Cone unrectifiable sets and non-differentiability of real-valued Lipschitz functions.

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Background

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz mapping.

- *Rademacher theorem* says that any such f is differentiable almost everywhere w.r.t. the Lebesgue measure.
  So the set $N_f = \{x \in \mathbb{R}^n : f \text{ is not differentiable at } x\}$ has Lebesgue measure 0.
- If $E \subset \mathbb{R}^n$ has measure 0, does there exist a Lipschitz f such that $E \subset N_f$?
  $\underline{n = m = 1 :}$ [Zahorski, 1946]: yes.
  $\underline{n \geq 2, m = 1 :}$
  - [Preiss, 1990] For $n \geq 2$ and $m = 1$, any G_δ set E containing a dense set of lines in $\mathbb{R}^n$ is a *universal differentiability set*, i.e. any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at some points in E.
  - [Doré - M, 2011, 2012] A universal differentiability set may be chosen to be compact of Hausdorff dimension 1,
  - [Dymond - M, 2016] ... and even of Minkowski dimension 1.
- But what conditions can guarantee that a given Lebesgue null set is not a universal differentiability set?
  ... and what about Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m > 1$?

Background - Advanced

Following Alberti, Csörnyei, Preiss [ACP], 2005–2017:

Width of a set

$e \in \mathbb{R}^n$ and $\alpha > 0$

(for practical reasons we can always imagine $\|e\| = 1$ and $\alpha \in (0,1)$)

$C_{e,\alpha}$ = all $v \in \mathbb{R}^n$ s.t. $\langle v, e \rangle \geq \alpha \|v\| \|e\|$,

$\Gamma_{e,\alpha}$ = all Lipschitz $\gamma : \mathbb{R} \to \mathbb{R}^n$ s.t. $\gamma'(t) \in C_{e,\alpha}$ for a.a. $t \in \mathbb{R}$.

1. $G \subset \mathbb{R}^n$ is open $\Rightarrow w_{e,\alpha}(G) = \sup\{\mathcal{H}^1(G \cap \gamma(\mathbb{R})) : \gamma \in \Gamma_{e,\alpha}\}$,
2. $E \subset \mathbb{R}^n \Rightarrow w_{e,\alpha}(E) = \inf\{w_{e,\alpha}(G) : G \supset E, G \text{ is open}\}$.

Máthé, 2017: In the above definition it is possible to replace "open G" by Borel G.

Tangent fields

A map τ defined on $E \subset \mathbb{R}^n$ with values in the Grassmanian $G(n,k)$ is said to be a k-dimensional tangent field of E if for all $e \in \mathbb{R}^n$ ($\|e\| = 1$) and $\alpha \in (0,1)$

$$\boxed{w_{e,\alpha}\Big\{x \in E : \tau(x) \cap C_{e,\alpha} = \{0\}\Big\} = 0}$$

Non-differentiability sets and how to measure nullness

Theorem

A set $E \subset \mathbb{R}^n$ is contained in a non-differentiability set N_f of a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ for some $m \geq n$ if and only if it is λ_n-Lebesgue null.

- Alberti, Csörnyei, Preiss, 2005–2017: $E \subset N_f$ is equiv. to existence of an $(n-1)$-dimensional tangent field.
- Alberti, Csörnyei, Preiss, 2005–2017: Lebesgue null, $n = 2$
- Csörnyei, Jones, 2015: Lebesgue null, all $n \geq 3$

- Preiss, Speight, 2015: $m < n \Rightarrow$ universal differentiability sets for Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$; poss. Hausd. dim. $< m + \varepsilon$ for any $\varepsilon > 0$.

Tangent and normal fields

Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ s.t. $E \subset N_f$ Looking for

- $(n-1)$-dim. tangent fields
- codim 1 tangent fields
- Use normal fields instead $\to$ Normal vectors defined pointwise

Directions of nullness and unrectifiability

Directions of nullness

For $x \in E \subset \mathbb{R}^n$ let
$\mathcal{N}(E, x) := \{e \in \mathbb{R}^n : \forall \eta > 0\, \exists r > 0 \text{ s.t. } w_{e,\eta}(B(x,r) \cap E) = 0\}$.

