Geometry of isotropic convex bodies and the slicing problem

Apostolos Giannopoulos

Abstract

Lecture notes for the introductory workshop of the program "Geometric Functional Analysis and Applications" at the Mathematical Sciences Research Institute, Fall 2017. We present the main results on the geometry of isotropic convex bodies. The emphasis is on the slicing problem, a well-known open question regarding the distribution of volume in high-dimensional convex bodies.

1 Introduction

Our starting point is the *slicing problem*, which asks if there exists an absolute constant $c > 0$ such that $\max_{\theta\in S^{n-1}} \mathrm{Vol}_n(K\cap\theta^{\perp}) \geqslant c$ for every convex body K of volume 1 in $\mathbb{R}^n$ that has barycenter at the origin. It turns out that a natural framework for the study of this problem is the *isotropic position* of a convex body. A convex body K in $\mathbb{R}^n$ is called isotropic if $\mathrm{Vol}_n(K) = 1$, its barycenter is at the origin and its inertia matrix is a multiple of the identity, that is, there exists a constant $L_K > 0$ such that

$$\int_K \langle x,\theta\rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$. The number L_K is then called the isotropic constant of K. We will see that the affine class of any convex body K contains a unique, up to orthogonal transformations, isotropic convex body; this is the isotropic position of K.

One of our first goals is to show that an affirmative answer to the slicing problem is equivalent to the following statement:

There exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leqslant C.$$

The notion of the isotropic constant can be reintroduced in the more general setting of finite log-concave measures, and a more general question can be posed in a way that is equivalent to the above when we consider uniform measures on convex bodies. We say that a finite log-concave measure μ in $\mathbb{R}^n$ is isotropic if μ is a probability measure, its barycenter is at the origin and the covariance matrix $\mathrm{Cov}(\mu)$ of μ is the identity matrix. The isotropic constant of μ is defined in an appropriate way, and a theorem of K. Ball shows that, in fact, for some absolute constant $c > 1$,

$$L_n \leqslant \sup\{L_\mu : \mu \text{ is isotropic in } \mathbb{R}^n\} \leqslant cL_n.$$

We present the best known upper bounds for L_n. Around 1985-6 (published in 1990), Bourgain obtained the upper bound $L_n \leqslant c\sqrt[4]{n}\log n$ and, in 2006, this estimate was improved by Klartag to $L_n \leqslant c\sqrt[4]{n}$. Actually, Klartag obtained a solution to the "isomorphic slicing problem", by showing that, for every convex body K in $\mathbb{R}^n$ and any $\varepsilon \in (0,1)$, one can find a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that $(1+\varepsilon)^{-1}T \subseteq K + x \subseteq (1+\varepsilon)T$ and $L_T \leqslant C/\sqrt{\varepsilon}$ for some absolute constant $C > 0$. Klartag's method relies on properties of the logarithmic Laplace transform of the uniform measure on a convex body.

Klartag's proof of the bound $L_n \leqslant c\sqrt[4]{n}$ combines his solution to the isomorphic slicing problem with the following very useful deviation inequality of Paouris: if μ is an isotropic log-concave measure in $\mathbb{R}^n$ then

$$\mu(\{x \in \mathbb{R}^n : |x| \geqslant ct\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right)$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant. We present the proof of this inequality, and we develop in parallel the basic theory of the L_q-centroid bodies of an isotropic log-concave measure.

Then, we discuss some recent approaches to the slicing problem. Among them are two reductions that rely heavily on the existence of convex bodies with maximal isotropic constant whose isotropic position is compatible with regular covering estimates, and an alternative approach of Klartag and E. Milman that combines the advantages of both the logarithmic Laplace transform and the theory of the L_q-centroid bodies.

Finally, we describe E. Milman's almost sharp estimate for the mean width $w(Z_q(K))$ of the L_q-centroid bodies $Z_q(K)$ of an isotropic convex body K in $\mathbb{R}^n$. This is the most recent important result of the theory, leading to the estimate $w(K) \leqslant C\sqrt{n}(\log n)^2 L_K$ for the mean width of any isotropic convex body K in $\mathbb{R}^n$. An interesting related question is to understand whether an isotropic convex body is sub-Gaussian in most directions. As a consequence of E. Milman's theorem, one can show that the answer is affirmative. More precisely, one has $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K$ for a random $\theta \in S^{n-1}$.

2 Notation and background from asymptotic convex geometry

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote the corresponding Euclidean norm by $|\cdot|$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by Vol_n. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also if $A, D \subseteq \mathbb{R}^n$ we will write $A \simeq D$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 A \subseteq D \subseteq c_2 A$.

Convex bodies

A *convex body* in $\mathbb{R}^n$ is a compact convex subset A of $\mathbb{R}^n$ with non-empty interior. We say that A is symmetric if $A = -A$, and that A is centered if it has barycenter at 0 i.e. if

$$\int_A \langle x, \theta \rangle \, dx = 0$$

for every $\theta \in S^{n-1}$.

The radial function $\rho_A : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ of a convex body A with $0 \in \mathrm{int}(A)$ is defined as follows:

$$\rho_A(x) = \max\{t > 0 : tx \in A\}.$$

The support function of A is defined for every $y \in \mathbb{R}^n$ by

$$h_A(y) = \max\{\langle x, y \rangle : x \in A\}.$$

Note that for every $\theta \in S^{n-1}$ one has $\rho_A(\theta) \leqslant h_A(\theta)$. The mean width of A is the quantity

$$w(A) = \int_{S^{n-1}} h_A(\theta) \, d\sigma(\theta).$$

The radius of A is

$$R(A) = \max\{|x| : x \in A\}.$$

If 0 is an interior point of A, we denote by $r(A)$ the largest $r>0$ for which $rB_2^n\subseteq A$. The volume radius of A is the quantity

$$\mathrm{vrad}(A)=\left(\frac{\mathrm{Vol}_n(A)}{\mathrm{Vol}_n(B_2^n)}\right)^{1/n}.$$

The polar body A° of a convex body A with $0\in\mathrm{int}(A)$ is defined as follows:

$$A^\circ=\{x\in\mathbb{R}^n:\langle x,y\rangle\leqslant 1\text{ for all }y\in A\}. \tag{2.1}$$

We write $\overline{A}$ for the multiple of $A\subseteq\mathbb{R}^n$ that has volume 1; in other words, $\overline{A}:=\mathrm{Vol}_n(A)^{-1/n}A$.

Geometric inequalities

We will often use the following basic inequalities for convex bodies.

(i) *Urysohn inequality.* If A is a convex body in $\mathbb{R}^n$ then

$$w(A)\geqslant\left(\frac{\mathrm{Vol}_n(A)}{\mathrm{Vol}_n(B_2^n)}\right)^{1/n}. \tag{2.2}$$

(ii) *Blaschke-Santaló inequality.* If A is a symmetric convex body in $\mathbb{R}^n$, and more generally if A is centered, then

$$\mathrm{Vol}_n(A)\,\mathrm{Vol}_n(A^\circ)\leqslant\mathrm{Vol}_n(B_2^n)^2. \tag{2.3}$$

(iii) *Bourgain–V. Milman inequality.* There exists an absolute constant $0<c<1$ with the following property: for every $n\geqslant 1$ and any convex body A in $\mathbb{R}^n$ with $0\in\mathrm{int}(A)$,

$$\mathrm{Vol}_n(A)\,\mathrm{Vol}_n(A^\circ)\geqslant c^n\mathrm{Vol}_n(B_2^n)^2. \tag{2.4}$$

(iv) *Rogers–Shephard inequality.* If A is a convex body in $\mathbb{R}^n$, then the volume of the *difference body* $A-A:=\{x-y:x,y\in A\}$ satisfies

$$\mathrm{Vol}_n(A-A)\leqslant\binom{2n}{n}\mathrm{Vol}_n(A). \tag{2.5}$$

(v) *Reverse Urysohn inequality.* From results of Lewis, Figiel and Tomczak-Jaegermann, combined with an inequality of Pisier, one has the following fact: If A is a centered convex body in $\mathbb{R}^n$ then there exists a symmetric and positive definite $T\in GL_n$ such that the position $\tilde{A}=T(A)$ of A satisfies

$$w(\tilde{A})\leqslant c\sqrt{n}\log n\,\mathrm{Vol}_n(\tilde{A})^{1/n}, \tag{2.6}$$

where $c>0$ is an absolute constant.

(vi) *M^*-inequality.* If A is a symmetric convex body in $\mathbb{R}^n$ then, for every $1\leqslant k\leqslant n$, a random subspace $F\in G_{n,k}$ satisfies

$$R(A\cap F)\leqslant c\sqrt{\frac{n}{n-k}}\,w(A)$$

with probability greater than $1-\exp(-c_2(n-k))$, where $c_1,c_2>0$ are absolute constants.

Covering numbers

Let A and B be two convex bodies in $\mathbb{R}^n$. The covering number $N(A,B)$ of A by B is the least integer N for which there exist N translates of B whose union covers A.

$$N(A,B)=\min\Big\{N\in\mathbb{N}:\ \exists x_1,\dots,x_N\in\mathbb{R}^n \text{ such that } A\subseteq\bigcup_{j=1}^{N}(x_j+B)\Big\}.$$

A variant of this notion is defined as follows:

$$\overline{N}(A,B)=\min\Big\{N\in\mathbb{N}:\ \exists x_1,\dots,x_N\in A \text{ such that } A\subseteq\bigcup_{j=1}^{N}(x_j+B)\Big\}.$$

From the definition we see that $N(A,B)\leqslant\overline{N}(A,B)$. One can also easily check that $\overline{N}(A,B-B)\leqslant N(A,B)$. In particular, if B is convex and symmetric, then $\overline{N}(A,2B)\leqslant N(A,B)$.

If A,B are convex bodies in $\mathbb{R}^n$, and B is symmetric, for every $t>0$ we define

$$S_t(A,B)=\max\{m\in\mathbb{N}:\ \exists x_1,\dots,x_m\in A \text{ such that } \|x_i-x_j\|_B>t \text{ for } i\neq j\}.$$

From the definition we easily check that

$$\overline{N}(A,tB)\leqslant S_t(A,B)\leqslant\overline{N}(A,\tfrac{t}{2}B),$$

where $\|\cdot\|_B$ is the norm induced by B on $\mathbb{R}^n$ (see the next subsection). We shall use some basic theorems for covering numbers. The first one is Sudakov's inequality.

Theorem 2.1 (Sudakov)**.** *If A is a convex body in $\mathbb{R}^n$ then for every $t>0$ one has*

$$N(A,tB_2^n)\leqslant 2\exp\left(cn\,(w(A)/t)^2\right), \tag{2.7}$$

where $c>0$ is an absolute constant.

The next theorem is due to Artstein-Avidan, V. Milman and Szarek.

Theorem 2.2 (duality of entropy)**.** *There exist absolute positive constants α and β such that for any $n\geqslant 1$ and any symmetric convex body A in $\mathbb{R}^n$*

$$N(B_2^n,\alpha^{-1}A^\circ)^{\frac{1}{\beta}}\leqslant N(A,B_2^n)\leqslant N(B_2^n,\alpha A^\circ)^{\beta} \tag{2.8}$$

V. Milman proved that there exists an absolute constant $\beta>0$ such that every centered convex body A in $\mathbb{R}^n$ has a linear image $\tilde{A}$ which satisfies $\mathrm{Vol}_n(\tilde{A})=\mathrm{Vol}_n(B_2^n)$ and

$$\max\big\{N(\tilde{A},B_2^n),N(B_2^n,\tilde{A}),N(\tilde{A}^\circ,B_2^n),N(B_2^n,\tilde{A}^\circ)\big\}\leqslant\exp(\beta n). \tag{2.9}$$

We say that a convex body A which satisfies this estimate is in M-position with constant β.

Pisier has proposed a different approach to this result, which allows one to find a whole family of M-positions and to give more detailed information on the behavior of the corresponding covering numbers. The precise statement is as follows.

Theorem 2.3 (Pisier)**.** *For every $0<\alpha<2$ and every symmetric convex body A in $\mathbb{R}^n$ there exists a linear image $\tilde{A}$ of A such that*

$$\max\big\{N(\tilde{A},tB_2^n),N(B_2^n,t\tilde{A}),N(\tilde{A}^\circ,tB_2^n),N(B_2^n,t\tilde{A}^\circ)\big\}\leqslant\exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $t\geqslant 1$, where $c(\alpha)$ depends only on α, and $c(\alpha)=O\big((2-\alpha)^{-\alpha/2}\big)$ as $\alpha\to 2$.

Some facts from the local theory of normed spaces

We will also need some basic facts from the local theory of normed spaces. Let A be a symmetric convex body in $\mathbb{R}^n$. The function $\|\cdot\|_A : \mathbb{R}^n \to \mathbb{R}^+$ defined by

$$\|x\|_A = \inf\{t > 0 : x \in tA\}$$

is a norm on $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_A)$ will be denoted by X_A. Conversely, if $X = (\mathbb{R}^n, \|\cdot\|)$ is a normed space, then the unit ball $A = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$ of X is a symmetric convex body.

Let X, Y be two n-dimensional normed spaces. The *Banach–Mazur distance* from X to Y is defined as follows:

$$d(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| \mid T : X \to Y \text{ linear isomorphism}\}. \tag{2.10}$$

In a geometric language, the Banach–Mazur distance has the following description: if $X = X_A$ and $Y = X_D$ (i.e. the unit balls of X, Y are the convex bodies A, D respectively) then the distance $d(X,Y)$ is the smallest $d > 0$ such that

$$A \subseteq T(D) \subseteq dA \tag{2.11}$$

for some $T \in GL_n$.

Besides the Banach-Mazur distance, we often use the geometric distance $d_G(A,D)$ of two symmetric convex bodies A and D in $\mathbb{R}^n$, or more generally two convex bodies having the origin as an interior point, which is the smallest $d > 0$ for which there exist $a, b > 0$ with $ab \leqslant d$ such that

$$\frac{1}{a}A \subseteq D \subseteq bA. \tag{2.12}$$

We define

$$M(A) := \int_{S^{n-1}} \|\theta\|_A d\sigma(\theta).$$

On observing that $\|x\|_A = h_{A^\circ}(x)$ for every $x \in \mathbb{R}^n$, we see that $M(A) = w(A^\circ)$ and that

$$M(A)^{-1} \leqslant \mathrm{vrad}(A) \leqslant w(A) = M(A^\circ).$$

The left hand side inequality is easily checked if we express the volume of A as an integral in polar coordinates and use the inequalities of Hölder and Jensen, while the right hand side inequality is an immediate consequence of Urysohn's inequality.

The dual Sudakov inequality of Pajor and Tomczak-Jaegermann provides an upper bound for the covering numbers $N(B_2^n, tA)$ in terms of the parameter $M(A)$.

Theorem 2.4 (Pajor-Tomczak)**.** *Let A be a symmetric convex body in $\mathbb{R}^n$. For every $t > 0$,*

$$\log N(B_2^n, tA) \leqslant cn\,(M(A)/t)^2, \tag{2.13}$$

where $c > 0$ is an absolute constant.

We write $k_*(A)$ for the largest integer $k \leqslant n$ which satisfies

$$\mu_{n,k}\left(F \in G_{n,k} : \frac{w(A)}{2}|x| \leqslant h_A(x) \leqslant 2w(A)|x|,\ x \in F\right) \geqslant \frac{n}{n+k}.$$

The next theorem shows that the dimension $k_*(A)$ is determined from the parameters $w(A)$ and $R(A)$ up to an absolute constant.

Theorem 2.5 (Milman-Schechtman)**.** *There exist* $c_1, c_2 > 0$ *such that*

$$c_1 n \frac{w(A)^2}{R(A)^2} \leqslant k_*(A) \leqslant c_2 n \frac{w(A)^2}{R(A)^2},$$

for every symmetric convex body A *in* $\mathbb{R}^n$.

For every $q \neq 0$ we define

$$w_q := w_q(A) = \left(\int_{S^{n-1}} h_A(\theta)^q d\sigma(\theta) \right)^{1/q}.$$

Note that $w_1(A) = w(A)$. The parameters w_q, $q \geqslant 1$ were studied by Litvak, Milman and Schechtman.

Theorem 2.6. *Let* A *be a symmetric convex body in* $\mathbb{R}^n$. *Then,*

$$\max\left\{ w(A), c_1 \frac{R(A)\sqrt{q}}{\sqrt{n}} \right\} \leqslant w_q(A) \leqslant \max\left\{ 2w(A), c_2 \frac{R(A)\sqrt{q}}{\sqrt{n}} \right\}$$

for all $q \in [1, n]$, *where* $c_1, c_2 > 0$ *are absolute constants.*

Note that the behavior of w_q changes when $q \simeq n(w/R)^2$. This value of q is roughly equal to the dual Dvoretzky dimension $k_*(A)$ of A. One also has $w_n \simeq R$, and since $w_q \leqslant R$ for every $q \geqslant 1$ we conclude that $w_q \simeq R$ for all $q \geqslant n$.

Let A be a symmetric convex body in $\mathbb{R}^n$. We define

$$d_*(A) = \min\left\{ -\log \sigma\left(\left\{ x \in S^{n-1} : h_A(x) \leqslant \frac{w(A)}{2} \right\} \right), n \right\}.$$

The parameter d_* was defined by Klartag and Vershynin, who also showed that $d_*(A)$ is always greater than $k_*(A)$:

Proposition 2.7 (Klartag-Vershynin)**.** *Let* A *be a symmetric convex body in* $\mathbb{R}^n$. *Then,*

$$d_*(A) \geqslant c k_*(A),$$

where $c > 0$ *is an absolute constant.*

The parameter $d_*(A)$ is closely related to estimates on the measure of the set of directions in which a norm is "much smaller" than its expectation on the sphere.

Theorem 2.8. *For every* $0 < \varepsilon < \frac{1}{2}$ *we have*

$$\sigma(\{x \in S^{n-1} : h_A(x) < \varepsilon w(A)\}) < \varepsilon^{c_1 d_*(A)} < \varepsilon^{c_1 k_*(A)},$$

where $c_1, c_2 > 0$ *are absolute constants.*

Theorem 2.8 implies reverse Hölder inequalities.

