac{1}{r!} \int_K \underbrace{x \odot \cdots \odot x}_{r} \, dx = \frac{1}{r!} \int_K x^r \, dx$

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- Applications in Stochastic Geometry, Medical Imaging, Material Sciences, ...

  M. Kiderlen and E. Vedel Jensen (eds.), *Tensor Valuations and their Applications in Stochastic Geometry and Imaging*, Lecture Notes in Math., 2177, Springer, Cham, 2017.

Classification of Tensor Valuations

Theorem (Alesker: Annals 1999, Geom. Dedicata 1999)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{T}^r$ *is a rotation equivariant, translation covariant, continuous valuation*

$$\Longleftrightarrow$$

Z *is a linear combination of* $Q^l\, \Phi_k^{m,s}$ *with* $2l + m + s = r$.

- $\Phi_k^{m,s}(K) = \int\limits_{\mathbb{R}^n \times \mathbb{S}^{n-1}} x^m\, u^s\, d\Theta_k(K,(x,u))$ Minkowski tensors

  McMullen 1997
- $\Theta_k(K,\cdot)$ k-th generalized curvature measure, Q metric tensor
- Steiner formula:

$$\mathrm{M}^r(K + s\,B^n) = \sum_{j=1}^{n+r} s^{n+1-j}\, v_{n+1-j} \sum_{k \in \mathbb{N}} \Phi_{j-r+k}^{r-k,k}(K)$$

Translation Covariance

$$\mathrm{M}^r(K+y) = \sum_{j=0}^{r} \mathrm{M}^{r-j}(K)\frac{y^j}{j!}$$

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Definition (McMullen 1997)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{T}^r$ is called **translation covariant** if there exist associated functions $\mathrm{Z}^j : \mathcal{K}^n \to \mathbb{T}^j$ for $j = 0, \dots, r$ such that

$$\mathrm{Z}(K+y) = \sum_{j=0}^{r} \mathrm{Z}^{r-j}(K)\frac{y^j}{j!}$$

for all $y \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$.

$\mathrm{SL_n}(\mathbb{Z})$ Equivariance

Definition

$\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is $\mathrm{SL_n}(\mathbb{Z})$ equivariant

$$\iff$$

$$\mathrm{Z}(\phi K) = \phi\, \mathrm{Z}\, K,$$

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$$\iff$$

If $\mathrm{Z}(K) = \sum v_{i_1} \odot \cdots \odot v_{i_r}$, then $\mathrm{Z}(\phi K) = \sum \phi v_{i_1} \odot \cdots \odot \phi v_{i_r}$.

$\mathrm{SL_n}(\mathbb{Z})$ Equivariance

Definition

$Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is $\mathrm{SL_n}(\mathbb{Z})$ equivariant

$$\iff$$

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If $Z(K) = \sum v_{i_1} \odot \cdots \odot v_{i_r}$, then $Z(\phi K) = \sum \phi v_{i_1} \odot \cdots \odot \phi v_{i_r}$.

Question.

Classification of $\mathrm{SL_n}(\mathbb{Z})$ equivariant and translation covariant tensor valuations on $\mathcal{P}(\mathbb{Z}^n)$.

Discrete Moment Tensors

$$\mathrm{L}^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} \underbrace{x \odot \cdots \odot x}_{r} = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r \qquad \text{discrete moment tensor of } P$$

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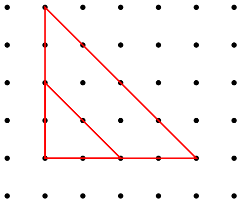

$$\mathrm{L}^r(kP) = \sum_{i=1}^{n+r} \mathrm{L}_i^r(P)\, k^i$$

Khovanskiĭ & Pukhlikov 1992

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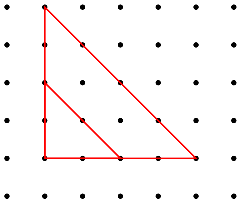

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- $\mathrm{L}_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is an $\mathrm{SL}_n(\mathbb{Z})$ equivariant valuation, which is homogeneous of degree i

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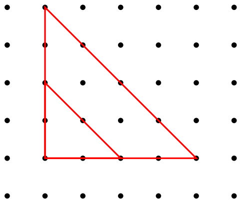

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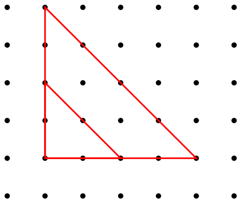

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- $\mathrm{L}_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is translation covariant
- $\mathrm{L}_{n+r}^r(P) = \frac{1}{r!} \int_P x^r \,\mathrm{d}x = \mathrm{M}^r(P)$ moment tensor of rank r of P

