Geometry of isotropic convex bodies

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Abstract

We present the main results on the geometry of isotropic convex bodies. Our starting point is the *slicing problem*, which asks if there exists an absolute constant $c > 0$ such that $\max_{\theta\in S^{n-1}} \mathrm{Vol}_n(K \cap \theta^{\perp}) \geqslant c$ for every convex body K of volume 1 in $\mathbb{R}^n$ that has barycenter at the origin. It turns out that a natural framework for the study of this problem is the *isotropic position* of a convex body. A convex body K in $\mathbb{R}^n$ is called isotropic if $\mathrm{Vol}_n(K) = 1$, its barycenter is at the origin and its inertia matrix is a multiple of the identity, that is, there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$. The number L_K is then called the isotropic constant of K. We will see that the affine class of any convex body K contains a unique, up to orthogonal transformations, isotropic convex body; this is the isotropic position of K.

One of our first goals is to show that an affirmative answer to the slicing problem is equivalent to the following statement:

There exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leqslant C.$$

The notion of the isotropic constant can be reintroduced in the more general setting of finite log-concave measures, and a more general question can be posed in a way that is equivalent to the above when we consider uniform measures on convex bodies. We say that a finite log-concave measure μ in $\mathbb{R}^n$ is isotropic if μ is a probability measure, its barycenter is at the origin and the covariance matrix $\mathrm{Cov}(\mu)$ of μ is the identity matrix. The isotropic constant of μ is defined in an appropriate way, and a theorem of K. Ball shows that, in fact, for some absolute constant $c > 1$,

$$L_n \leqslant \sup\{L_\mu : \mu \text{ is isotropic on } \mathbb{R}^n\} \leqslant cL_n.$$

We present the best known upper bounds for L_n. Around 1985-6 (published in 1990), Bourgain obtained the upper bound $L_n \leqslant c\sqrt[4]{n}\log n$ and, in 2006, this estimate was improved by Klartag to $L_n \leqslant c\sqrt[4]{n}$. Actually, Klartag obtained a solution to the "isomorphic slicing problem", by showing that, for every convex body K in $\mathbb{R}^n$ and any $\varepsilon \in (0, 1)$, one can find a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that $(1 + \varepsilon)^{-1}T \subseteq K + x \subseteq (1 + \varepsilon)T$ and $L_T \leqslant C/\sqrt{\varepsilon}$ for some absolute constant $C > 0$. Klartag's method relies on properties of the logarithmic Laplace transform of the uniform measure on a convex body.

Klartag's proof of the bound $L_n \leqslant c\sqrt[4]{n}$ combines his solution to the isomorphic slicing problem with the following very useful deviation inequality of Paouris: if μ is an isotropic log-concave probability measure in $\mathbb{R}^n$ then

$$\mu(\{x \in \mathbb{R}^n : |x| \geqslant ct\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right)$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant. We present the proof of this inequality, and we develop in parallel the basic theory of the L_q-centroid bodies of an isotropic log-concave measure.

Then, we discuss some recent approaches to the slicing problem. Among them are two reductions that rely heavily on the existence of convex bodies with maximal isotropic constant whose isotropic position is compatible with regular covering estimates, and an alternative approach of Klartag and E. Milman that combines the advantages of both the logarithmic Laplace transform and the theory of the L_q-centroid bodies.

Finally, we describe E. Milman's almost sharp estimate for the mean width $w(Z_q(K))$ of the L_q-centroid bodies $Z_q(K)$ of an isotropic convex body K in $\mathbb{R}^n$. This is the most recent important result of the theory, leading to the estimate $w(K) \leqslant C\sqrt{n}(\log n)^2 L_K$ for any isotropic convex body in $\mathbb{R}^n$. An interesting related question is to understand whether an isotropic convex body is sub-Gaussian in most directions. As a consequence of E. Milman's theorem, one can show that the answer is affirmative. More precisely, one has $\|\langle \cdot, \theta \rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K$ for a random $\theta \in S^{n-1}$.