Theorem 2.9. *Let* A *be a symmetric convex body in* $\mathbb{R}^n$. *Then, for every* $0 < q < c_1 d_*(A)$,

$$c_2 w(A) \leqslant \left(\int_{S^{n-1}} \frac{1}{h_A^q(x)} d\sigma(x) \right)^{-1/q} \leqslant c_3 w(A).$$

In other words, for every $0 < q < c_1 d_*(A)$ *one has*

$$w_{-q}(A) \simeq w(A).$$

Since $d_*(A) \geqslant c k_*(A)$, combining the above we get:

Theorem 2.10. *Let* A *be a symmetric convex body in* $\mathbb{R}^n$. *Then,* $w_q(A) \simeq w_{-q}(A)$ *for every* $1 \leqslant q \leqslant c k_*(A)$.

Indeed, from Theorem 2.6 we have $w_q(A) \simeq w(A)$ for all $q \leqslant k_*(A)$, while from Theorem 2.9 we see that $w_{-q}(A) \simeq w(A)$ for all $q \leqslant c k_*(A)$.

We refer the reader to the book [5] for the theory of convex bodies and to the books [1], [3] and [4] for the local theory of normed spaces and asymptotic convex geometry.

3 Isotropic position and the slicing problem

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $\mathrm{Vol}_n(K)=1$, it is centered, and there is a constant $\alpha>0$ such that

$$\int_K \langle x,y\rangle^2 dx = \alpha^2|y|^2 \tag{3.1}$$

for all $y\in\mathbb{R}^n$. Let $\{e_1,\dots,e_n\}$ be any orthonormal basis of $\mathbb{R}^n$. Note that if K satisfies the isotropic condition (3.1) then

$$\int_K |x|^2dx = \sum_{i=1}^n \int_K \langle x,e_i\rangle^2 dx = n\alpha^2.$$

Also, it is easily checked that if K is an isotropic convex body in $\mathbb{R}^n$ then $U(K)$ is also isotropic for every $U\in O(n)$.

It is useful to check that the *isotropic condition* (3.1) is equivalent to the fact that for every $i,j=1,\dots,n$,

$$\int_K x_ix_jdx = \alpha^2\delta_{ij}, \tag{3.2}$$

where $x_j=\langle x,e_j\rangle$ are the coordinates of x with respect to some orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$. This is in turn equivalent to the fact that for every $T\in L(\mathbb{R}^n)$,

$$\int_K \langle x,Tx\rangle dx = \alpha^2(\mathrm{tr}T). \tag{3.3}$$

The next proposition shows that every centered convex body has a linear image which satisfies the isotropic condition.

Proposition 3.1. *Let K be a centered convex body in $\mathbb{R}^n$. There exists $T\in GL_n$ such that $T(K)$ is isotropic.*

Proof. The operator $M\in L(\mathbb{R}^n)$ defined by $M(y)=\int_K\langle x,y\rangle xdx$ is symmetric and positive definite; therefore, it has a symmetric and positive definite square root S. Consider the linear image $\tilde{K}=S^{-1}(K)$ of K. Then, for every $y\in\mathbb{R}^n$ we have

$$\begin{aligned}\int_{\tilde{K}}\langle x,y\rangle^2dx &= |\mathrm{det}S|^{-1}\int_K\langle S^{-1}x,y\rangle^2dx = |\mathrm{det}S|^{-1}\int_K\langle x,S^{-1}y\rangle^2dx \\ &= |\mathrm{det}S|^{-1}\Big\langle \int_K\langle x,S^{-1}y\rangle xdx, S^{-1}y\Big\rangle = |\mathrm{det}S|^{-1}\langle MS^{-1}y,S^{-1}y\rangle = |\mathrm{det}S|^{-1}|y|^2.\end{aligned} \tag{3.4}$$

Normalizing the volume of $\tilde{K}$ we get the result. □

Proposition 3.1 shows that every centered convex body K in $\mathbb{R}^n$ has a *position* $\tilde{K}$ which is isotropic. The next theorem shows that the isotropic position of a convex body is uniquely determined up to orthogonal transformations, and arises as a solution of a minimization problem.

Theorem 3.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Define*

$$\Delta(K)=\inf\Big\{\int_{TK}|x|^2dx : T\in SL_n\Big\}. \tag{3.5}$$

Then, a position K_1 of K, of volume 1, is isotropic if and only if

$$\int_{K_1}|x|^2dx=\Delta(K). \tag{3.6}$$

Furthermore, if K_1 and K_2 are isotropic positions of K then there exists $U\in O(n)$ such that $K_2=U(K_1)$.

Proof. Fix an isotropic position K_1 of K. We know that there exists $\alpha > 0$ such that

$$\int_{K_1} \langle x, Tx \rangle dx = \alpha^2 (\mathrm{tr} T)$$

for every $T \in L(\mathbb{R}^n)$. Then, for every $T \in SL_n$ we have

$$\int_{TK_1} |x|^2 dx = \int_{K_1} |Tx|^2 dx = \int_{K_1} \langle x, T^*Tx \rangle dx = \alpha^2 \mathrm{tr}(T^*T) \geqslant n\alpha^2 = \int_{K_1} |x|^2 dx, \tag{3.7}$$

where we have used the arithmetic-geometric means inequality in the form

$$\mathrm{tr}(T^*T) \geqslant n[\det(T^*T)]^{1/n}.$$

This shows that K_1 satisfies (3.6). In particular, the infimum in (3.5) is a minimum.

Note also that if we have equality in (3.7) then $T^*T = \mathrm{Id}$, and hence $T \in O(n)$. This shows that any other position $\tilde{K}$ of K which satisfies (3.6) is an orthogonal image of K_1, therefore it is isotropic.

Finally, if K_2 is some other isotropic position of K then the first part of the proof shows that K_2 satisfies (3.6). By the previous step, we must have $K_2 = U(K_1)$ for some $U \in O(n)$. □

We can now give the following definition for the isotropic constant of a general convex body K in $\mathbb{R}^n$.

Definition 3.3. Let $K \subset \mathbb{R}^n$ be a convex body. Its isotropic constant L_K is defined by

$$L_K^2 = \frac{1}{n} \min \left\{ \frac{1}{\mathrm{Vol}_n(T\tilde{K})^{1+\frac{2}{n}}} \int_{T\tilde{K}} |x|^2 dx \;\Big|\; T \in GL_n \right\},$$

where $\tilde{K} = K - \mathrm{bar}(K)$ is the centered translate of K.

Note that L_K depends only on the affine class of K. Note also that if K is isotropic then for all $\theta \in S^{n-1}$ we have

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2.$$

The main problem in these notes is the following.

Problem 3.4 (isotropic constant problem)**.** *There exists an absolute constant $C > 0$ such that for any $n \geqslant 1$ and any convex body $K \subset \mathbb{R}^n$ we have*

$$L_K \leqslant C.$$

Equivalently, if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\int_K \langle x, \theta \rangle^2 dx \leqslant C^2$$

for every $\theta \in S^{n-1}$.

The moments of inertia of a centered convex body are closely related with the volume of its hyperplane sections that pass through the origin. In the isotropic case this relation takes the following form.

Theorem 3.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $\theta \in S^{n-1}$ we have*

$$\frac{c_1}{L_K} \leqslant \mathrm{Vol}_{n-1}(K \cap \theta^\perp) \leqslant \frac{c_2}{L_K}, \tag{3.8}$$

where $c_1, c_2 > 0$ are absolute constants.

For the proof, given $\theta \in S^{n-1}$ we consider the function $f(t) = f_{K,\theta}(t) = \mathrm{Vol}_{n-1}(K \cap \{x : \langle x, \theta\rangle = t\})$, $t \in \mathbb{R}$. We restrict our attention to the symmetric case. Then, f is even and $\|f\|_\infty = f(0)$. For the proof in the general case, which is more or less the same, we need an additional fact (due to Fradelizi) which shows that hyperplane sections through the center of mass are, up to an absolute constant, maximal: If K is a centered convex body of volume 1 in $\mathbb{R}^n$ then, for every $\theta \in S^{n-1}$,

$$\|f_{K,\theta}\|_\infty \leqslant e f(0) = e\mathrm{Vol}_{n-1}(K \cap \theta^\perp).$$

Proof of Theorem 3.5 (symmetric case). Let $f := f_{K,\theta}$. To prove the left hand side of (3.8) we set $\beta = \int_0^{+\infty} f(t)dt = \frac{1}{2}$ and define

$$g(t) = \|f\|_\infty \mathbf{1}_{[0,\beta/\|f\|_\infty]}(t).$$

Since $g \geqslant f$ on the support of g, we have

$$\int_0^s f(t)dt \leqslant \int_0^s g(t)dt$$

for every $0 \leqslant s \leqslant \beta/\|f\|_\infty$. The integrals of f and g on $[0, +\infty)$ are both equal to β. So,

$$\int_s^\infty g(t)dt \leqslant \int_s^\infty f(t)dt$$

for every $s \geqslant 0$. It follows that

$$\begin{aligned}\int_0^\infty t^2 f(t)dt &= \int_0^\infty \int_0^t 2sf(t)dsdt = \int_0^\infty 2s\left(\int_s^\infty f(t)dt\right)ds\\ &\geqslant \int_0^\infty 2s\left(\int_s^\infty g(t)dt\right)ds = \int_0^\infty t^2 g(t)dt\\ &= \int_0^{\beta/\|f\|_\infty} t^2\|f\|_\infty dt = \frac{\beta^3}{3\|f\|_\infty^2}.\end{aligned}$$

It follows that

$$\int_K \langle x, \theta\rangle^2 dx = 2\int_0^\infty t^2 f(t)\,dt \geqslant \frac{2\beta^3}{3\|f\|_\infty^2} = \frac{1}{12f(0)^2}.$$

To prove the right hand side inequality of (3.8) we distinguish two cases. Assume first that there exists $s > 0$ such that $f(s) = f(0)/2$. Then,

$$\frac{1}{2} = \int_0^\infty f(t)dt \geqslant \int_0^s f(t)dt \geqslant sf(s) = sf(0)/2,$$

because, since f is log-concave, we easily see that $f(t) \geqslant f(0)^{1-t/s} f(s)^{t/s} \geqslant f(s)$ on $[0, s]$. On the other hand, if $t > s$, then

$$f(s) \geqslant [f(0)]^{1-\frac{s}{t}}[f(t)]^{\frac{s}{t}},$$

which implies that $f(t) \leqslant f(0)2^{-t/s}$. We now write

$$\begin{aligned}\int_0^\infty t^2 f(t)dt &= \int_0^s t^2 f(t)dt + \int_s^\infty t^2 f(t)dt \leqslant f(0)\int_0^s t^2 dt + \int_s^\infty t^2 f(0)2^{-t/s}dt\\ &= f(0)\left(e\frac{s^3}{3} + s^3\int_1^\infty u^2 2^{-u}du\right) \leqslant c_0 f(0)s^3 \leqslant c_0/[f(0)]^2.\end{aligned}$$

Now, assume that, for every $s > 0$ on the support of f, we have $f(s) > f(0)/2$. Then, the role of s is played by $s_0 = \sup\{s > 0 : f(s) > 0\}$. We have $\frac{1}{2} \geqslant f(0)s_0/2$ and

$$\int_{-\infty}^\infty t^2 f(t)dt = 2\int_0^\infty t^2 f(t)dt = 2\int_0^{s_0} t^2 f(t)dt \leqslant \frac{2f(0)s_0^3}{3} \leqslant \frac{2}{3[f(0)]^2}.$$

Thus, we get the same estimate as before, without using the fact that $\log f$ is concave. $\square$

Theorem 3.5 reveals a close connection between the isotropic constant problem and the *slicing problem*:

Problem 3.6 (The slicing problem)**.** There exists an absolute constant $c > 0$ with the following property: if K is a convex body in $\mathbb{R}^n$ with volume 1 and barycenter at the origin, there exists $\theta \in S^{n-1}$ such that

$$\mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \geqslant c. \tag{3.9}$$

We will show that the two problems are equivalent. One direction is simple, by the previous discussion; assume that the slicing problem has an affirmative answer. If K is isotropic, Theorem 3.5 shows that *all* sections $K \cap \theta^{\perp}$ have volume bounded from above by c_2/L_K. Since (3.9) must be true for at least one $\theta \in S^{n-1}$, we get $L_K \leqslant c_2/c$.

Conversely, we will show that if there exists an absolute bound C for the isotropic constant, then the slicing problem has an affirmative answer. One way to see this is through the Binet ellipsoid of inertia.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $M(K) = (m_{ij})_{i,j=1}^n$ be the matrix of inertia of K, which is defined by $m_{ij} = \int_K x_i x_j dx$. As we saw in the proof of Proposition 3.1, $M(K)$ has a symmetric and positive definite square root S. Consider the ellipsoid $\mathcal{E}_B(K) := S^{-1}(B_2^n)$; then

$$\|y\|_{\mathcal{E}_B(K)}^2 = |Sy|^2 = \langle Sy, Sy\rangle = \langle My, y\rangle = \int_K \langle x, y\rangle^2 dx.$$

$\mathcal{E}_B(K)$ is called the *Binet ellipsoid* of K. Observe that K is in isotropic position if and only if $\mathcal{E}_B(K) = L_K^{-1} B_2^n$.

The next proposition shows that the volume of $\mathcal{E}_B(K)$ is invariant under the action of SL_n.

Proposition 3.7. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\mathrm{Vol}_n(\mathcal{E}_B(K)) = \omega_n L_K^{-n}.$$

Proof. If K is an isotropic convex body in $\mathbb{R}^n$ then $\mathcal{E}_B(K) = L_K^{-1} B_2^n$, and hence $\mathrm{Vol}_n(\mathcal{E}_B(K)) = \omega_n L_K^{-n}$. It is easily checked that if $T \in SL_n$ then $M_{T(K)} = T M_K T^*$, and hence $|\det M_K| = |\det M_{T(K)}|$; furthermore, by definition we have $\mathcal{E}_B(T(K)) = S^{-1}(B_2^n)$ where $S^2 = M_{T(K)}$. It follows that

$$\mathrm{Vol}_n(\mathcal{E}_B(TK)) = \omega_n |\det M_{T(K)}|^{-1/2} = \omega_n |\det M_K|^{-1/2} = \mathrm{Vol}_n(\mathcal{E}_B(K))$$

for every $T \in SL_n$. $\square$

Corollary 3.8. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. There exists $\theta \in S^{n-1}$ such that*

$$\int_K \langle x, \theta\rangle^2 dx \leqslant L_K^2.$$

Proof. Note that by integration in polar coordinates

$$L_K^{-n} = \frac{\mathrm{Vol}_n(\mathcal{E}_B(K))}{\omega_n} = \int_{S^{n-1}} \|\theta\|_{\mathcal{E}_B(K)}^{-n} d\sigma(\theta).$$

It follows that $\min_{\theta \in S^{n-1}} \|\theta\|_{\mathcal{E}_B(K)} \leqslant L_K$. $\square$

Assume that the isotropic constant problem has an affirmative answer and let K be a centered convex body of volume 1 in $\mathbb{R}^n$. According to Corollary 3.8, there exists a direction $\theta \in S^{n-1}$ such that

$$\int_K \langle x, \theta\rangle^2 dx \leqslant L_K^2 \leqslant C^2.$$

Then, the proof of Theorem 3.5 shows that

$$\mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \geqslant c := \frac{1}{2\sqrt{3}eC}.$$

Simple bounds for the isotropic constant

We start with a lower bound L_K; in fact, it is quite simple to check that the Euclidean ball is the extremal body.

Proposition 3.9 (lower bound)**.** *For every isotropic convex body K in $\mathbb{R}^n$*

$$L_K \geqslant L_{B_2^n} \geqslant c,$$

where $c > 0$ is an absolute constant.

Proof. If $r_n = \omega_n^{-1/n}$, then $\mathrm{Vol}_n(r_n B_2^n) = 1$ and $r_n B_2^n$ is isotropic. Let K be an isotropic convex body. Observe that $|x| > r_n$ on $K \setminus r_n B_2^n$ and $|x| \leqslant r_n$ on $r_n B_2^n \setminus K$. Since $K \setminus r_n B_2^n$ and $r_n B_2^n \setminus K$ have the same volume, it follows that

$$\begin{aligned} nL_K^2 &= \int_K |x|^2 dx = \int_{K \cap r_n B_2^n} |x|^2 dx + \int_{K \setminus r_n B_2^n} |x|^2 dx \\ &\geqslant \int_{K \cap r_n B_2^n} |x|^2 dx + \int_{r_n B_2^n \setminus K} |x|^2 dx = \int_{r_n B_2^n} |x|^2 dx = nL_{B_2^n}^2. \end{aligned}$$

A simple computation shows that

$$L_{B_2^n}^2 = \frac{1}{n} \int_{r_n B_2^n} |x|^2 dx = \frac{1}{n} \frac{n\omega_n}{n+2} r_n^{n+2} = \frac{\omega_n^{-2/n}}{n+2} \geqslant c^2,$$

where $c > 0$ is an absolute constant, therefore $L_K \geqslant L_{B_2^n} \geqslant c$. $\square$

Remark 3.10 (radius and inradius)**.** It is useful to note that the inradius $r(K)$ and the radius $R(K)$ of an isotropic convex body K in $\mathbb{R}^n$ satisfy the bounds

$$c_1 L_K \leqslant r(K) \leqslant R(K) \leqslant c_2 n L_K, \tag{3.10}$$

where $c_1, c_2 > 0$ are absolute constants. The following simple argument proves the right hand side inequality: given $\theta \in S^{n-1}$, one knows that

$$\mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \simeq \frac{1}{L_K}. \tag{3.11}$$

Let $x_\theta \in K$ such that $\langle x_\theta, \theta \rangle = h_K(\theta)$ and consider the cone

$$C(\theta) = \mathrm{conv}(K \cap \theta^{\perp}, x_\theta).$$

Then $C(\theta) \subseteq K$, and hence

$$1 = \mathrm{Vol}_n(K) \geqslant \mathrm{Vol}_n(C(\theta)) = \frac{\mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \cdot h_K(\theta)}{n}.$$

It follows that $h_K(\theta) \leqslant c_2 n L_K$.