1. $\mathcal{N}(E, x)$ is a linear subspace for each $x \in E$.
2. If $\mathcal{N}(E, x) \neq \{0\}$ for each $x \in E$, then E is null.
3. Máthé, 2017: If E is Borel and is 1-purely unrectifiable, i.e. *for any Lipschitz $\gamma : \mathbb{R} \to \mathbb{R}^n$ we have $\mathcal{H}^1(E \cap \gamma(\mathbb{R})) = 0$,* then E is uniformly purely unrectifiable (upu): $w_{e,\varepsilon}(E) = 0$ for every unit vector $e \in \mathbb{R}^n$ and every $\varepsilon > 0$.
4. Hence any 1-purely unrectifiable ($\iff$ upu) set is null in every direction, i.e. $\mathcal{N}(E, x) = \mathbb{R}^n$.

Cone unrectifiable sets

$E \subset \mathbb{R}^n$ s.t. $\mathcal{N}(E, x) \neq \{0\}\ \forall x \in E$ are called cone unrectifiable sets (cu).

Sufficient condition for existence of non-differentiable Lipschitz function

Cone unrectifiable sets

$E \subset \mathbb{R}^n$ s.t. $\mathcal{N}(E,x) \neq \{0\}$ $\forall x \in E$ are called cone unrectifiable sets (cu).

Examples

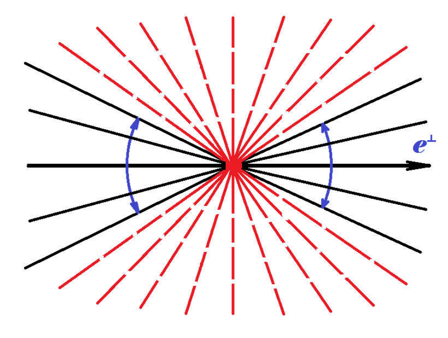
1. If E admits a continuous $(n-1)$-dimensional tangent, then $\mathcal{N}(E,x) \supset \tau(x)^{\perp} \neq \{0\}$, so the set is cone unrectifiable.
2. If E is 1-pu $\iff$ upu, then E is cu (cone unrectifiable).

Theorem 1 (M-Preiss, 2017).

Suppose $E \subset \mathbb{R}^n$ is cone unrectifiable.
Then there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ that is non-differentiable at every point of E. $\to$ Example 1.

Example 1:

Let us compare the composition of classes of upu and cu sets.
For each fixed $x \in E$ and [all $e \in S$ or some $e \in S$], S a unit sphere $\forall \varepsilon > 0$ there is an $r > 0$ s.t. $w_{e,\varepsilon}(E \cap B(x,r)) = 0$.
However for upu we can replace "all $\varepsilon > 0$" by "there exists $\varepsilon > 0$":
E is upu $\iff$ there exists $\varepsilon \in (0,1)$ s.t. $w_{e,\varepsilon}(E) = 0$ for all $e \in S$.
In contrast with this, for cu we need (in general) all $\varepsilon > 0$:
there exists a **UDS** N with $w_{\varepsilon,e}(N) = 0$ for a fixed pair $e \in S$, $\varepsilon \in (0,1)$.



Force a G_δ set E containing a dense set of lines "around" $e^{\perp}$ to intersect all curves $\gamma \in \Gamma_{e,\varepsilon}$ by a set of measure 0,

E is a UDS of $w_{e,\varepsilon}$ width zero.

Theorem 1 (M-Preiss, 2017).

Suppose $E \subset \mathbb{R}^n$ is cone unrectifiable.
Then $\forall \varepsilon > 0$ there is a $(1+\varepsilon)$-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ s.t.