Discrete Moment Tensors

$$\mathrm{L}^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r \quad \text{discrete moment tensor of } P$$

- $\mathrm{L}^r(P)(e_1[r]) = \dfrac{1}{r!} \displaystyle\sum_{x \in P \cap \mathbb{Z}^n} (x \cdot e_1)^r$
- $\mathrm{L}^r(kT_1)(e_1[r]) = \dfrac{1}{r!} \displaystyle\sum_{i=1}^{k} i^r$

$$= \frac{1}{r+1} \sum_{l=0}^{r} (-1)^l \binom{r+1}{l} B_l k^{r+1-l}$$

Faulhaber's sum

Johann Faulhaber
(1580 - 1635)

Classification of Tensor Valuations

Theorem (L. & Silverstein: AiM 2017+)

$\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^2$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant, translation covariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_1, \dots, c_{n+2} \in \mathbb{R}$:

$$\mathrm{Z} = c_1\, \mathrm{L}_1^2 + \cdots + c_{n+2}\, \mathrm{L}_{n+2}^2$$

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$$\Longleftrightarrow$$

Z *is a linear combination of* $Q^l\, \Phi_k^{m,s}$ *with* $2l + m + s = 2$.

- Dimension of space of such valuations for $r = 2$ is $3n + 1$.

Classification of Tensor Valuations

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Theorem (Ludwig: DMJ 2003; Haberl & Parapatits: AiM 2017)

$\mathrm{Z} : \mathcal{P}_{(0)}^n \to \mathbb{T}^2$ *is an* $\mathrm{SL_n}(\mathbb{R})$ *equivariant and measurable valuation*

$$\iff$$

$\exists c_1, c_2 \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_1\, \mathrm{M}^2(P) + c_2\, \mathrm{M}^{0,2}(P^*)$$

for every $P \in \mathcal{P}_{(0)}^n$.

- $\mathrm{M}^{0,2}(P^*)$ LYZ tensor of P^* (Lutwak, Yang, Zhang: DMJ 2000)

Classification of Tensor Valuations

Theorem (L. & Silverstein: AiM 2017+)

For $1 \leqslant r \leqslant 8$, *a function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant and translation covariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_1, \dots, c_{n+r} \in \mathbb{R}$*:* $\quad \mathrm{Z} = c_1\,\mathrm{L}_1^r + \cdots + c_{n+r}\,\mathrm{L}_{n+r}^r$

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New Tensor Valuations

- New $\mathrm{SL}_2(\mathbb{Z})$ equivariant, translation invariant tensor valuation for $n = 2$ and $r = 9$:

$$\mathrm{N}^9(T_2) = \mathrm{L}_1^3(T_2) \odot \mathrm{L}_1^3(T_2) \odot \mathrm{L}_1^3(T_2)$$

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$$P = (\phi_1 T_2 + x_1) \sqcup \cdots \sqcup (\phi_m T_2 + x_m) \text{ with } \phi_i \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } x_i \in \mathbb{Z}^2$$

$$\mathrm{N}^9(P) = \sum_{i=1}^{m} \mathrm{L}_1^3(T_2)^3 \circ \phi_i^t$$

New Tensor Valuations

- New $\mathrm{SL}_2(\mathbb{Z})$ equivariant, translation invariant tensor valuation for $n = 2$ and $r = 9$:

$$\mathrm{N}^9(T_2) = \mathrm{L}_1^3(T_2) \odot \mathrm{L}_1^3(T_2) \odot \mathrm{L}_1^3(T_2)$$

$$P = (\phi_1 T_2 + x_1) \sqcup \cdots \sqcup (\phi_m T_2 + x_m) \text{ with } \phi_i \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } x_i \in \mathbb{Z}^2$$

$$\mathrm{N}^9(P) = \sum_{i=1}^{m} \mathrm{L}_1^3(T_2)^3 \circ \phi_i^t$$

- For $n = 2$ and $r \geqslant 9$ odd:

$$\mathrm{L}_1^{s_1}(T_2) \odot \cdots \odot \mathrm{L}_1^{s_k}(T_2)$$

with $s_1 + \cdots + s_k = r$ and $s_i \geqslant 3$ odd

Ehrhart Tensor Polynomial

$$\mathrm{L}^r(kP) = \sum_{i=0}^{n+r} L_i^r(P)\, k^i$$

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- $$\mathrm{L}^r(kP) = \sum_{j=0}^{n+r} h_j^r(P) \binom{k+n+r-j}{n+r}$$

  $\dim P = 2$ and $r = 2 \Rightarrow h_j^r(P)$ positive definite

- Berg, Jochemko & Silverstein 2017

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