For the left hand side inequality, let $\theta \in S^{n-1}$. By a classical lemma of Grünbaum's we know that

$$\mathrm{Vol}_n(\{x : \langle x, \theta \rangle \geqslant 0\}) \geqslant \frac{1}{e}.$$

This implies that $e^{-1} \leqslant \|f_{K,\theta}\|_\infty h_K(\theta)$ and we get that

$$e^{-1} \leqslant e\mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) h_K(\theta).$$

Taking into account (3.11) we see that $h_K(\theta) \geqslant c_1 L_K$, and since θ was arbitrary, this gives $r(K) \geqslant c_1 L_K$. In the symmetric case one actually has the bound $r(K) \geqslant L_K$, because $|\langle x, \theta\rangle| \leqslant h_K(\theta)$, and hence

$$h_K(\theta) \geqslant \left(\int_K \langle x, \theta\rangle^2 dx\right)^{1/2} = L_K$$

for every $\theta \in S^{n-1}$.

We can now easily prove a first general upper bound for the isotropic constant of any convex body K in $\mathbb{R}^n$.

Proposition 3.11 (simple upper bound)**.** *For every isotropic convex body K in $\mathbb{R}^n$*

$$L_K \leqslant c\sqrt{n},$$

where $c > 0$ is an absolute constant.

Proof. Assume that K is in isotropic position. Since $r(K)B_2^n \subseteq K$ and $r(K) \geqslant c_1 L_K$, we get

$$\omega_n(c_1 L_K)^n \leqslant \omega_n(r(K))^n = \mathrm{Vol}_n(r(K)B_2^n) \leqslant \mathrm{Vol}_n(K) = 1.$$

It follows that $L_K \leqslant c_1^{-1}\omega_n^{-1/n} \leqslant c\sqrt{n}$ for an absolute constant $c > 0$. $\square$

4 Log-concave measures and tail estimates

We denote by $\mathcal{P}_n$ the class of all Borel probability measures in $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of a measure $\mu \in \mathcal{P}_n$ is denoted by f_μ.

We say that a measure $\mu \in \mathcal{P}_n$ has barycenter at $x_0 \in \mathbb{R}^n$, and we write $x_0 = \mathrm{bar}(\mu)$, if

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle \, d\mu(x) = \langle x_0, \theta\rangle$$

for all $\theta \in S^{n-1}$. Equivalently, if

$$x_0 = \int_{\mathbb{R}^n} x \, d\mu(x).$$

The subclass $\mathcal{CP}_n$ of $\mathcal{P}_n$ consists of all *centered* $\mu \in \mathcal{P}_n$. These are the measures $\mu \in \mathcal{P}_n$ that have barycenter at the origin; so, $\mu \in \mathcal{CP}_n$ if

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle d\mu(x) = 0$$

for all $\theta \in S^{n-1}$.

The subclass $\mathcal{SP}_n$ of $\mathcal{P}_n$ consists of all *even* measures $\mu \in \mathcal{P}_n$; μ is called even (or symmetric) if $\mu(A) = \mu(-A)$ for every Borel subset A of $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function with finite, positive integral. As in the case of measures, the barycenter of f is defined as

$$\mathrm{bar}(f) = \frac{\int_{\mathbb{R}^n} x f(x) \, dx}{\int_{\mathbb{R}^n} f(x) \, dx}.$$

In particular, f has barycenter at the origin if

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle f(x) \, dx = 0$$

for all $\theta \in S^{n-1}$. If so, we will say that f is centered.

Definition 4.1. (i) A measure $\mu \in \mathcal{P}_n$ is called *log-concave* if for all compact subsets A, B of $\mathbb{R}^n$ and all $0 < \lambda < 1$ we have

$$\mu((1-\lambda)A + \lambda B) \geqslant \mu(A)^{1-\lambda}\mu(B)^{\lambda}.$$

(ii) A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if

$$f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$$

for all $x, y \in \mathbb{R}^n$ and any $0 < \lambda < 1$.

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $\int_{\mathbb{R}^n} f(x)\, dx = 1$ (then we say that f is a *log-concave density*). From the Prékopa-Leindler inequality it follows that the measure μ with density f is log-concave. The next theorem of Borell shows that, conversely, any non-degenerate log-concave probability measure in $\mathbb{R}^n$ belongs to the class $\mathcal{P}_n$ and has a log-concave density.

Theorem 4.2 (Borell)**.** *Let μ be a log-concave probability measure in $\mathbb{R}^n$ such that $\mu(H) < 1$ for any hyperplane H. Then, μ is absolutely continuous with respect to the Lebesgue measure and has a log-concave density f, that is $d\mu(x) = f(x)\, dx$.*

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\Phi : \mathbb{R} \to [0, +\infty)$ be an even convex function satisfying $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = +\infty$ (we say that Φ is an Orlicz function). The Orlicz space $L_\Phi(\mu)$ that corresponds to Φ consists of all the $\mathcal{A}$-measurable functions f for which there is a constant $\kappa > 0$ such that $\int_\Omega \Phi(f/\kappa) d\mu < \infty$. The norm of any such function f is defined to be the infimum of all $\kappa > 0$ such that $\int_\Omega \Phi(f/\kappa) d\mu \leqslant 1$.

One can check that $L_\Phi(\mu) \subseteq L_1(\mu)$: if a measurable function f has finite $\Phi(\mu)$-norm then f is integrable with respect to μ.

The family of ψ_α-norms, which is a subclass of Orlicz norms, will play a central role in these notes.

Definition 4.3 (ψ_α-norm)**.** Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $f : \Omega \to \mathbb{R}$ be an $\mathcal{A}$-measurable function. For any $\alpha \geqslant 1$ we define the ψ_α-norm of f as follows:

$$\|f\|_{\psi_\alpha} := \inf\left\{ t > 0 : \int_\Omega \exp\left((|f(\omega)|/t)^\alpha\right)\, d\mu(\omega) \leqslant 2 \right\},$$

provided that the set on the right hand side is non-empty. Note that the ψ_α-norm is exactly the Orlicz norm corresponding to the function $t \in \mathbb{R} \to e^{|t|^\alpha} - 1$.

The next lemma gives an equivalent expression for the ψ_α-norm in terms of the L_q-norms.

Lemma 4.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\alpha \geqslant 1$ and let $f : \Omega \to \mathbb{R}$ be an $\mathcal{A}$-measurable function. Then,*

$$\|f\|_{\psi_\alpha} \simeq \sup_{p \geqslant \alpha} \frac{\|f\|_{L_p(\mu)}}{p^{1/\alpha}},$$

up to some absolute constants.

Definition 4.5. Let $\mu \in \mathcal{P}_n$, $\alpha \geqslant 1$ and $\theta \in S^{n-1}$. We say that μ satisfies a *ψ_α-estimate with constant* $b_\alpha = b_\alpha(\theta)$ in the direction of θ if we have

$$\|\langle \cdot, \theta\rangle\|_{\psi_\alpha} \leqslant b_\alpha \|\langle \cdot, \theta\rangle\|_2.$$

We say that μ is a ψ_α-measure with constant $B_\alpha > 0$ if

$$\sup_{\theta \in S^{n-1}} \frac{\|\langle \cdot, \theta\rangle\|_{\psi_\alpha}}{\|\langle \cdot, \theta\rangle\|_2} \leqslant B_\alpha.$$

Using Lemma 4.4 we see that μ satisfies a ψ_α-estimate with constant $b'_\alpha \simeq b_\alpha$ in the direction of $\theta \in S^{n-1}$ if

$$\|\langle \cdot, \theta\rangle\|_q \leqslant b_\alpha q^{1/\alpha}\|\langle \cdot, \theta\rangle\|_2$$

for all $q \geqslant \alpha$.

Remark 4.6. Let $\mu \in \mathcal{P}_n$ and let $\alpha \geqslant 1$ and $\theta \in S^{n-1}$.

(i) If μ satisfies a ψ_α-estimate with constant b in the direction of θ then for all $t > 0$ we have $\mu(\{x : |\langle x, \theta\rangle| \geqslant t\|\langle \cdot, \theta\rangle\|_2\}) \leqslant 2e^{-t^a/b^\alpha}$.

(ii) If we have $\mu(\{x : |\langle x, \theta\rangle| \geqslant t\|\langle \cdot, \theta\rangle\|_2\}) \leqslant 2e^{-t^a/b^\alpha}$ for some $b > 0$ and for all $t > 0$ then μ satisfies a ψ_α-estimate with constant $\leqslant cb$ in the direction of θ, where $c > 0$ is an absolute constant.

Lemma 4.7 (Borell)**.** *Let μ be a log-concave measure in $\mathcal{P}_n$. Then, for any symmetric convex set A in $\mathbb{R}^n$ with $\mu(A) = \alpha \in (0, 1)$ and any $t > 1$ we have*

$$1 - \mu(tA) \leqslant \alpha\left(\frac{1-\alpha}{\alpha}\right)^{\frac{t+1}{2}}. \tag{4.1}$$

Proof. Using the symmetry and convexity of A we check that

$$\frac{2}{t+1}(\mathbb{R}^n \setminus (tA)) + \frac{t-1}{t+1}A \subseteq \mathbb{R}^n \setminus A.$$

for every $t > 1$. Then, we apply the log-concavity of μ to get the result. □

Using Borell's lemma we see that there exists an absolute constant $C > 0$ such that every log-concave measure $\mu \in \mathcal{P}_n$ is a ψ_1-measure with constant C.

Theorem 4.8. *Let $\mu \in \mathcal{P}_n$ be log-concave. If $f : \mathbb{R}^n \to \mathbb{R}$ is a seminorm then, for any $q > p \geqslant 1$, we have*

$$\left(\int_{\mathbb{R}^n} |f|^p\, d\mu\right)^{1/p} \leqslant \left(\int_{\mathbb{R}^n} |f|^q\, d\mu\right)^{1/q} \leqslant c\frac{q}{p}\left(\int_{\mathbb{R}^n} |f|^p\, d\mu\right)^{1/p},$$

where $c > 0$ is an absolute constant.

Proof. We write $\|f\|_p^p := \int |f|^p\, d\mu$. Then, the set

$$A = \{x \in \mathbb{R}^n : |f(x)| \leqslant 3\|f\|_p\}$$

is symmetric and convex. Also, for any $t > 0$ we get

$$tA = \{x \in \mathbb{R}^n : |f(x)| \leqslant 3t\|f\|_p\},$$

while $\mu(A) \geqslant 1 - 3^{-p}$. So, in our case $\frac{1}{\alpha} - 1 \leqslant \frac{3^{-p}}{1-3^{-p}} \leqslant e^{-p/2}$. Using Borell's lemma we see that

$$\mu(x : |f(x)| \geqslant 3t\|f\|_p) \leqslant e^{-c_1 p(t-1)}$$

for any $t > 1$, with $c_1 = \frac{1}{4}$. Now, we write

$$\begin{aligned}\int_{\mathbb{R}^n} |f|^q\, d\mu &= \int_0^\infty q s^{q-1}\mu(\{x : |f(x)| \geqslant s\})\, ds \\ &\leqslant (3\|f\|_p)^q + (3\|f\|_p)^q\int_1^\infty q t^{q-1}e^{-c_1 p(t-1)}\, dt \\ &\leqslant (3\|f\|_p)^q + e^{c_1 p}(3\|f\|_p)^q\int_0^\infty q t^{q-1}e^{-c_1 pt}\, dt \\ &\leqslant (3\|f\|_p)^q + e^{c_1 p}\left(\frac{3\|f\|_p}{c_1 p}\right)^q\Gamma(q+1).\end{aligned}$$

Stirling's formula and the fact that $(a+b)^{1/q} \leqslant a^{1/q} + b^{1/q}$ for all $a, b > 0$ and $q \geqslant 1$, imply that $\|f\|_{L^q(\mu)} \leqslant c\frac{q}{p}\|f\|_{L^p(\mu)}$. □

Remark 4.9. Let K be a convex body of volume 1 in $\mathbb{R}^n$. We define a probability measure μ_K in $\mathbb{R}^n$, setting

$$\mu_K(A) = \mathrm{Vol}_n(K \cap A) = \int_A \mathbf{1}_K(x)dx$$

for every Borel $A \subseteq \mathbb{R}^n$. From the convexity of K we easily check that $\mathbf{1}_K$ is a log-concave function, and hence μ_K is a log-concave probability measure.

For every $\theta \in S^{n-1}$ the function $x \mapsto |\langle x, \theta\rangle|$, $x \in K$, satisfies the hypothesis of Theorem 4.8. Therefore,

$$\|\langle \cdot, \theta\rangle\|_q \leqslant cq\|\langle \cdot, \theta\rangle\|_1$$

for all $\theta \in S^{n-1}$ and $q \geqslant 1$, where $c > 0$ is an absolute constant. It follows that

$$\|\langle \cdot, \theta\rangle\|_{\psi_1} \leqslant c\|\langle \cdot, \theta\rangle\|_1$$

for all $\theta \in S^{n-1}$.

The next result provides a small ball probability estimate for log-concave probability measures.

Theorem 4.10 (Latała)**.** *Let μ be a log-concave probability measure in $\mathbb{R}^n$. For any norm $\|\cdot\|$ on $\mathbb{R}^n$ and any $0 \leqslant t \leqslant 1$ one has*

$$\mu(\{x : \|x\| \leqslant t\mathbb{E}_\mu(\|x\|)\}) \leqslant Ct, \tag{4.2}$$

where $C > 0$ is an absolute constant.

A consequence of Theorem 4.10 is the next Kahane-Khintchine inequality for negative exponents.

Theorem 4.11. *Let μ be a log-concave probability measure in $\mathbb{R}^n$. For any norm $\|\cdot\|$ on $\mathbb{R}^n$ and any $-1 < q < 0$ one has*

$$\mathbb{E}_\mu(\|x\|) \leqslant \frac{C}{1+q}\big(\mathbb{E}_\mu(\|x\|^q)\big)^{1/q}, \tag{4.3}$$

where $C > 0$ is an absolute constant.

5 Bourgain's upper bound for the isotropic constant

In this section we present Bourgain's $O(\sqrt[4]{n}\log n)$ bound for the isotropic constant.

Theorem 5.1 (Bourgain)**.** *If K is an isotropic convex body in $\mathbb{R}^n$ then*

$$L_K \leqslant c\sqrt[4]{n}\log n,$$

where $c > 0$ is an absolute constant.

We need some auxiliary facts.

Proposition 5.2. *Let $\alpha \geqslant 1$ and assume that the random variables $\{X_i\}_{i=1}^N$, $N \geqslant 2$, satisfy the ψ_α-estimate*

$$\|X_i\|_{\psi_\alpha} \leqslant b$$

for all $i = 1, \dots, N$. Then

$$\mathbb{E}\max_{1\leqslant i\leqslant N}|X_i| \leqslant Cb(\log N)^{1/\alpha},$$

where $C > 0$ is an absolute constant.

From Remark 4.9 we know that the random variables $\langle\cdot,\theta\rangle$ on K satisfy the ψ_1-estimate

$$\|\langle\cdot,\theta\rangle\|_{\psi_1} \leqslant c\|\langle\cdot,\theta\rangle\|_2 = cL_K$$

for all $\theta \in S^{n-1}$, where $c > 0$ is an absolute constant. Therefore, we get:

Proposition 5.3. *Let K be an isotropic convex body in $\mathbb{R}^n$, and let $N \geqslant 2$ and $\theta_1, \ldots, \theta_N \in S^{n-1}$. Then*

$$\int_K \max_{1\leqslant i\leqslant N} |\langle x, \theta_i\rangle| \, dx \leqslant CL_K(\log N),$$

where $C > 0$ is an absolute constant.

Next, we state our second tool, known as Dudley-Fernique decomposition, in a straightforward geometric way.

Proposition 5.4. *Let K be a convex body in $\mathbb{R}^n$, with $0 \in K$ and $K \subset RB_2^n$. There exist $Z_j \subseteq (3R/2^j)B_2^n$, $j \in \mathbb{N}$, with cardinality*

$$\log|Z_j| \leqslant cn\left(\frac{2^j w(K)}{R}\right)^2,$$

which satisfy the following: for every $x \in K$ and any $m \in \mathbb{N}$ we can find $z_j \in Z_j$, $j = 1, \ldots, m$, and $w_m \in (R/2^m)B_2^n$ such that $x = z_1 + \cdots + z_m + w_m$.

Proof. We use elementary properties of covering numbers and Sudakov's inequality. For every $j \in \mathbb{N}$ we may find a subset N_j of K with cardinality

$$|N_j| = N(K, (R/2^j)B_2^n)$$

such that

$$K \subseteq \bigcup_{y\in N_j} (y + (R/2^j)B_2^n).$$

From Sudakov's inequality we have

$$\log|N_j| \leqslant cn\left(\frac{2^j w(K)}{R}\right)^2.$$

We set $N_0 = \{0\}$ and

$$W_j = N_j - N_{j-1} = \{y - y' : y \in N_j, y' \in N_{j-1}\}$$

for every $j \geqslant 1$. We define $Z_j = W_j \cap (3R/2^j)B_2^n$. Thus $\log|Z_j| \leqslant \log|W_j| \leqslant c'n\left(\frac{2^j w(K)}{R}\right)^2$. We need to show that for every $x \in K$ and any $m \in \mathbb{N}$ we can find $z_j \in W_j \cap (3R/2^j)B_2^n$, $j = 1, \ldots, m$, and $w_m \in (R/2^m)B_2^n$ such that

$$x = z_1 + \cdots + z_m + w_m.$$

Given such x, by the definition of N_j, we can find $y_j \in N_j$, $j = 1, \ldots, m$, such that

$$|x - y_j| \leqslant \frac{R}{2^j}.$$

We write

$$x = (y_1 - 0) + (y_2 - y_1) + \cdots + (y_m - y_{m-1}) + (x - y_m).$$

We set $y_0 = 0$ and $w_m = x - y_m$, $z_j = y_j - y_{j-1}$ for $j = 1, \ldots, m$. Then, $|w_m| = |x - y_m| \leqslant R/2^m$, and $z_j \in N_j - N_{j-1} = W_j$. Also,

$$|z_j| \leqslant |x - y_j| + |x - y_{j-1}| \leqslant \frac{R}{2^j} + \frac{R}{2^{j-1}} = \frac{3R}{2^j}.$$

Finally, $x = z_1 + \cdots + z_m + w_m$ as claimed. $\square$

We are now ready to give S. Dar's version of the proof of Bourgain's bound.