$\forall x \in E$, $y \in \mathbb{R}^n$
$$\boxed{D^+ f(x;y) - D_+ f(x;y) \geq 2 \sup_{e \in \mathcal{N}(E,x)} \langle e, y \rangle}$$
$\to$ Example 2:

Theorem 2 (M-Preiss, 2017), [ACP]

For every (uniformly) purely unrectifiable set $E \subset \mathbb{R}^n$, there is a real valued 1-Lipschitz function f on $\mathbb{R}^n$ such that $\liminf_{r \searrow 0} \sup_{\|y\| \leq r} |f(x+y) - f(x) - \langle e, y \rangle|/r = 0$ for every $x \in E$ and $e \in \mathbb{R}^n$ with $\|e\| \leq 1$.
In particular, $D^+ f(x;y) = \|y\|$ and $D_+ f(x;y) = -\|y\|$ for every $x \in E$ and $y \in \mathbb{R}^n$.

Example 2:

It is not possible to have $\varepsilon = 0$ in Theorem 1:
There is a compact set $E \subset \mathbb{R}^2$ and a continuous mapping $x \in E \mapsto e_x \in \mathbb{R}^2$ such that $\mathcal{N}(E,x) = \langle e_x \rangle$ for every $x \in E$ and whenever $f : \mathbb{R}^2 \to \mathbb{R}$ has $Lip(f) \leq 1$, there is $x \in E$ s.t. $D^+ f(x, e_x) < 1$.

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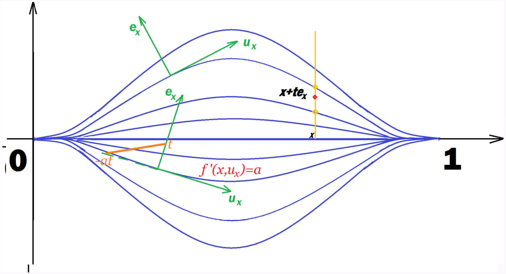

$f : \mathbb{R}^2 \to \mathbb{R}$.
Assume $Lip(f) \leq 1$.
Suppose $x \neq (\pm 1, 0)$ is s.t. $D^+ f(x, e_x) = 1$.
We show: if $a = f'(x, u_x)$ exists, then $a = 0$.

$\limsup_{t \to 0} \frac{f(x+te_x) - f(x-atu_x)}{t} = 1 + a^2 \Rightarrow$

$1 = Lip(f) \geq \limsup_{t \to 0} \frac{|f(x+te_x) - f(x-atu_x)|}{\|(x+te_x) - (x-atu_x)\|} = \sqrt{1 + a^2} \Rightarrow a = 0.$

f Lipschitz $\Rightarrow a$ exists a.e.

$f'(x, u_x) = 0$ a.e. $\Rightarrow f = const$ along $\phi_k \Rightarrow f|_E = const$

$\frac{\|f(x+te_x) - f(x)\|}{t} \leq \frac{dist(x+te_x, E)}{t} \underset{t \to 0}{\to} 0 \Rightarrow$ contradiction with $D^+ f(x, e_x) = 1$.

Sufficient condition for existence of non-differentiable Lipschitz function: union of cone unrectifiable sets

* For a countable collection of cone unrectifiable sets, we may try to add the Lipschitz functions non-differentiable on individual sets, to get a function not differentiable at every point from the union.
  - addition of non-differentiability may result in differentiability,
  - try to force the Lipschitz function not differentiable on a cone unrectifiable set to be differentiable everywhere outside. However this may not be possible...

Theorem 3 (M-Preiss, 2017)

$E = \bigcup_k E_k \subset \mathbb{R}^n$, E_k are $\mathbf{F}_\sigma$ cone unrectifiable sets.

Then there is a Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ such that

- f is non-differentiable at every $x \in E$, and
- f is differentiable at every $x \in \mathbb{R}^n \setminus E$: $N_f = E$, $D_f = \mathbb{R}^n \setminus E$.