Proof of Theorem 5.1. By the reverse Urysohn inequality (see Section 2) there exists a symmetric and positive definite $T \in SL_n$ such that

$$w(TK) \leqslant c\sqrt{n}\log n.$$

Using the elementary properties of the isotropic position we write

$$nL_K^2 = \int_K |x|^2 dx \leqslant \frac{\mathrm{tr}T}{n}\int_K |x|^2 = \int_K \langle x, Tx\rangle dx.$$

Therefore,

$$nL_K^2 \leqslant \int_K \max_{y\in TK} |\langle y, x\rangle| dx.$$

If $TK \subset RB_2^n$, we can use Proposition 5.4 to find $Z_j \subset (3R/2^j)B_2^n$ such that

$$\log|Z_j| \leqslant cn\left(\frac{w(TK)2^j}{R}\right)^2, \tag{5.1}$$

and so that for every $m\in\mathbb{N}$, every $y \in TK$ can be written in the form $y = z_1 + \cdots + z_m + w_m$ with $z_j \in Z_j$ and $w_m \in (R/2^m)B_2^n$. This implies that

$$\begin{aligned}\max_{y\in TK}|\langle y, x\rangle| &\leqslant \sum_{j=1}^m \max_{z\in Z_j}|\langle z, x\rangle| + \max_{w\in (R/2^m)B_2^n}|\langle w, x\rangle| \\ &\leqslant \sum_{j=1}^m \frac{3R}{2^j}\max_{z\in Z_j}|\langle \overline{z}, x\rangle| + \frac{R}{2^m}|x|,\end{aligned}$$

where $\overline{z}$ denotes the unit vector in the direction of z. Noting that (by Cauchy-Schwarz inequality) $\int_K |x|dx \leqslant \sqrt{n}L_K$ and using the above, we see that

$$\begin{aligned}nL_K^2 &\leqslant \sum_{j=1}^m \frac{3R}{2^j}\int_K \max_{z\in Z_j}|\langle \overline{z}, x\rangle|dx + \frac{R}{2^m}\int_K |x|dx \\ &\leqslant \sum_{j=1}^m \frac{3R}{2^j}\int_K \max_{z\in Z_j}|\langle \overline{z}, x\rangle|dx + \frac{R}{2^m}\sqrt{n}L_K.\end{aligned}$$

From Proposition 5.3 and (5.1) we get

$$nL_K^2 \leqslant \sum_{j=1}^m \frac{3R}{2^j}c_1 nL_K\left(\frac{w(TK)2^j}{R}\right)^2 + \frac{R}{2^m}\sqrt{n}L_K. \tag{5.2}$$

The sum on the right hand side is bounded by

$$c_2 L_K n w^2(TK)\frac{2^m}{R}.$$

Solving the equation

$$\frac{nw^2(TK)2^s}{R} = \frac{R\sqrt{n}}{2^s}$$

(where s here can be non-integer), we see that the optimal (integer) value of m satisfies the "equation"

$$\frac{R}{2^m} \simeq \sqrt[4]{n}w(TK).$$

Going back to (5.2), we obtain

$$nL_K^2 \leqslant c_3\sqrt{n}\sqrt[4]{n}w(TK)L_K.$$

Since $w(TK) \leqslant c_4\sqrt{n}\log n$, we get the result. $\square$

6 Alesker and Bobkov-Nazarov

In this section we discuss two results from the 1990's that have greatly influenced subsequent developments. The first one is a theorem of Alesker.

Theorem 6.1 (Alesker)**.** *There exists an absolute constant $c > 0$ such that: if K is an isotropic convex body in $\mathbb{R}^n$ then*

$$\mathrm{Vol}_n(\{x \in K : |x| \geqslant c\sqrt{n}L_K t\}) \leqslant 2\exp(-t^2)$$

for every $t > 0$.

It is useful to consider the q-th moment of the function $x \mapsto |x|$ on K,

$$I_q(K) := \left(\int_K |x|^q \, dx\right)^{1/q}$$

for $q \geqslant 1$. Theorem 6.1 is a direct consequence of Lemma 4.6 and of the next statement.

Theorem 6.2 (Alesker)**.** *Let K be an isotropic convex body in $\mathbb{R}^n$. For all $q \geqslant 1$ we have*

$$I_q(K) \leqslant c\sqrt{q} I_2(K),$$

and letting $f(x) = |x|$, we have

$$\|f\|_{\psi_2} \leqslant c\sqrt{n}L_K,$$

where $c > 0$ is an absolute constant.

For the proof we first note the following simple formula:

Lemma 6.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$,*

$$\left(\int_{S^{n-1}} \int_K |\langle x, \theta\rangle|^q dx d\sigma(\theta)\right)^{1/q} \simeq \sqrt{\frac{q}{q+n}} I_q(K).$$

Proof. For every $q \geqslant 1$ and $x \in \mathbb{R}^n$, we check that

$$\left(\int_{S^{n-1}} |\langle x, \theta\rangle|^q d\sigma(\theta)\right)^{1/q} \simeq \frac{\sqrt{q}}{\sqrt{q+n}}|x|. \tag{6.1}$$

To see this, using polar coordinates we first see that

$$\int_{B_2^n} |\langle x, y\rangle|^q dy = n\omega_n \int_0^1 r^{n+q-1} dr \int_{S^{n-1}} |\langle x, \theta\rangle|^q d\sigma(\theta) = \frac{n\omega_n}{n+q} \int_{S^{n-1}} |\langle x, \theta\rangle|^q d\sigma(\theta).$$

But we can also write the left hand side as

$$\begin{aligned}\int_{B_2^n} |\langle x, y\rangle|^q dy &= |x|^q \int_{B_2^n} \left|\left\langle \frac{x}{|x|}, y\right\rangle\right|^q dy = |x|^q \int_{B_2^n} |\langle e_1, y\rangle|^q dy \\ &= 2\omega_{n-1}|x|^q \int_0^1 t^q(1-t^2)^{(n-1)/2} dt = \omega_{n-1}|x|^q \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+q+2}{2}\right)}.\end{aligned}$$

Comparing the two expressions and using Stirling's formula we get (6.1). A simple application of Fubini's theorem gives the result. □

Proof of Theorem 6.2. By Lemma 4.4, the first assertion implies the second. Thus we concentrate on proving that for every $q > 1$

$$\left(\int_K |x|^q dx\right)^{1/q} \leqslant c_1 \sqrt{q}\sqrt{n} L_K \tag{6.2}$$

for some absolute constant $c_1 > 0$. We know that for every $\theta \in S^{n-1}$

$$\int_K |\langle x, \theta\rangle|^q dx \leqslant c_2^q q^q L_K^q.$$

Integrating on the sphere we get

$$\int_{S^{n-1}} \int_K |\langle x, \theta\rangle|^q dx\, d\sigma(\theta) \leqslant c_2^q q^q L_K^q.$$

Taking into account Lemma 6.3, we see that

$$\left(\int_K |x|^q dx\right)^{1/q} \leqslant c_3 q \sqrt{\frac{n+q}{q}} L_K \leqslant c_4 \sqrt{q}\sqrt{n} L_K,$$

provided that $q \leqslant n$. On the other hand, if $q > n$, using the fact that $K \subset cnL_K B_2^n$, we get

$$\left(\int_K |x|^q dx\right)^{1/q} \leqslant cnL_K \leqslant c\sqrt{q}\sqrt{n} L_K.$$

Combining the above we see that (6.2) holds true for all $q > 1$. □

The second result of this section is due to Bobkov and Nazarov and concerns the case of symmetric convex bodies which generate a norm with unconditional basis. After a linear transformation, we may assume that the standard orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ is an unconditional basis for $\|\cdot\|_K$. That is, for every choice of real numbers $t_1, \dots, t_n$ and every choice of signs $\varepsilon_i = \pm 1$,

$$\big\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\big\|_K = \big\|t_1 e_1 + \cdots + t_n e_n\big\|_K.$$

Geometrically, this means that if $x = (x_1, \dots, x_n) \in K$ then the whole rectangle $\prod_{i=1}^n [-|x_i|, |x_i|]$ is contained in K.

Note that the matrix of inertia of such a body is diagonal, therefore one can bring it to the isotropic position by a diagonal operator. This explains that for every unconditional convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K}$ of K which has the following properties:

1. The volume of $\tilde{K}$ is equal to 1.
2. If $x = (x_1, \dots, x_n) \in \tilde{K}$ then $\prod_{i=1}^n [-|x_i|, |x_i|] \subseteq \tilde{K}$.
3. For every $j = 1, \dots, n$,
$$\int_{\tilde{K}} x_j^2 dx = L_K^2.$$

This last condition implies that $\tilde{K}$ is in isotropic position, because

$$\int_{\tilde{K}} x_i x_j dx = 0 \text{ for all } i \neq j$$

by Property 2.

We assume that K has these three properties. It will be convenient to consider the normalized part

$$K^+ = 2K \cap \mathbb{R}_+^n$$

of K in $\mathbb{R}_+^n = [0, +\infty)^n$. So, if $x = (x_1, \dots, x_n)$ is uniformly distributed in K, then $(2|x_1|, \dots, 2|x_n|)$ is uniformly distributed in K^+. It is easy to check that K^+ has the following three properties:

4. The volume of K^+ is equal to 1.

5. If $x=(x_1,\dots,x_n)\in K^+$ and $0\leqslant y_j\leqslant x_j$ for all $1\leqslant j\leqslant n$, then $y=(y_1,\dots,y_n)\in K^+$.
6. For every $j=1,\dots,n$,
$$\int_{K^+} x_j^2dx = 4L_K^2.$$

It is not difficult to show that the isotropic constants of unconditional convex bodies are uniformly bounded. One way to see this is to use the Loomis-Whitney inequality

$$1=\mathrm{Vol}_n(K)^{n-1}\leqslant\prod_{i=1}^n\mathrm{Vol}_{n-1}(P_{e_i^\perp}(K))=\prod_{i=1}^n\mathrm{Vol}_{n-1}(K\cap e_i^\perp),$$

where the last equality comes from the fact that $P_{e_i^\perp}(K)=K\cap e_i^\perp$. This shows that $\mathrm{Vol}_{n-1}(K\cap e_i)\geqslant 1$ for some $i\leqslant n$, and then Theorem 3.5 shows that $\mathrm{Vol}_{n-1}(K\cap\theta^\perp)\geqslant c$ for some absolute constant $c>0$ and for all $\theta\in S^{n-1}$. In fact, Bobkov and Nazarov provide a different direct argument which gives:

Theorem 6.4. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$|K\cap\theta^\perp|\geqslant\frac{1}{\sqrt{6}}$$

for every $\theta\in S^{n-1}$.

Our main interest is in the next distributional inequality from the same work.

Theorem 6.5 (Bobkov-Nazarov)**.** *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{Vol}_n(\{x\in K^+: x_1\geqslant\alpha_1,\dots,x_n\geqslant\alpha_n\})\leqslant\left(1-\frac{\alpha_1+\cdots+\alpha_n}{\sqrt{6}n}\right)^n,$$

for all $(\alpha_1,\dots,\alpha_n)\in K^+$.

Proof. We define a function $u:K^+\to[0,\infty)$ by

$$u(\alpha_1,\dots,\alpha_n)=\mathrm{Vol}_n(\{x\in K^+: x_1\geqslant\alpha_1,\dots,x_n\geqslant\alpha_n\}).$$

The Brunn-Minkowski inequality shows that the function $h=u^{\frac1n}$ is concave on K^+. Observe that $u(0)=1$ and

$$\frac{\partial u}{\partial\alpha_j}(0)=-\mathrm{Vol}_{n-1}(K\cap e_j^\perp)\leqslant-\frac{1}{\sqrt{6}}, \tag{6.3}$$

where the last inequality comes from Theorem 6.4. Let $\alpha\in K^+$ and consider the function $h_\alpha:[0,1]\to\mathbb{R}$ defined by $h_\alpha(t)=h(\alpha t)$. Note that

$$h'_\alpha(0)=\sum_{j=1}^n\alpha_j\frac{\partial h}{\partial\alpha_j}(0)=\sum_{i=1}^n\alpha_i\cdot\frac1n\frac{\partial u}{\partial\alpha_j}(0)\leqslant-\frac{\alpha_1+\cdots+\alpha_n}{\sqrt{6}n}$$

by (6.3). Since h is concave, h_α is concave on $[0,1]$. This implies that h'_α is decreasing on $[0,1]$, and hence,

$$h(\alpha)-1=h_\alpha(1)-h_\alpha(0)\leqslant h'_\alpha(0)\leqslant-\frac{\alpha_1+\cdots+\alpha_n}{\sqrt{6}n}$$

for all $\alpha\in K^+$. This proves the theorem. □

As a direct consequence we get the following statement, which is valid for all $\alpha_j\geqslant 0$.

Corollary 6.6. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{Vol}_n(\{x \in K^+ : x_1 \geqslant \alpha_1, \dots, x_n \geqslant \alpha_n\}) \leqslant \exp(-c(\alpha_1 + \cdots + \alpha_n)),$$

for all $\alpha_1, \dots, \alpha_n \geqslant 0$, where $c = 1/\sqrt{6}$.

Proof. If $(\alpha_1, \dots, \alpha_n) \in K^+$ we apply Theorem 6.5 and then just use the fact that $1 - x \leqslant e^{-x}$ for all $x \geqslant 0$. If not, then the left hand side is equal to zero. $\square$

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$. We write $x_1^*, \dots, x_n^*$ for the coordinates of x in decreasing order. That is,

$$\max x_j = x_1^* \geqslant x_2^* \geqslant \cdots \geqslant x_n^* = \min x_j.$$

Let μ_K^+ denote the uniform distribution on K^+. Corollary 6.6 has the following consequence.

Proposition 6.7. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$\mu_K^+(\{x \in \mathbb{R}^n_+ : x_k^* \geqslant \alpha\}) \leqslant \binom{n}{k} e^{-ck\alpha}$$

for all $\alpha \geqslant 0$ and $1 \leqslant k \leqslant n$, where $c = 1/\sqrt{6}$.

Proof. Let $1 \leqslant j_1 < \cdots < j_k \leqslant n$. From Corollary 6.6 we have

$$\mu_K^+(\{x \in \mathbb{R}^n_+ : x_{j_1} \geqslant \alpha, \dots, x_{j_k} \geqslant \alpha\}) \leqslant \exp(-ck\alpha).$$

Since

$$\{x \in \mathbb{R}^n_+ : x_k^* \geqslant \alpha\} = \bigcup_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \{x \in \mathbb{R}^n_+ : x_{j_1} \geqslant \alpha, \dots, x_{j_k} \geqslant \alpha\},$$

we get

$$\begin{aligned} \mu_K^+(\{x : x_k^* \geqslant \alpha\}) &\leqslant \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \mu_K^+(\{x : x_{j_1} \geqslant \alpha, \dots, x_{j_k} \geqslant \alpha\}) \\ &\leqslant \binom{n}{k} e^{-ck\alpha} \end{aligned}$$

as claimed. $\square$

This leads to the next improved version of Alesker's theorem in the unconditional case.

Theorem 6.8 (Bobkov-Nazarov)**.** *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then, for every $t \geqslant 4$,*

$$\mathrm{Vol}_n(\{x \in K : |x| \geqslant c_2 t\sqrt{n}\}) \leqslant \exp\left(-\frac{t\sqrt{n}}{2}\right),$$

where $c_2 = \sqrt{6}$.

Proof. Let $\alpha_1, \dots, \alpha_n \geqslant 0$. From Proposition 6.7 we have

$$\begin{aligned} \mathrm{Vol}_n\Big(\Big\{x \in K : |x|^2 \geqslant \sum_{k=1}^n \alpha_k^2\Big\}\Big) &= \mu_K^+\left(\Big\{x \in \mathbb{R}^n_+ : \sum_{k=1}^n x_k^2 \geqslant 4\sum_{k=1}^n \alpha_k^2\Big\}\right) \\ &= \mu_K^+\left(\Big\{x \in \mathbb{R}^n_+ : \sum_{k=1}^n (x_k^*)^2 \geqslant 4\sum_{k=1}^n \alpha_k^2\Big\}\right) \\ &\leqslant \sum_{k=1}^n \mu_K^+(\{x \in \mathbb{R}^n_+ : x_k^* \geqslant 2\alpha_k\}) \\ &\leqslant \sum_{k=1}^n \binom{n}{k} \exp(-2ck\alpha_k). \end{aligned}$$

This shows that

$$\text{Vol}_n\left(\left\{x \in K : |x|^2 \geqslant \sum_{k=1}^{n} \frac{\alpha_k^2}{c^2}\right\}\right) \leqslant \sum_{k=1}^{n} \exp\left(-k\left(2\alpha_k - \log\frac{en}{k}\right)\right), \tag{6.4}$$

where $c = 1/\sqrt{6}$. Given $t > 0$ we choose

$$\alpha_k = \frac{1}{2}\log\frac{en}{k} + t\frac{\sqrt{n}}{k}.$$

We check that if $t \geqslant 2$ then $\sum_{k=1}^{n} \alpha_k^2 \leqslant 4nt^2$, and going back to (6.4) we have

$$\text{Vol}_n\left(\left\{x \in K : |x| \geqslant 2\sqrt{6}t\sqrt{n}\right\}\right) \leqslant n\exp(-2t\sqrt{n}) \leqslant \exp(-t\sqrt{n})$$

for every $t \geqslant 2$. This proves the theorem. $\square$

It was after this result that people started thinking whether an analogous estimate might be true in full generality.

7 Isotropic log-concave measures

Definition 7.1. Let μ be a Borel probability measure in $\mathbb{R}^n$ which is absolutely continuous with respect to Lebesgue measure. We shall say that μ is *isotropic* if it is centered and satisfies the isotropic condition

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle^2 \, d\mu(x) = 1$$

for all $\theta \in S^{n-1}$. Similarly, we shall say that a centered log-concave function $f : \mathbb{R}^n \to [0, \infty)$ is isotropic if $\int f = 1$ and the measure $d\mu = f(x)dx$ is isotropic.

As in the case of convex bodies, we easily check that a centered measure μ as above is isotropic if and only if for any $T \in L(\mathbb{R}^n)$ one has

$$\int_{\mathbb{R}^n} \langle x, Tx\rangle \, d\mu(x) = \text{tr}(T),$$

or equivalently if $\int_{\mathbb{R}^n} x_i x_j \, d\mu(x) = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

Note that if μ is isotropic, then

$$\int_{\mathbb{R}^n} |x|^2 \, d\mu(x) = n,$$

and more generally,

$$\int_{\mathbb{R}^n} |Tx|^2 d\mu(x) = \|T\|_{\text{HS}}^2$$

for any $T \in L(\mathbb{R}^n)$.

Following the proof of Proposition 3.1, we can check that every non-degenerate absolutely continuous probability measure μ has an isotropic image $\nu = \mu \circ S$, where $S : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map. Similarly, every log-concave $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < \infty$ has an isotropic image: there exist an affine isomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$ and a positive number a such that $af \circ S$ is isotropic.

Remark 7.2. It is useful to compare the definition of an isotropic convex body with the definition of an isotropic log-concave measure. Note that a convex body K of volume 1 in $\mathbb{R}^n$ being isotropic implies that the covariance matrix of the measure $\mathbf{1}_K dx$ is $L_K\,\text{Id}$. So, we see that a convex body K of volume 1 is isotropic if and only if the function $f_K := L_K^n \mathbf{1}_{\frac{1}{L_K}K}$ is an isotropic log-concave function.

Definition 7.3 (general definition of the isotropic constant)**.** Let f be a log-concave function with finite, positive integral. We define its *inertia* – or *covariance* – matrix $\mathrm{Cov}(f)$ as the matrix with entries

$$[\mathrm{Cov}(f)]_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx}.$$

Note that if f is isotropic then $\mathrm{Cov}(f)$ is the identity matrix. If f is the density of a measure μ we denote this matrix also by $\mathrm{Cov}(\mu)$. The *isotropic constant* of f is defined by

$$L_f := \left(\frac{\sup_{x\in\mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(f)]^{\frac{1}{2n}}. \tag{7.1}$$

(and given a log-concave measure μ with density f_μ we let $L_\mu := L_{f_\mu}$).

With the above definition it is easy to check that the isotropic constant L_μ is an affine invariant; we have $L_\mu = L_{\lambda\mu\circ A}$, $L_f = L_{\lambda f\circ A}$ for every invertible affine transformation A of $\mathbb{R}^n$ and every positive number λ.

The following characterization of the isotropic constant holds and is completely analogous to the one in Theorem 3.2: if $f : \mathbb{R}^n \to [0,\infty)$ is a log-concave density, then

$$nL_f^2 = \inf_{\substack{T\in SL_n\\ y\in\mathbb{R}^n}} \left(\sup_{x\in\mathbb{R}^n} f(x)\right)^{2/n} \int_{\mathbb{R}^n} |Tx+y|^2 f(x)\,dx.$$

A very useful inequality of Fradelizi, that will be frequently used in these notes, asserts that if $f : \mathbb{R}^n \to [0,\infty)$ is a centered log-concave function, then

$$f(0) \leqslant \|f\|_\infty \leqslant e^n f(0).$$

The hyperplane conjecture for log-concave measures can now be stated as follows:

Problem 7.4 (main problem)**.** *Let $f : \mathbb{R}^n \to [0,\infty)$ be an isotropic log-concave density. Then*

$$\|f\|_\infty^{1/n} \leqslant C,$$

where $C > 0$ is an absolute constant.

One can prove that the isotropic constants of all log-concave measures are uniformly bounded from below by a constant $c > 0$ which is independent of the dimension. If $f : \mathbb{R}^n \to [0,\infty)$ is an isotropic log-concave density, then

$$L_f = \|f\|_\infty^{1/n} \simeq [f(0)]^{1/n} \geqslant c, \tag{7.2}$$

where $c > 0$ is an absolute constant.

8 Convex bodies associated with log-concave measures

In this section we discuss a family of sets $K_p(f)$ associated with any given log-concave function f. The bodies $K_p(f)$ were introduced by K. Ball who also established their convexity. They play a very important role as they allow us to study properties of log-concave measures through those of convex bodies and vice versa.

Definition 8.1 (Ball)**.** Let $f : \mathbb{R}^n \to [0,\infty)$ be a measurable function such that $f(0) > 0$. For any $p > 0$ we define the set $K_p(f)$ as follows:

$$K_p(f) = \left\{x\in\mathbb{R}^n : \int_0^\infty r^{p-1} f(rx)\,dr \geqslant \frac{f(0)}{p}\right\}.$$

If f_μ is the density of a Borel probability measure μ and $f_\mu(0) > 0$, then we define

$$K_p(\mu) := K_p(f_\mu).$$

From the definition it follows that the radial function of $K_p(f)$ is given by

$$\rho_{K_p(f)}(x) = \left(\frac{1}{f(0)}\int_0^\infty pr^{p-1}f(rx)\,dr\right)^{1/p}. \tag{8.1}$$

Lemma 8.2. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in K$. Then, we have $K_p(\mathbf{1}_K) = K$ for all $p > 0$.*

Proof. For every $\theta \in S^{n-1}$ we have

$$\rho^p_{K_p(\mathbf{1}_K)}(\theta) = \frac{1}{\mathbf{1}_K(0)}\int_0^{+\infty} pr^{p-1}\,\mathbf{1}_K(r\theta)\,dr = \int_0^{\rho_K(\theta)} pr^{p-1}\,dr = \rho^p_K(\theta).$$

It follows that $K_p(\mathbf{1}_K) = K$. □

The next proposition describes some basic properties of the sets $K_p(f)$.

Proposition 8.3. *Let $f, g : \mathbb{R}^n \to [0,\infty)$ be two integrable functions with $f(0) = g(0) > 0$, and set*

$$m = \inf\left\{\frac{f(x)}{g(x)} : g(x) > 0\right\} \quad \text{and} \quad M^{-1} = \inf\left\{\frac{g(x)}{f(x)} : f(x) > 0\right\}.$$

Then, for every $p > 0$ we have the following:

(i) $0 \in K_p(f)$.

(ii) *$K_p(f)$ is a star-shaped set.*

(iii) *$K_p(f)$ is symmetric if f is even.*

(iv) $m^{1/p}K_p(g) \subseteq K_p(f) \subseteq M^{1/p}K_p(g)$.

(v) *For any $\theta \in S^{n-1}$ we have*

$$\int_{K_{n+1}(f)} \langle x, \theta\rangle\,dx = \frac{1}{f(0)}\int_{\mathbb{R}^n}\langle x,\theta\rangle f(x)\,dx.$$

In particular, f is centered if and only if $K_{n+1}(f)$ is centered.

(vi) *For any $\theta \in S^{n-1}$ and $p > 0$ we have*

$$\int_{K_{n+p}(f)} |\langle x,\theta\rangle|^p\,dx = \frac{1}{f(0)}\int_{\mathbb{R}^n}|\langle x,\theta\rangle|^p\,f(x)\,dx.$$

(vii) *If $p > -n$ and V is a star-shaped body with gauge function $\|\cdot\|_V$ then*

$$\int_{K_{n+p}(f)} \|x\|_V^p\,dx = \frac{1}{f(0)}\int_{\mathbb{R}^n}\|x\|_V^p f(x)\,dx. \tag{8.2}$$

Assuming the log-concavity of f one can prove that the sets $K_p(f)$, $p > 0$, are convex. The proof is based on a variant of the Prékopa-Leindler inequality.

Theorem 8.4 (Ball)**.** *Let $f : \mathbb{R}^n \to [0,\infty)$ be a log-concave function such that $f(0) > 0$. For every $p > 0$, $K_p(f)$ is a convex set$K_p(f)$.*

To show that $K_n(f)$ is indeed a convex body, namely is compact and with non-empty interior, one simply computes its volume to see that it is non-zero and finite:

Lemma 8.5. *For every measurable function $f:\mathbb{R}^n\to[0,\infty)$ such that $f(0)>0$ we have*

$$\mathrm{Vol}_n(K_n(f))=\frac{1}{f(0)}\int_{\mathbb{R}^n}f(x)dx.$$

In particular, if f is log-concave and such that $0<\int_{\mathbb{R}^n}f<\infty$, then $K_n(f)$ is a convex body.

Proof. We can write

$$\begin{aligned}\mathrm{Vol}_n(K_n(f))&=\int_{K_n(f)}\mathbf{1}\,dx=n\omega_n\int_{S^{n-1}}\int_0^{\rho_{K_n(f)}(\phi)}r^{n-1}drd\sigma(\phi)\\&=\frac{n\omega_n}{f(0)}\int_{S^{n-1}}\int_0^{\infty}r^{n-1}f(r\phi)drd\sigma(\phi)=\frac{1}{f(0)}\int_{\mathbb{R}^n}f(x)dx\end{aligned}$$

using (8.1) and integration in polar coordinates. □

The fact that all of the convex sets $K_p(f)$, $p>0$, are indeed convex bodies, namely they are compact and have non-empty interior, whenever the log-concave function f has finite, positive integral, is a consequence of the next proposition.

Proposition 8.6. *Let $f:\mathbb{R}^n\to[0,\infty)$ be a centered log-concave function. For every $0<p\leqslant q$,*

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}}K_q(f)\subseteq K_p(f)\subseteq e^{\frac{n}{p}-\frac{n}{q}}K_q(f). \tag{8.3}$$

As a consequence we obtain an approximate formula for the volume of $K_{n+p}(f)$ when $p>0$.

Corollary 8.7. *Let $f:\mathbb{R}^n\to[0,\infty)$ be a centered log-concave density. Then, for every $p>0$ we have*

$$e^{-1}\leqslant f(0)^{\frac{1}{n}+\frac{1}{p}}\mathrm{Vol}_n(K_{n+p}(f))^{\frac{1}{n}+\frac{1}{p}}\leqslant e\frac{n+p}{n}, \tag{8.4}$$

while for $-n<p<0$ we have

$$e^{-1}\leqslant f(0)^{\frac{1}{-p}-\frac{1}{n}}\mathrm{Vol}_n(K_{n+p}(f))^{\frac{1}{-p}-\frac{1}{n}}\leqslant e. \tag{8.5}$$

A flavor of the applications of the bodies $K_p(f)$ may be given by the next propositions which relate the isotropic constants of convex bodies with those of log-concave functions.

Proposition 8.8 (Ball)**.** *Let $f:\mathbb{R}^n\to[0,\infty)$ be an even log-concave function with finite, positive integral. Then, the body $T=K_{n+2}(f)$ is a centrally symmetric convex body with*

$$c_1L_f\leqslant L_T\leqslant c_2L_f,$$

where $c_1,c_2>0$ are absolute constants. Furthermore, if f is isotropic, then $\overline{T}=\mathrm{Vol}_n(T)^{-1/n}T$ is an isotropic convex body.

Proof. Since f is even and log-concave, T is a centrally symmetric convex body; we also have $f(x)\leqslant f(0)$ for all $x\in\mathbb{R}^n$. Hence, $f(0)>0$. From Proposition 8.3 (vi)

$$\int_T\langle x,\theta\rangle^2\,dx=\frac{1}{f(0)}\int_{\mathbb{R}^n}\langle x,\theta\rangle^2f(x)\,dx,$$

and more generally

$$\int_T\langle x,\theta\rangle\langle x,\phi\rangle\,dx=\frac{1}{f(0)}\int_{\mathbb{R}^n}\langle x,\theta\rangle\langle x,\phi\rangle f(x)\,dx$$

for all $\theta, \phi \in S^{n-1}$. It follows that

$$\mathrm{Vol}_n(T)\mathrm{Cov}(\mathbf{1}_T) = \frac{\int f}{f(0)}\mathrm{Cov}(f).$$

By the definition of the isotropic constant we obtain

$$L_T = L_{\mathbf{1}_T} = \frac{1}{\mathrm{Vol}_n(T)^{\frac{1}{2}+\frac{1}{n}}}\left(\frac{1}{f(0)}\int f\right)^{\frac{1}{2}+\frac{1}{n}} L_f.$$

On the other hand, applying Corollary 8.7 with $p = 2$ we see that

$$\mathrm{Vol}_n(T)^{\frac{1}{2}+\frac{1}{n}} = \mathrm{Vol}_n(K_{n+2}(f))^{\frac{1}{2}+\frac{1}{n}} \simeq \left(\frac{1}{f(0)}\int_{\mathbb{R}^n} f(x)\,dx\right)^{\frac{1}{2}+\frac{1}{n}}.$$

This shows that $L_T \simeq L_f$. Finally, note that if f is isotropic then

$$\int_{\overline{T}} \langle x,\theta\rangle^2\,dx = \frac{1}{\mathrm{Vol}_n(T)^{1+\frac{2}{n}}}\int_T \langle x,\theta\rangle^2\,dx = \frac{1}{f(0)\mathrm{Vol}_n(T)^{1+\frac{2}{n}}}$$

for every $\theta \in S^{n-1}$, which shows that $\overline{T}$ is in isotropic position. $\square$

The next proposition, which is due to Klartag, shows that we can further reduce our study of the behavior of the isotropic constant to the class of symmetric convex bodies.

Proposition 8.9. *For every convex body K we can find a symmetric convex body T with the property that*

$$L_K \leqslant cL_T,$$

where $c > 0$ is an absolute constant.

Proof. Without loss of generality, we may assume that K has volume 1 and barycenter at the origin. We define a function f supported on $K - K$ as follows:

$$f(x) = (\mathbf{1}_K * \mathbf{1}_{-K})(x) = \int_{\mathbb{R}^n} \mathbf{1}_K(y)\mathbf{1}_{-K}(x-y)\,dy = \mathrm{Vol}_n(K \cap (x+K)).$$

Using the Brunn-Minkowski inequality one can see that f is an even and log-concave function with $\int_{\mathbb{R}^n} f = 1$ and that $f(x) \leqslant f(0) = \mathrm{Vol}_n(K) = 1$. Therefore,

$$L_f = [\det \mathrm{Cov}(f)]^{\frac{1}{2n}}.$$

Next, since one easily checks that for any h and g with barycenter at 0 and total mass 1 one has

$$\mathrm{Cov}(h * g) = \mathrm{Cov}(h) + \mathrm{Cov}(g),$$

we get that

$$\mathrm{Cov}(f) = \mathrm{Cov}(K) + \mathrm{Cov}(-K).$$

As these are positive definite matrices it follows that

$$[\det \mathrm{Cov}(f)]^{1/n} \geqslant [\det \mathrm{Cov}(K)]^{1/n} + [\det \mathrm{Cov}(-K)]^{1/n} = 2[\det \mathrm{Cov}(K)]^{1/n},$$

and hence

$$L_K = [\det \mathrm{Cov}(K)]^{\frac{1}{2n}} \leqslant \frac{1}{\sqrt{2}}[\det \mathrm{Cov}(f)]^{\frac{1}{2n}} = \frac{1}{\sqrt{2}}L_f.$$

It is easy now to check that the body $T := K_{n+2}(f)$ has the desired properties: T is symmetric because f is even, and in addition $L_T \simeq L_f \geqslant L_K$. $\square$

Assuming that the function $f : \mathbb{R}^n \to [0,\infty)$ is centered, but not necessarily even, we prefer to work with the centered body $K_{n+1}(f)$ instead of $K_{n+2}(f)$.

Proposition 8.10. *Let $f : \mathbb{R}^n \to [0,\infty)$ be a centered log-concave function with finite, positive integral. Then, $T = K_{n+1}(f)$ is a centered convex body in $\mathbb{R}^n$ with*

$$c_1 L_f \leqslant L_T \leqslant c_2 L_f,$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Note that, f being centered implies $f(0) > 0$; thus, $K_{n+1}(f)$ is well-defined and by Proposition 8.3 (v) and Theorem 8.4 we know that it is a centered convex body. Without loss of generality we may assume that f is log-concave with $\int f = 1$, otherwise we work with $f_1 = \frac{f}{\int f}$ using the fact that $K_{n+1}(\lambda f) = K_{n+1}(f)$ and $L_{\lambda f} = L_f$ for any $\lambda > 0$. By Proposition 8.3 we have

$$\int_T |\langle x, \theta\rangle|\, dx = \frac{1}{f(0)} \int |\langle x, \theta\rangle| f(x)\, dx.$$

Borell's lemma implies that for every $y \in \mathbb{R}^n$

$$\begin{aligned}\left(\frac{1}{\mathrm{Vol}_n(T)} \int_T \langle x, y\rangle^2\, dx\right)^{1/2} &\simeq \frac{1}{\mathrm{Vol}_n(T)} \int_T |\langle x, y\rangle|\, dx = \frac{1}{f(0)\mathrm{Vol}_n(T)} \int |\langle x, y\rangle| f(x)\, dx \\ &\simeq \frac{1}{f(0)\mathrm{Vol}_n(T)} \left(\int \langle x, y\rangle^2 f(x)\, dx\right)^{1/2},\end{aligned}$$

which, combined with the fact that T and f are both centered, implies that there exist absolute constants $c_1, c_2 > 0$ such that as positive definite matrices

$$c_2 \mathrm{Cov}(\mathbf{1}_T) \leqslant (\mathrm{Vol}_n(T) f(0))^{-2} \mathrm{Cov}(f) \leqslant c_1 \mathrm{Cov}(\mathbf{1}_T).$$

Therefore

$$[\det \mathrm{Cov}(\mathbf{1}_T)]^{1/n} \simeq (\mathrm{Vol}_n(T) f(0))^{-2} [\det \mathrm{Cov}(f)]^{1/n}. \tag{8.6}$$

From the definition of the isotropic constant it follows that

$$\begin{aligned}L_T &= \frac{1}{\mathrm{Vol}_n(T)^{1/n}} [\det \mathrm{Cov}(T)]^{\frac{1}{2n}} \simeq \mathrm{Vol}_n(T)^{-1/n} (f(0)\mathrm{Vol}_n(T))^{-1} [\det \mathrm{Cov}(f)]^{\frac{1}{2n}} \\ &\simeq (f(0)\mathrm{Vol}_n(T))^{-1-\frac{1}{n}} L_f,\end{aligned}$$

where we have also used the fact that one has $\|f\|_\infty^{1/n} \simeq f(0)^{1/n}$. Finally, applying Proposition 8.7 with $p = 1$ we get that

$$e^{-1} \leqslant (f(0)\mathrm{Vol}_n(T))^{1+\frac{1}{n}} \leqslant e\frac{n+1}{n} \leqslant 2e. \tag{8.7}$$

This completes the proof. $\square$

9 Centroid bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ we define the L_q-centroid body $Z_q(K)$ of K to be the symmetric convex body with support function

$$h_{Z_q(K)}(y) = \|\langle \cdot, y\rangle\|_{L^q(K)} = \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}.$$

From Hölder's inequality it is clear that if $1 \leqslant p \leqslant q \leqslant \infty$ then

$$Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K) := \mathrm{conv}(K \cup (-K)).$$

Note that $Z_q(T(K)) = T(Z_q(K))$ for every $T \in SL(n)$ and any $q \geqslant 1$. Also, a centered convex body K of volume 1 is isotropic if $Z_2(K)$ is a multiple of the Euclidean unit ball.