- $x \in D \setminus A \Rightarrow f + g$ also not diff at x!

Non-differentiability w.r.t. measure

Let μ be a finite measure on $\mathbb{R}^n$.
The *decomposability bundle* of μ is a map
$V(\mu, x) : \mathbb{R}^n \to Gr(\mathbb{R}^n) = \bigcup_{k=0}^n G(n, k)$, whose values are vector subspaces of $\mathbb{R}^n$ (possibly of varying dimensions), such that:
(a) for every measure $\nu \ll \mu$ with a 1-rectifiable representation $\nu = \int_I \nu_t dt$ (where ν_t is the restriction of $\mathcal{H}^1$ to a 1-rectifiable set E_t) it holds $Tan(E_t, x) \subset V(\mu, x)$ for ν_t-a.e. x and a.e. $t \in I$;
(b) for every other map $W : \mathbb{R}^n \to Gr(\mathbb{R}^n)$ satisfying (a), it holds $V(\mu, x) \subseteq W(x)$ for μ-a.e. x.

Alberti–Marchese, 2016: μ: finite measure on $\mathbb{R}^n$,
$V(\mu, x) : \mathbb{R}^n \to Gr(\mathbb{R}^n)$: decomposability bundle.
Then there exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ s.t. f is not diff. at x in any direction $v \notin V(\mu, x)$, for μ-a.e. $x \in \mathbb{R}^n$.
More precisely: $D^+ f(x, v) - D_+ f(x, v) > 0$ for any such v.

Alberti–Marchese, 2016: Let μ be a finite measure on $\mathbb{R}^n$ and $V(\mu, x) : \mathbb{R}^n \to Gr(\mathbb{R}^n)$ be a decomposability bundle.
Then there exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that, for μ-a.e. $x \in \mathbb{R}^n$, f is not differentiable at x in any direction $v \notin V(\mu, x)$, and, more precisely, $D^+ f(x, v) - D_+ f(x, v) > 0$ for any such v.

Theorem 4 (M-Preiss, 2017)

μ: a Radon measure on $\mathbb{R}^n$, $T : \mathbb{R}^n \to Gr(\mathbb{R}^n)$ a μ-measurable map s.t. for every unit vector e and $0 < \alpha < 1$,

$$\Big\{x : C_{e,\alpha} \cap T(x) = \{0\}\Big\} = N_\mu \cup E \text{ s.t. } \mu(N_\mu) = 0 \text{ and } w_{e,\alpha}(E) = 0. \quad (\star)$$

Then there is a Lipschitz function f on $\mathbb{R}^n$ such that for μ-a.e. $x \in \mathbb{R}^n$ there is $c_x > 0$ such that $D^+ f(x, v) - D_+ f(x, v) \geq c_x \mathrm{dist}(v, T(x))$ for every $v \in \mathbb{R}^n$.

Alberti–Marchese, 2016: Decomposability bundle satisfies $(\star)$.

Theorem 3 (M-Preiss, 2017)

$E = \bigcup_k E_k \subset \mathbb{R}^n$, E_k are $\mathbf{F}_\sigma$ cone unrectifiable sets.

Then there is a Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ such that $N_f = E$, $D_f = \mathbb{R}^n \setminus E$.

Merlo, 2017: Let E be a bounded analytic set and let $m \geq n$. Then the following are equivalent:

1. E is contained in a countable union of closed pu sets,
2. $S = \left\{ f \in Lip_1(n, m) : f \text{ is fully non-differentiable on } E \right\}$ is residual.

Theorem (Preiss – Tišer, 1995) Let E be a bounded analytic set. Then the following statements are equivalent:

1. $\left\{ f \in Lip_1 : f \text{ is differentiable at no point of } E \right\}$ is residual in Lip_1,
2. The set E is contained in an F_σ subset of measure zero.