Analogously, if μ is a log-concave probability measure on $\mathbb{R}^n$, we define

$$h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^q \, d\mu(x)\right)^{1/q}.$$

Basic properties

Let K be a convex body of volume 1 in $\mathbb{R}^n$. Using the standard Khintchine-type inequalities for seminorms we see that if $1 \leqslant p < q$ then

$$Z_p(K) \subseteq Z_q(K) \subseteq \frac{c_1 q}{p} Z_p(K), \tag{9.1}$$

where $c_1 > 0$ is an absolute constant. If K has its barycenter at the origin, then

$$Z_q(K) \supseteq c_2 Z_\infty(K) \tag{9.2}$$

for every $q \geqslant n$, where $c_2 > 0$ is an absolute constant. This is a consequence of the inequality

$$\int_K |\langle x, \theta\rangle|^q dx \geqslant \frac{\Gamma(q+1)\Gamma(n)}{2e\Gamma(q+n+1)} \max\left\{h_K^q(\theta), h_K^q(-\theta)\right\},$$

which holds true for all $\theta \in S^{n-1}$ and $q \geqslant 1$. Then, if $q \geqslant n$ we see that

$$\|\langle \cdot, \theta\rangle\|_q \simeq \max\{h_K(\theta), h_K(-\theta)\},$$

and hence $Z_q(K) \supseteq c Z_\infty(K)$. In particular,

$$c \leqslant \mathrm{Vol}_n(Z_n(K))^{1/n} \leqslant \mathrm{Vol}_n(K-K)^{1/n} \leqslant 4 \tag{9.3}$$

for some absolute constant $c > 0$.

We have similar results in the context of log-concave measures. If μ is a log-concave probability measure in $\mathbb{R}^n$ with density f then for every $1 \leqslant p < q$ we have

$$Z_p(f) \subseteq Z_q(f) \subseteq \frac{cq}{p} Z_p(f), \tag{9.4}$$

where $c > 0$ is an absolute constant.

A first basic observation of Paouris is the next asymptotic formula.

Theorem 9.1 (Paouris). *Let f be a centered log-concave density on $\mathbb{R}^n$. Then,*

$$\frac{c_1}{f(0)^{1/n}} \leqslant \mathrm{Vol}_n(Z_n(f))^{1/n} \leqslant \frac{c_2}{f(0)^{1/n}}, \tag{9.5}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Using Proposition 8.3 (vi) we check that, for every $q \geqslant 1$,

$$\mathrm{Vol}_n(K_{n+q}(f))^{1+\frac{q}{n}} \int_{\overline{K_{n+q}(f)}} |\langle x, \theta\rangle|^q \, dx = \int_{K_{n+q}(f)} |\langle x, \theta\rangle|^q \, dx = \frac{1}{f(0)} \int_{\mathbb{R}^n} |\langle x, \theta\rangle|^q f(x) \, dx$$

for all $\theta \in S^{n-1}$, and hence

$$Z_q(\overline{K_{n+q}}(f))\mathrm{Vol}_n(K_{n+q}(f))^{\frac{1}{q}+\frac{1}{n}} f(0)^{1/q} = Z_q(f). \tag{9.6}$$

Now, let $1 \leqslant q \leqslant n$. Using also Corollary 8.7 we see that

$$\frac{1}{e} Z_q(\overline{K_{n+q}}(f)) \subseteq f(0)^{1/n} Z_q(f) \subseteq e\frac{n+q}{n} Z_q(\overline{K_{n+q}}(f)) \subseteq 2e Z_q(\overline{K_{n+q}}(f)). \tag{9.7}$$

On the other hand, using the inclusions in Proposition 8.6, we write

$$\begin{aligned}
h_{Z_q(\overline{K_{n+q}}(f))}(\theta) &= \frac{1}{\mathrm{Vol}_n(K_{n+q}(f))^{\frac{1}{q}+\frac{1}{n}}} \left(\int_{K_{n+q}(f)} |\langle x, \theta\rangle|^q dx \right)^{1/q} \\
&\leqslant \frac{1}{\mathrm{Vol}_n(K_{n+q}(f))^{\frac{1}{q}+\frac{1}{n}}} \left(\int_{\frac{\Gamma(n+q+1)^{1/n+q}}{\Gamma(n+2)^{1/n+1}} K_{n+1}(f)} |\langle x, \theta\rangle|^q dx \right)^{1/q} \\
&= \left(\frac{\mathrm{Vol}_n(K_{n+1}(f))}{\mathrm{Vol}_n(K_{n+q}(f))} \right)^{\frac{1}{q}+\frac{1}{n}} \left(\frac{\Gamma(n+q+1)^{\frac{1}{n+q}}}{\Gamma(n+2)^{\frac{1}{n+1}}} \right)^{1+\frac{n}{q}} h_{Z_q(\overline{K_{n+1}}(f))}(\theta) \\
&\leqslant \left(e^{\frac{n^2}{n+1} - \frac{n^2}{n+q}} \right)^{\frac{1}{q}+\frac{1}{n}} \left(\frac{\Gamma(n+q+1)^{\frac{1}{n+q}}}{\Gamma(n+2)^{\frac{1}{n+1}}} \right)^{1+\frac{n}{q}} h_{Z_q(\overline{K_{n+1}}(f))}(\theta)
\end{aligned}$$

for every $\theta \in S^{n-1}$. After estimating the constant we get that $Z_q(\overline{K_{n+q}}(f)) \subseteq c_1 Z_q(\overline{K_{n+1}}(f))$, and in the same way we establish an analogous inverse inclusion. Therefore, for all $1 \leqslant q \leqslant n$ we get

$$c_1 f(0)^{1/n} Z_q(f) \subseteq Z_q(\overline{K_{n+1}}(f)) \subseteq c_2 f(0)^{1/n} Z_q(f) \tag{9.8}$$

where $c_1, c_2 > 0$ are absolute constants.

Now recall that, since f is centered, the body $K_{n+1}(f)$ is also centered. Applying (9.3) for the body $\overline{K_{n+1}}(f)$ we see that

$$\mathrm{Vol}_n(Z_n(\overline{K_{n+1}}(f)))^{1/n} \simeq 1$$

and hence, by (9.8),

$$f(0)^{1/n}\mathrm{Vol}_n(Z_n(f))^{1/n} \simeq \mathrm{Vol}_n(Z_n(\overline{K_{n+1}}(f)))^{1/n} \simeq 1.$$

This completes the proof. $\square$

Marginals and projections

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function. Let $1 \leqslant k < n$ and $F \in G_{n,k}$. The marginal $\pi_F(f) : F \to [0, \infty)$ of f with respect to F is defined by

$$\pi_F(f)(x) := \int_{x+F^\perp} f(y)dy. \tag{9.9}$$

More generally, for every $\mu \in \mathcal{P}_n$ we define the marginal of μ with respect to a k-dimensional subspace F setting

$$\pi_F(\mu)(A) := \mu(P_F^{-1}(A))$$

for every Borel subset A of F. If μ has a log-concave density f_μ then the two definitions agree. We can check that

$$f_{\pi_F(\mu)} = \pi_F(f_\mu)$$

almost everywhere. Indeed, for every Borel subset A of F we have

$$\pi_F(\mu)(A) = \mu(P_F^{-1}(A)) = \int f_\mu(x)\mathbf{1}_A(P_F x)\,dx = \int_F \int_{F^\perp} f_\mu(x+y)\mathbf{1}_A(x)\,dy\,dx, \tag{9.10}$$

from Fubini's theorem. A change of variables shows that

$$\pi_F(\mu)(A) = \int_A \left(\int_{x+F^\perp} f_\mu(y)\,dy \right) dx = \int_A \pi_F(f_\mu)(x)\,dx.$$

In the next proposition we collect some basic properties of marginals.

Proposition 9.2. *Let $f : \mathbb{R}^n \to [0,\infty)$ be an integrable function and let $F \in G_{n,k}$.*

1. *If f is even then $\pi_F(f)$ is also even.*
2. *We have*
$$\int_F \pi_F(f)(x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx.$$
3. *For every measurable function $g : F \to \mathbb{R}$ we have*
$$\int_{\mathbb{R}^n} g(P_F x) f(x)\,dx = \int_F g(x)\pi_F(f)(x)\,dx.$$
4. *For every $\theta \in S_F$,*
$$\int_F \langle x,\theta\rangle \pi_F(f)(x)dx = \int_{\mathbb{R}^n} \langle x,\theta\rangle f(x)dx. \tag{9.11}$$
In particular, if f is centered then, for every $F \in G_{n,k}$ we have that $\pi_F(f)$ is also centered.
5. *For every $p > 0$ and any $\theta \in S_F$,*
$$\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^p f(x)dx = \int_F |\langle x,\theta\rangle|^p \pi_F(f)(x)dx.$$
In particular, if f is isotropic then $\pi_F(f)$ is also isotropic.
6. *If f is log-concave then $\pi_F(f)$ is also log-concave.*

Similar results are valid for any measure $\mu \in \mathcal{P}_n$.

A second basic observation of Paouris is that any projection of the L_q-centroid body of a density f coincides with the L_q-centroid body of the corresponding marginal of f. The proof is a direct application of Fubini's theorem.

Theorem 9.3 (Paouris)**.** *Let $f : \mathbb{R}^n \to [0,\infty)$ be a density in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n$ and any $F \in G_{n,k}$ and $q \geqslant 1$, we have*

$$P_F(Z_q(f)) = Z_q(\pi_F(f)). \tag{9.12}$$

Proof. Given $q \geqslant 1$ and $\theta \in S_F$, we write

$$\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^q f(x)dx = \int_F |\langle x,\theta\rangle|^q \pi_F(f)(x)dx,$$

because $\langle x,\theta\rangle = \langle P_F(x),\theta\rangle$ for every $x \in \mathbb{R}^n$. Equivalently,

$$h_{Z_q(f)}(\theta) = h_{Z_q(\pi_F f)}(\theta),$$

$\theta \in S_F$, and the result follows from the observation that $h_{P_F(Z_q(f))}(\theta) = h_{Z_q(f)}(\theta) \quad \theta \in S_F$. $\square$

Let f be a centered log-concave density in $\mathbb{R}^n$. Then, for every $F \in G_{n,k}$, the function $\pi_F(f)$ is a centered log-concave density on F. Therefore, we may apply Theorem 9.1 for $\pi_F(f)$ to get

$$\frac{c_1}{\pi_F(f)(0)^{1/k}} \leqslant \mathrm{Vol}_k(Z_k(\pi_F(f)))^{1/k} \leqslant \frac{c_2}{\pi_F(f)(0)^{1/k}}.$$

Combining this inequality with (9.12) we have the following.

Theorem 9.4. *Let f be a log-concave density with* $\mathrm{bar}(f) = 0$ *in $\mathbb{R}^n$. Then, for every $1 \leqslant k < n$ and any $F \in G_{n,k}$ we have*

$$c_1 \leqslant [\pi_F(f)(0)]^{\frac{1}{k}} \mathrm{Vol}_k(P_F(Z_k(f)))^{1/k} \leqslant c_2, \tag{9.13}$$

where $c_1, c_2 > 0$ are absolute constants.

Marginals and sections

The next proposition gives some very useful expressions for the volume of central sections of an isotropic convex body.

Proposition 9.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. We denote by μ_K the isotropic log-concave measure with density $L_K^n \mathbf{1}_{\frac{K}{L_K}}$. Then, for every $1 \leqslant k < n$ and $F \in G_{n,k}$, the body $\overline{K_{k+1}}(\pi_F(\mu_K))$ is almost isotropic and*

$$\mathrm{Vol}_{n-k}(K \cap F^{\perp})^{1/k} \simeq \frac{L_{\overline{K_{k+1}}(\pi_F(\mu_K))}}{L_K}, \tag{9.14}$$

Also, for all $1 \leqslant q \leqslant k$,

$$Z_q\big(\overline{K_{k+1}}(\pi_F(\mu_K))\big) \simeq \mathrm{Vol}_{n-k}(K \cap F^{\perp})^{1/k} P_F(Z_q(K)). \tag{9.15}$$

Proof. Fix $1 \leqslant k < n$ and $F \in G_{n,k}$. Let f_K be the density of μ_K. Since f_K is isotropic, Proposition 9.2 shows that $\pi_F(f_K)$ is isotropic. Hence, by Proposition 8.10 we get that $\overline{K_{k+1}}(\pi_F(f_K))$ is almost isotropic with some absolute constant $C > 0$. Using (9.8) (with $q = 2$) we get:

$$\begin{aligned} L_{\overline{K_{k+1}}(\pi_F(f_K))} &= \left(\frac{\mathrm{Vol}_k(Z_2(\overline{K_{k+1}}(\pi_F(f_K))))}{\mathrm{Vol}_k(B_F)}\right)^{1/k} \simeq \pi_F(f_K)(0)^{1/k} \left(\frac{\mathrm{Vol}_k(Z_2(\pi_F(f_K)))}{\mathrm{Vol}_k(B_F)}\right)^{1/k} \\ &= \pi_F(f_K)(0)^{1/k} \left(\frac{\mathrm{Vol}_k(P_F(Z_2(f_K)))}{\mathrm{Vol}_k(B_F)}\right)^{1/k}, \end{aligned}$$

where we have used the fact that $Z_2(\pi_F(f)) = P_F(Z_2(f))$ for any log-concave function f. Note that, since K is isotropic, we get

$$Z_2(f_K) = L_K^{-1} Z_2(K) = B_2^n \ \text{ and hence } \ P_F(Z_2(f_K)) = B_F.$$

Moreover, we have

$$\pi_F(f_K)(0) = \int_{F^{\perp}} f_K(y)\, dy = L_K^n \mathrm{Vol}_{n-k}\left(\tfrac{1}{L_K} K \cap F^{\perp}\right) = L_K^k \mathrm{Vol}_{n-k}(K \cap F^{\perp}).$$

Combining the above we conclude that

$$L_{\overline{K_{k+1}}(\pi_F(f_K))} \simeq L_K \mathrm{Vol}_{n-k}(K \cap F^{\perp})^{1/k}.$$

The second assertion follows immediately from (9.8) and the equalities $\pi_F(\mu_K)(0)^{1/k} = L_K |K \cap F^{\perp}|^{1/k}$ and $Z_q(\pi_F(\mu_K)) = L_K^{-1} P_F(Z_q(K))$. $\square$

Volume of the centroid bodies

A lower bound for the volume of L_q-centroid bodies follows from the L_q-affine isoperimetric inequality of Lutwak, Yang and Zhang. Using our normalization we can write it in the following form.

Proposition 9.6 (Lutwak-Yang-Zhang). *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\mathrm{Vol}_n(Z_q(K))^{1/n} \geqslant \mathrm{Vol}_n(Z_q(\overline{B_2^n}))^{1/n} \geqslant c\sqrt{q/n}$$

for every $1 \leqslant q \leqslant n$, where $c > 0$ is an absolute constant.

We will see that a reverse inequality holds true (up to the isotropic constant).

Theorem 9.7 (Paouris). *If μ is an isotropic log-concave measure on $\mathbb{R}^n$, then for every $2 \leqslant q \leqslant n$ we have that*

$$\mathrm{Vol}_n(Z_q(\mu))^{1/n} \leqslant c\sqrt{q/n}. \tag{9.16}$$

Moreover, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then for every $2 \leqslant q \leqslant n$ we have that

$$\mathrm{Vol}_n(Z_q(K))^{1/n} \leqslant c\sqrt{q/n}\, L_K, \tag{9.17}$$

where $c > 0$ is an absolute constant.

For the proof we will use Steiner's formula: recall that for every convex body C in $\mathbb{R}^n$ we have

$$\mathrm{Vol}_n(C + tB_2^n) = \sum_{k=0}^{n} \binom{n}{k} W_k(C) t^k$$

for all $t > 0$, where $W_k(C) = V_{n-k}(C) = V(C; n-k, B_2^n; k)$ is the k-th quermassintegral of C. Also, the Alexandrov-Fenchel inequality implies the log-concavity of the sequence $(W_0(C), \ldots, W_n(C))$, and in particular we have that

$$\left(\frac{W_{n-i}(C)}{\omega_n}\right)^{1/i} \geqslant \left(\frac{W_{n-j}(C)}{\omega_n}\right)^{1/j}, \tag{9.18}$$

for all $1 \leqslant i < j \leqslant n$. We will also use Kubota's integral formula:

$$W_{n-m}(C) = \frac{\omega_n}{\omega_m} \int_{G_{n,m}} \mathrm{Vol}_m(P_F(C))\, d\nu_{n,m}(F), \quad (1 \leqslant m \leqslant n). \tag{9.19}$$

Proof of Theorem 9.7. It is enough to prove (9.16) for integer values of $1 \leqslant q \leqslant n-1$. Observe that for any $F \in G_{n,q}$ we have

$$\mathrm{Vol}_q(P_F(Z_q(\mu)))^{1/q} = \mathrm{Vol}_q(Z_q(\pi_F(\mu)))^{1/q} \leqslant \frac{c_1}{[f_{\pi_F(\mu)}(0)]^{1/q}} \leqslant c_2,$$

where we have used Theorem 9.3, Theorem 9.1 and (7.2) respectively, for the isotropic function $f_{\pi_F(\mu)} = \pi_F(f_\mu)$. Applying (9.19) we get

$$W_{n-q}(Z_q(\mu)) \leqslant \frac{\omega_n}{\omega_q} c_2^q.$$

Now, we apply (9.18) for $C = Z_q(\mu)$ with $j = n$ and $i = q$; this gives

$$W_{n-q}^{1/q}(Z_q(\mu)) \geqslant \mathrm{Vol}_n(Z_q(\mu))^{1/n} \omega_n^{1/q-1/n}.$$

Combining the above, we get

$$\mathrm{Vol}_n(Z_q(\mu))^{1/n} \leqslant \frac{\omega_n^{1/n}}{\omega_q^{1/q}} c_2.$$

Since $\omega_k^{1/k} \simeq 1/\sqrt{k}$, we get (9.16). For the second assertion of the theorem we may assume that K is isotropic (because the volume of $Z_q(T(K))$ is the same for all $T \in SL(n)$). Consider the measure μ with density $f_\mu = L_K^n \mathbf{1}_{\frac{K}{L_K}}$. Then, μ is isotropic and $Z_q(\mu) = L_K^{-1} Z_q(K)$. Thus, the result follows immediately from (9.16). $\square$

10 Paouris' inequality

We are now ready to prove a very useful inequality of Paouris.

Theorem 10.1 (Paouris). *Let μ be an isotropic log-concave probability measure in $\mathbb{R}^n$. Then,*

$$\mu(\{x \in \mathbb{R}^n : |x| \geqslant ct\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right) \tag{10.1}$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant.

The proof of Theorem 10.1 is reduced to the behavior of the moments of the function $x \mapsto |x|$. For every $q \geqslant 1$ we define

$$I_q(\mu) = \left(\int_{\mathbb{R}^n} |x|^q d\mu(x)\right)^{1/q}.$$

Theorem 4.8 shows that for all $y \in \mathbb{R}^n$ and $p, q \geqslant 1$ we have

$$\|\langle \cdot, y\rangle\|_{pq} \leqslant c_1 q \|\langle \cdot, y\rangle\|_p,$$

where $c_1 > 0$ is an absolute constant. Moreover, since $|x|$ is a norm, for every $p, q \geqslant 1$ we have

$$I_{pq}(K) \leqslant c_1 q I_p(K).$$

In particular, we have

$$I_q(\mu) \leqslant c_1 q I_2(\mu) \tag{10.2}$$

for all $q \geqslant 2$. Paouris proved the following.

Theorem 10.2 (Paouris). *There exist absolute constants $c_3, c_4 > 0$ such that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ then*

$$I_q(\mu) \leqslant c_4 I_2(\mu) \tag{10.3}$$

for all $q \leqslant c_3\sqrt{n}$.

Assuming that we have proved Theorem 10.2, we obtain Theorem 10.1 as follows: we consider an isotropic log-concave probability measure μ in $\mathbb{R}^n$. From Markov's inequality, for every $q \geqslant 2$ we have

$$\mu(\{|x| \geqslant e^3 I_q(\mu)\}) \leqslant e^{-3q}.$$

Then, Borell's lemma gives

$$\begin{aligned}\mu(\{|x| \geqslant e^3 I_q(\mu) s\}) &\leqslant (1 - e^{-3q}) \left(\frac{e^{-3q}}{1 - e^{-3q}}\right)^{(s+1)/2} \\ &\leqslant e^{-qs}\end{aligned}$$

for every $s \geqslant 1$. Choosing $q = c_3\sqrt{n}$, and using (10.3), we see that

$$\mu(\{|x| \geqslant c_4 e^3 I_2(\mu) s\}) \leqslant \exp(-c_3\sqrt{n}s)$$

for all $s \geqslant 1$. Since μ is isotropic, we have $I_2(\mu) = \sqrt{n}$. This proves the theorem.

We pass to the proof of Theorem 10.2. We will actually prove a stronger statement.

Theorem 10.3. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For every $q \geqslant 1$,*

$$I_q(\mu) \leqslant C\left(I_2(\mu) + R(Z_q(\mu))\right). \tag{10.4}$$

Note that if μ is isotropic then $R(Z_q(\mu)) \leqslant cq$, and hence the right hand side of (10.4) is bounded by $C_1 \max\{I_2(\mu), q\}$. Since $I_2(\mu) = \sqrt{n}$, for all $q \leqslant \sqrt{n}$ we get

$$I_q(\mu) \leqslant C_1 \max\{I_2(\mu), q\} = C_1 I_2(\mu),$$

which is exactly the statement of Theorem 10.2.

We start with the next lemma, which relates the q-moment of the Euclidean norm with respect to μ with the parameters w_q and the L_q-centroid bodies of μ through the next lemma.

Lemma 10.4. *Let μ be a log-concave probability measure in $\mathbb{R}^n$. For every $q \geqslant 1$ we have*

$$w_q(Z_q(\mu)) = a_{n,q}\sqrt{\frac{q}{q+n}}I_q(\mu)$$

where $a_{n,q} \simeq 1$.

Proof. This is in fact a different way to express Alsker's computation: for every $x \in \mathbb{R}^n$ we have

$$\left(\int_{S^{n-1}} |\langle x, \theta\rangle|^q d\sigma(\theta)\right)^{1/q} = a_{n,q}\frac{\sqrt{q}}{\sqrt{q+n}}|x|,$$

where $a_{n,q} \simeq 1$. Since

$$w_q(Z_q(\mu)) = \left(\int_{S^{n-1}}\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^q d\mu(x)\sigma(d\theta)\right)^{1/q},$$

the lemma follows. □

Proof of Theorem 10.3. We start with the formula

$$I_q(\mu) = c_{n,q} w_q(Z_q(\mu)), \tag{10.5}$$

where $c_{n,q} \simeq \max\{1, \sqrt{n/q}\}$. Therefore, we need to show that

$$w_q(Z_q(\mu)) \leqslant C \min\{1, \sqrt{q/n}\} \left(I_2(\mu) + R(Z_q(\mu))\right).$$

Since $w_q(Z_q(\mu)) \leqslant R(Z_q(\mu))$, we clearly have the result when $q \geqslant n$, and hence in the sequel we may assume that q is an integer and $1 \leqslant q \leqslant n$.

Recall the result of Litvak, Milman and Schechtman (Theorem 2.6): we have

$$w_q(Z_q(\mu)) \leqslant c_1 \max\{w(Z_q(\mu)), \sqrt{q/n}R(Z_q(\mu))\}. \tag{10.6}$$

Therefore, the theorem will follow if we show that, for all $1 \leqslant q \leqslant n$,

$$w(Z_q(\mu)) \leqslant C\sqrt{q/n}(I_2(\mu) + R(Z_q(\mu))). \tag{10.7}$$

If $q \geqslant k_*(Z_q(\mu))$ then we have

$$w(Z_q(\mu)) \leqslant c_2\sqrt{q/n}R(Z_q(\mu)) \tag{10.8}$$

by the definition of $k_*(Z_q(\mu))$. If $q \leqslant k_*(Z_q(\mu))$ then Theorem 2.6 (Dvoretzky theorem for $Z_q(\mu)$) shows that a random $F \in G_{n,q}$ satisfies

$$\int |P_F(x)|^2 \, d\mu(x) \leqslant c_3(q/n)I_2^2(\mu)$$

(this is justified by averaging over all $F \in G_{n,k}$ and then applying Markov's inequality) and

$$w(Z_q(\mu))B_F \subseteq c_4 P_F(Z_q(\mu)). \tag{10.9}$$

Since $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$ (by Theorem 9.3) and $\pi_F(\mu)$ is a q-dimensional centered log-concave probability measure, from Theorem 9.4 we get

$$\mathrm{vrad}(Z_q(\pi_F(\mu))) \simeq \frac{\sqrt{q}}{\|\pi_F(\mu)\|_\infty^{1/q}} = \sqrt{q}\frac{|\det \mathrm{Cov}(\pi_F(\mu))|^{\frac{1}{2q}}}{L_{\pi_F(\mu)}}. \tag{10.10}$$

Using the the fact that $L_{\pi_F(\mu)} \geqslant c > 0$, we see that

$$\mathrm{vrad}(Z_q(\pi_F(\mu))) \leqslant c_5 \frac{\left(\int |x|^2 d\pi_F\mu(x)\right)^{1/2}}{L_{\pi_F(\mu)}} \leqslant c_6 \left(\int |P_F(x)|^2 \, d\mu(x)\right)^{1/2} \leqslant c_7\sqrt{q/n}I_2(\mu). \tag{10.11}$$

Combining (10.9), (10.10) and (10.11) we have

$$w(Z_q(\mu)) \leqslant c_8\sqrt{q/n}I_2(\mu). \tag{10.12}$$

This completes the proof. $\square$

We end this section with a basic application of the previous results.

Theorem 10.5. *Let μ be an isotropic log-concave probability measure in $\mathbb{R}^n$. If $1 \leqslant q \leqslant \sqrt{n}$, then*

$$w(Z_q(\mu)) \simeq \sqrt{q}. \tag{10.13}$$

For Theorem 10.5 we write $w(Z_q(\mu)) \simeq w_q(Z_q(\mu)) \simeq \sqrt{q/n}I_q(\mu) \simeq \sqrt{q}$, where the first equality holds because $\sqrt{n} \leqslant q_*(\mu)$, the second comes from Lemma 10.4 and the third follows from Theorem 10.2.

11 The isomorphic slicing problem

In this section we describe Klartag's affirmative answer to the isomorphic slicing problem

Theorem 11.1 (Klartag)**.** *Let K be a convex body in $\mathbb{R}^n$. For every $\varepsilon \in (0,1)$ we can find a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that*

$$\frac{1}{1+\varepsilon}T \subseteq K + x \subseteq (1+\varepsilon)T \tag{11.1}$$

and

$$L_T \leqslant \frac{C}{\sqrt{\varepsilon}}$$

for some absolute constant $C > 0$.

The body T will be of the form $K_{n+1}(g)$ for some function g on the centered translate of K, which is not much different from $\mathbf{1}_K$. More precisely, g will be chosen from a family of functions proportional to $e^{\langle x,\xi\rangle}\mathbf{1}_K$. We start with the next lemma which will allow us to compare the isotropic constant of a function f and of $K_{n+1}(g)$ when g is the centered translate of f.

Lemma 11.2. *Let K be a convex body in $\mathbb{R}^n$ and let $f : K \to (0,\infty)$ be a log-concave function such that*

$$\sup_{x\in K} f(x) \leqslant m^{n+1} \inf_{x\in K} f(x)$$

for some $m > 1$. We write $x_0 = \mathrm{bar}(f)$ and denote the centered translate of f by g, namely $g(x) := f(x+x_0)$ for all $x \in K - x_0$. Then the body $T := K_{n+1}(g)$ is centered,

$$L_f = L_g \simeq L_T$$

and

$$\frac{1}{m}T \subseteq K - x_0 \subseteq mT. \tag{11.2}$$

Proof. It is easy to check that f seen as a function on all of $\mathbb{R}^n$ (where we set $f(x) = 0$ for $x \notin K$) is log-concave as well, and that $x_0 = \mathrm{bar}(f) \in K$. The facts that T is centered and that $L_T \simeq L_g = L_f$ follow easily from Proposition 8.10, so it remains to prove (11.2). Since $K_{n+1}(\lambda g) = K_{n+1}(g)$ for every $\lambda > 0$, we may assume without loss of generality that $g(0) = \mathbf{1}_{K-x_0}(0) = 1$. Then,

$$\inf\left\{\frac{g(x)}{\mathbf{1}_{K-x_0}(x)} : \mathbf{1}_{K-x_0}(x) > 0\right\} = \inf_{x\in K-x_0} g(x) = \inf_{y\in K} f(y) \geqslant m^{-(n+1)}$$

and

$$\inf\left\{\frac{\mathbf{1}_{K-x_0}(x)}{g(x)} : g(x) > 0\right\} = \left(\sup_{y\in K} f(y)\right)^{-1} \geqslant m^{-(n+1)}.$$

From Lemma 8.2 and Proposition 8.3 (iv) we obtain

$$\frac{1}{m}K_{n+1}(g) \subseteq K_{n+1}\big(\mathbf{1}_{K-x_0}\big) = K - x_0 \subseteq mK_{n+1}(g),$$

and this completes the proof. $\square$

We consider the uniform measure on K, which we denote by $\mu = \mathbf{1}_K dx$, and a family of measures $\{\mu_\xi\}_{\xi\in\mathbb{R}^n}$ which will be probability measures with density proportional to $e^{\langle x,\xi\rangle}\mathbf{1}_K(x)$. The properties of these measures are closely related to the logarithmic Laplace transform of the measure μ, which is defined by

$$\Lambda_\mu(\xi) = \log\left(\frac{1}{\mu(\mathbb{R}^n)}\int_{\mathbb{R}^n} e^{\langle \xi,x\rangle} d\mu(x)\right). \tag{11.3}$$

We summarize some of its basic properties in the next proposition.

Proposition 11.3. *If $\mu = \mu_K$ is Lebesgue measure on some convex body K in $\mathbb{R}^n$, then*

$$(\nabla\Lambda_\mu)(\mathbb{R}^n) = \mathrm{int}(K). \tag{11.4}$$

If μ_ξ is the probability measure in $\mathbb{R}^n$ with density proportional to $e^{\langle\xi,x\rangle}\mathbf{1}_K(x)$, then

$$bar(\mu_\xi) = \nabla\Lambda_\mu(\xi) \tag{11.5}$$

and

$$(\mathrm{Hess}\Lambda_\mu)(\xi) = \mathrm{Cov}(\mu_\xi). \tag{11.6}$$

Moreover, the map $\nabla\Lambda_\mu$ transports the measure ν with density $\det(\mathrm{Hess}\Lambda_\mu)(\xi)$ to μ. Equivalently, for every continuous non-negative function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_K \phi(x)\,dx = \int_{\mathbb{R}^n} \phi(\nabla\Lambda_\mu(\xi))\det(\mathrm{Hess}(\Lambda_\mu)(\xi))\,d\xi = \int \phi(\nabla\Lambda_\mu(\xi))d\nu(\xi). \tag{11.7}$$

Proof. Let $F = \Lambda_\mu$, that is

$$F(x) = \log\left(\frac{1}{\mathrm{Vol}_n(K)}\int_K e^{\langle x,y\rangle}\,dy\right).$$

Observe that F is a C^2-smooth, strictly convex function. Smoothness is clear, as we are integrating a smooth function on a compact set. The strict convexity follows from Cauchy-Schwarz inequality. Differentiating under the integral sign we get:

$$\nabla F(\xi) = \frac{\int_K y e^{\langle \xi,y\rangle}\,dy}{\int_K e^{\langle \xi,z\rangle}\,dz} = \int y\,d\mu_\xi(y) = \mathrm{bar}(\mu_\xi). \tag{11.8}$$

Since μ_ξ is supported on the compact, convex set K we obtain that $\nabla F(\xi) = \mathrm{bar}(\mu_\xi) \in K$ for all $\xi \in \mathbb{R}^n$. This shows that $\nabla F(\mathbb{R}^n) \subseteq K$. In fact, this is what we really need, although one can check that $\nabla F(\mathbb{R}^n) = \mathrm{int}(K)$.

To compute the Hessian we differentiate twice to get:

$$\begin{aligned}\frac{\partial^2 F(\xi)}{\partial\xi_j\partial\xi_i} &= \frac{\int_K x_i x_j e^{\langle\xi,x\rangle}\,dx - \int_K x_i e^{\langle\xi,x\rangle}\,dx\int_K x_j e^{\langle\xi,x\rangle}\,dx}{\left(\int_K e^{\langle\xi,x\rangle}\,dx\right)^2} \\ &= \int x_i x_j\,d\mu_\xi(x) - \int x_i\,d\mu_\xi(x)\int x_j\,d\mu_\xi(x) \\ &= \mathrm{Cov}(\mu_\xi)_{ij}\end{aligned} \tag{11.9}$$

For the last assertion note that since F is strictly convex, ∇F is one-to-one. So, for any continuous function $g : \mathbb{R}^n \to \mathbb{R}$, changing variables $y = \nabla F(\xi)$ we get

$$\int_{\nabla F(\mathbb{R}^n)} g(y)\,dy = \int_{\mathbb{R}^n} g(\nabla F(\xi))\det(\mathrm{Hess}F)(\xi)\,d\xi = \int_{\mathbb{R}^n} g(\nabla F(\xi))\,d\nu(\xi). \tag{11.10}$$

This completes the proof of the proposition. $\square$

Proof of Theorem 11.1. Without loss of generality we may assume that K is centered and that $\mathrm{Vol}_n(K) = 1$. We denote again by $\mu = \mu_K$ the Lebesgue measure restricted on K, and

$$d\nu(\xi) = \det(\mathrm{Hess}\,\Lambda_\mu)(\xi) \equiv \det\mathrm{Cov}(\mu_\xi)d\xi$$

as in Proposition 11.3. Using (11.7) with $\phi = 1$ we get that

$$\nu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \mathbf{1}\det(\mathrm{Hess}\,\Lambda_\mu)(\xi)\,d\xi = \int_K \mathbf{1}\,dx = \mathrm{Vol}_n(K) = 1.$$

Thus, for every $\varepsilon > 0$ we may write

$$\mathrm{Vol}_n(\varepsilon n(K-K)^\circ)\min_{\xi\in\varepsilon n(K-K)^\circ}\det\mathrm{Cov}(\mu_\xi) \leqslant\leqslant \int_{\varepsilon n(K-K)^\circ}\det\mathrm{Cov}(\mu_\xi)\,d\xi = \nu(\varepsilon n(K-K)^\circ) \leqslant 1.$$

By the Bourgain-Milman inequality we have $\mathrm{Vol}_n(\varepsilon n(K-K)^\circ)^{1/n} \simeq \varepsilon$. Therefore, there exists $\xi_0 \in \varepsilon n(K-K)^\circ$ such that

$$\det\mathrm{Cov}(\mu_{\xi_0}) = \min_{\xi\in\varepsilon n(K-K)^\circ}\det\mathrm{Cov}(\mu_\xi) \leqslant \mathrm{Vol}_n(\varepsilon n(K-K)^\circ)^{-1}\left(\frac{c_1}{\varepsilon}\right)^n.$$

From the definition of μ_{ξ_0} and of the isotropic constant we have that

$$L_{\mu_{\xi_0}} = \left(\frac{\sup_{x\in K} e^{\langle\xi_0,x\rangle}}{\int_K e^{\langle\xi_0,x\rangle}dx}\right)^{\frac{1}{n}}[\det\mathrm{Cov}(\mu_{\xi_0})]^{\frac{1}{2n}}.$$

Since $\xi_0 \in \varepsilon n(K-K)^\circ$ and $K \cup (-K) \subset K-K$, we know that $|\langle \xi_0, x\rangle| \leqslant \varepsilon n$ for all $x \in K$, therefore

$$\sup_{x\in K} e^{\langle \xi_0, x\rangle} \leqslant e^{\varepsilon n} \qquad \text{and} \qquad \sup_{x\in K} e^{\langle \xi_0, x\rangle} \geqslant e^{-\varepsilon n}.$$

On the other hand, since K is centered, from Jensen's inequality we have that

$$\int_K e^{\langle \xi_0, x\rangle} dx \geqslant e^{\left(\int_K \langle \xi_0, x\rangle\, dx\right)} = 1.$$

Combining the above we get

$$L_{\mu_{\xi_0}} \leqslant \frac{c_2}{\sqrt{\varepsilon}}. \tag{11.11}$$

Finally, we note that the function $f_{\xi_0}(x) = e^{\langle \xi_0, x\rangle}\mathbf{1}_K(x)$ (which is proportional to the density of μ_{ξ_0}) is obviously log-concave and satisfies

$$\sup_{x\in \mathrm{supp}(f_{\xi_0})} f_{\xi_0}(x) \leqslant e^{2\varepsilon n} \inf_{x\in \mathrm{supp}(f_{\xi_0})} f_{\xi_0}(x).$$

Therefore, applying Lemma 11.2, we can find a centered convex body T_{ξ_0} in $\mathbb{R}^n$ such that

$$L_{T_{\xi_0}} \simeq L_{f_{\xi_0}} = L_{\mu_{\xi_0}} \leqslant \frac{c_2}{\sqrt{\varepsilon}}$$

and

$$\frac{1}{e^{2\varepsilon}} T_{\xi_0} \subseteq K - b_{\xi_0} \subseteq e^{2\varepsilon} T_{\xi_0}$$

where b_{ξ_0} is the barycenter of f_{ξ_0}. Since $e^{2\varepsilon} \leqslant 1 + c\varepsilon$ when $\varepsilon \in (0,1)$, the result follows. $\square$

12 Klartag's upper bound for the isotropic constant

Using Theorem 11.1 and Paouris' distributional inequality, Klartag was also able to slightly improve Bourgain's upper bound for the isotropic constant.

Theorem 12.1 (Klartag)**.** *Let K be a convex body in $\mathbb{R}^n$. Then*

$$L_K \leqslant C\sqrt[4]{n}, \tag{12.1}$$

where $C > 0$ is an absolute constant.

Theorem 12.1 will follow from Theorem 11.1 and the next lemma.

Lemma 12.2. *Let K, T be two convex bodies in $\mathbb{R}^n$ and $t \geqslant 1$. Suppose that*

$$\frac{1}{1 + \frac{t}{\sqrt{n}}}(T + y) \subseteq K + x \subseteq \left(1 + \frac{t}{\sqrt{n}}\right)(T + y) \tag{12.2}$$

for some $x, y \in \mathbb{R}^n$. Then

$$L_K \leqslant ctL_T,$$

where $c > 0$ is an absolute constant.

Proof. We may assume that $t < \sqrt{n}$, otherwise the conclusion of the lemma is trivial since by Proposition 3.11 and Proposition 3.9 we have $L_K < c\sqrt{n} \leqslant c'\sqrt{n}L_T$. Note that (12.2) continues to hold (with possibly different $x, y \in \mathbb{R}^n$) if we translate either K or T or if we apply an invertible linear transformation to both of them, thus we may assume that T is in isotropic position. Then by Paouris's inequality we have that

$$\mathrm{Vol}_n(T \setminus Ct\sqrt{n}L_T B_2^n) \leqslant \exp(-4t\sqrt{n})$$

for some absolute constant $C \geqslant 1$. We set

$$K_1 = \Big(1 + \frac{t}{\sqrt{n}}\Big)^{-1}(K + x) - y,$$

and by (12.2) we have $K_1 \subseteq T$, and hence

$$\mathrm{Vol}_n(K_1 \setminus Ct\sqrt{n}L_T B_2^n) \leqslant \exp(-4t\sqrt{n}). \tag{12.3}$$

By (12.2) we also see that

$$\mathrm{Vol}_n(K_1) = \Big(1 + \frac{t}{\sqrt{n}}\Big)^{-n}\mathrm{Vol}_n(K) \geqslant \Big(1 + \frac{t}{\sqrt{n}}\Big)^{-2n}\mathrm{Vol}_n(T) > e^{-2t\sqrt{n}}, \tag{12.4}$$

which combined with (12.3) gives

$$\mathrm{Vol}_n(K_1 \cap (Ct\sqrt{n}L_T B_2^n)) > \frac{\mathrm{Vol}_n(K)}{2}.$$

Therefore the median of the Euclidean norm on K_1, with respect to the uniform measure on K_1, is not larger than $Ct\sqrt{n}L_T$. Since K_1 is convex, and hence the uniform measure on K_1 is a log-concave probability measure, using Theorem 4.11 we obtain

$$\left(\frac{1}{\mathrm{Vol}_n(K_1)}\int_{K_1}|x|^2\,dx\right)^{1/2} \leqslant C't\sqrt{n}L_T, \tag{12.5}$$

for some absolute constant $C' > 0$. Recall that by Theorem 3.2

$$\sqrt{n}L_K = \sqrt{n}L_{K_1} = \min\left\{\left(\frac{1}{\mathrm{Vol}_n(S(K_0))^{1+\frac{2}{n}}}\int_{S(K_0)}|x|^2\,dx\right)^{1/2} \,\Big|\, S \in GL_n\right\},$$

where K_0 is the centered translate of K_1, that is,

$$K_0 = K_1 - \mathrm{bar}(K_1) = K_1 - \frac{1}{\mathrm{Vol}_n(K_1)}\int_{K_1} x dx.$$

It is also not hard to check that

$$\begin{aligned}\frac{1}{\mathrm{Vol}_n(K_0)^{1+\frac{2}{n}}}\int_{K_0}|x|^2\,dx &= \frac{1}{\mathrm{Vol}_n(K_1)^{1+\frac{2}{n}}}\int_{K_1}|x|^2\,dx - \frac{1}{\mathrm{Vol}_n(K_1)^{\frac{2}{n}}}|\mathrm{bar}(K_1)|^2\\ &\leqslant \frac{1}{\mathrm{Vol}_n(K_1)^{1+\frac{2}{n}}}\int_{K_1}|x|^2\,dx,\end{aligned}$$

and thus

$$\sqrt{n}L_K \leqslant \left(\frac{1}{\mathrm{Vol}_n(K_1)^{1+\frac{2}{n}}}\int_{K_1}|x|^2\,dx\right)^{1/2} \leqslant \frac{C't\sqrt{n}L_T}{\mathrm{Vol}_n(K_1)^{1/n}} \leqslant C''t\sqrt{n}L_T$$

by (12.4) and (12.5). This proves the lemma. $\square$

Proof of Theorem 12.1. Let K be a convex body in $\mathbb{R}^n$. According to Theorem 11.1, given $\varepsilon \in (0,1)$ we can find a centered convex body $T = T_\varepsilon$ such that

$$L_T \leqslant \frac{C}{\sqrt{\varepsilon}}$$

and

$$\frac{1}{(1+\varepsilon)} T \subseteq K + x \subseteq (1+\varepsilon) T$$

for some $x = x_\varepsilon \in \mathbb{R}^n$. If we choose $\varepsilon = \frac{1}{\sqrt{n}}$, Lemma 12.2 shows that

$$L_K \leqslant c L_T \leqslant c \frac{C}{\sqrt{\varepsilon}} = C' \sqrt[4]{n},$$

which was the assertion of the theorem. $\square$

13 Negative moments and small ball probability estimates

A parameter which was originally central in the work of Paouris is $q_*(\mu)$, which is defined for every centered log-concave probability measure μ in $\mathbb{R}^n$, as follows:

$$q_*(\mu) = \max\{q \geqslant 2 : k_*(Z_q(\mu)) \geqslant q\}.$$

We shall need a lower bound for $q_*(\mu)$.

Proposition 13.1. *There exists an absolute constant $c > 0$ with the following property: if μ is a centered log-concave probability measure in $\mathbb{R}^n$ then*

$$q_*(\mu) \geqslant c \sqrt{k_*(Z_2(\mu))}.$$

Proof. We set $q_* := q_*(\mu)$. From Theorem 2.6 (i), Lemma 10.4, Hölder's inequality and the simple observation $I_2(\mu) = w_2(Z_2(\mu))$ we get

$$\begin{aligned} (13.1) \qquad w(Z_{q_*}(\mu)) &\geqslant c_1 w_{q_*}(Z_{q_*}(\mu)) = c_1 a_{n,q_*} \sqrt{\frac{q_*}{n+q_*}} I_{q_*}(\mu) \geqslant c_1 a_{n,q_*} \sqrt{\frac{q_*}{n+q_*}} I_2(\mu) \\ &= c_1 a_{n,q_*} \sqrt{\frac{q_*}{n+q_*}} \sqrt{n} w_2(Z_2(\mu)). \end{aligned}$$

In other words,

$$(13.2) \qquad w(Z_{q_*}(\mu)) \geqslant c_2 \sqrt{q_*} w(Z_2(\mu)).$$

Since $R(Z_{q_*}(\mu)) \leqslant C q_* R(Z_2(\mu))$, using the definition of q_* and Theorem 2.5 we write

$$(13.3) \qquad 2q_* \geqslant k_*(Z_{q_*}(\mu)) \geqslant c_3 n \left(\frac{w(Z_{q_*}(\mu))}{R(Z_{q_*}(\mu))} \right)^2 \geqslant c_3 n \frac{c_2^2 q_*}{C^2 q_*^2} \frac{w^2(Z_2(\mu))}{R^2(Z_2(\mu))} = c_5 \frac{k_*(Z_2(\mu))}{q_*}.$$

This shows that $q_*(\mu) \geqslant c \sqrt{k_*(Z_2(\mu))}$ for some absolute constant $c > 0$. $\square$

Note that if μ is isotropic then $k_*(Z_2(\mu)) = n$. Therefore, in the isotropic case we have:

Corollary 13.2. *There exists an absolute constant $c > 0$ with the following property: for every isotropic log-concave probability measure μ in $\mathbb{R}^n$,*

$$q_*(\mu) \geqslant c\sqrt{n}.$$

If μ is a centered log-concave probability measure in $\mathbb{R}^n$, we extend the definition of $I_q(\mu)$, allowing negative values of q, in the obvious way: for every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} |x|^q d\mu(x) \right)^{1/q}.$$

The main result of this section is the next theorem.

Theorem 13.3 (Paouris)**.** *Let μ be a centered log-concave probability measure in $\mathbb{R}^n$. For every integer $1 \leqslant k \leqslant q_*(\mu)$ we have*

$$I_{-k}(\mu) \simeq I_k(\mu).$$

In particular, this theorem shows that for every $k \leqslant q_*(\mu)$ we have $I_k(\mu) \leqslant C I_2(\mu)$, where $C > 0$ is an absolute constant. This was precisely the assertion of Theorem 10.2.

The proof of Theorem 13.3 is based on two identities:

(i) If f is a centered log-concave density on $\mathbb{R}^n$ and $1 \leqslant k < n$ is a positive integer, then

$$I_{-k}(f) = c_{n,k} \left(\int_{G_{n,k}} \pi_F(f)(0) d\nu_{n,k}(F) \right)^{-1/k}, \tag{13.4}$$

where

$$c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}.$$

Proof. Let $1 \leqslant k < n$. Then, we have

$$\begin{aligned}
\int_{G_{n,k}} \pi_F(f)(0)\, d\nu_{n,k}(F) &= \int_{G_{n,n-k}} \pi_{E^\perp}(f)(0)\, d\nu_{n,n-k}(E) = \int_{G_{n,n-k}} \int_E f(y)\, dy\, d\nu_{n,n-k}(E) \\
&= \int_{G_{n,n-k}} (n-k)\omega_{n-k} \int_{S_E} \int_0^\infty r^{n-k-1} f(r\theta)\, dr\, d\sigma_E(\theta)\, d\nu_{n,n-k}(E) \\
&= \frac{(n-k)\omega_{n-k}}{n\omega_n} n\omega_n \int_{S^{n-1}} \int_0^\infty r^{n-k-1} f(r\theta)\, dr\, d\sigma(\theta) \\
&= \frac{(n-k)\omega_{n-k}}{n\omega_n} \int_{\mathbb{R}^n} |x|^{-k} f(x)\, dx = \frac{(n-k)\omega_{n-k}}{n\omega_n} I_{-k}^{-k}(f).
\end{aligned}$$

It follows that

$$I_{-k}(f) = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \left(\int_{G_{n,k}} \pi_F(f)(0)\, d\nu_{n,k}(F) \right)^{-1/k}.$$

Check that $c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}$. $\square$

(ii) If C is a symmetric convex body in $\mathbb{R}^n$ and $1 \leqslant k < n$ is a positive integer, then

$$w_{-k}(C) \simeq \sqrt{k} \left(\int_{G_{n,k}} \mathrm{Vol}_k(P_F(C))^{-1} d\nu_{n,k}(F) \right)^{-\frac{1}{k}}. \tag{13.5}$$

Proof. Using the Blaschke-Santaló and the Bourgain-Milman inequality, we write

$$\begin{aligned}
w_{-k}^{-1}(C) &= \left(\int_{S^{n-1}} \frac{1}{h_C^k(\theta)} d\sigma(\theta) \right)^{1/k} = \left(\int_{G_{n,k}} \int_{S_F} \frac{1}{\|\theta\|_{(P_F C)^\circ}^k} d\sigma(\theta) d\nu_{n,k}(F) \right)^{1/k} \\
&= \left(\int_{G_{n,k}} \frac{\mathrm{Vol}_k(P_F(C))^\circ}{\mathrm{Vol}_k(B_2^k)} d\nu_{n,k}(F) \right)^{1/k} \simeq \left(\int_{G_{n,k}} \frac{\mathrm{Vol}_k(B_2^k)}{\mathrm{Vol}_k(P_F(C))} d\nu_{n,k}(F) \right)^{1/k},
\end{aligned}$$

and the result follows. $\square$

Now, consider a centered log-concave density f in $\mathbb{R}^n$, an integer $1 \leqslant k < n$ and some $F \in G_{n,k}$. From Theorem 9.4 we have

$$\frac{1}{\mathrm{Vol}_k(P_F(Z_k(f)))^{1/k}} \simeq \pi_F(f)(0)^{1/k}.$$

Combining the two identities we get:

Theorem 13.4. *Let f be a centered log-concave density in $\mathbb{R}^n$. For every integer $1 \leqslant k < n$ we have*

$$w_{-k}(Z_k(f)) \simeq \sqrt{k}\left(\int_{G_{n,k}} \pi_F(f)(0)d\nu_{n,k}(F)\right)^{-\frac{1}{k}}$$

and

$$I_{-k}(f) \simeq \sqrt{\frac{n}{k}}w_{-k}(Z_k(f)). \tag{13.6}$$

Proof of Theorem 13.3. Recall that, for every $1 \leqslant k < n$,

$$w_k(Z_k(\mu)) \simeq \sqrt{k/n}\, I_k(\mu). \tag{13.7}$$

On the other hand, from (13.6) we see that

$$w_{-k}(Z_k(\mu)) \simeq \sqrt{k/n}\, I_{-k}(\mu).$$

We set $k_0 = \lfloor q_* \rfloor,\ \ q_* = q_*(\mu)$. Then,

$$k_*(Z_{k_0}(\mu)) \simeq k_*(Z_{q_*}(\mu)) \geqslant c_1 q_* \geqslant c_1 k_0. \tag{13.8}$$

From Theorem 2.10 we have

$$w_{-k}(Z_{k_0}(\mu)) \simeq w_k(Z_{k_0}(\mu)) \tag{13.9}$$

for every $1 \leqslant k \leqslant c_2 k_*(Z_{k_0}(\mu))$, and (13.8) shows that (13.9) holds for every $k \leqslant c_3 q_*(\mu)$. Setting $k_1 = \lfloor c_3 q_*(\mu) \rfloor \simeq k_0$, and using the fact that $Z_{k_0}(\mu) \simeq Z_{k_1}(\mu)$, we get

$$w_{-k_1}(Z_{k_1}(\mu)) \simeq w_{k_1}(Z_{k_1}(\mu)). \tag{13.10}$$

It is now clear that $I_{-k_1}(\mu) \simeq I_{k_1}(\mu)$ and since $k_1 \simeq q_*(\mu)$ we see that $q \mapsto I_q(\mu)$ is “constant” in the range $1 \leqslant |q| \leqslant c q_*(\mu)$. $\square$

A useful consequence of Theorem 13.3 is the next small ball probability estimate:

Theorem 13.5. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then, for every $0 < \varepsilon < \varepsilon_0$ we have*

$$\mu(\{x \in \mathbb{R}^n : |x| < \varepsilon\sqrt{n}\}) \leqslant \varepsilon^{c\sqrt{n}}, \tag{13.11}$$

where $\varepsilon_0, c > 0$ are absolute constants.

Proof. Let $1 \leqslant k \leqslant q_*(\mu)$. We write

$$\begin{aligned}\mu(\{x \in \mathbb{R}^n : |x| < \varepsilon I_2(\mu)\}) &\leqslant \mu(\{x : |x| < c_1\varepsilon I_{-k}(\mu)\}) \\ &\leqslant (c_1\varepsilon)^k \leqslant \varepsilon^{k/2},\end{aligned}$$

for every $0 < \varepsilon < c_1^{-2}$ and $k \leqslant c_2 q_*(\mu)$. Since $q_*(\mu) \geqslant c_3\sqrt{n}$, the result follows with $\varepsilon_0 = c_1^{-2}$ $\ c = c_2c_3/2$. $\square$

14 Reduction to the negative moments

In this section we describe the work of Dafnis and Paouris: they proved that a positive answer to the hyperplane conjecture is equivalent to some very strong small probability estimates for the Euclidean norm on isotropic convex bodies. Recall that, for $-n < p \leqslant \infty$, $p \neq 0$,

$$I_p(K) := \left(\int_K |x|^p dx \right)^{1/p}$$

and, given any $\zeta \geqslant 1$, consider the parameter

$$q_{-c}(K, \zeta) := \max\{p \geqslant 1 : I_2(K) \leqslant \zeta I_{-p}(K)\}. \tag{14.1}$$

The results in this section imply that the hyperplane conjecture is equivalent to the following statement:

There exist absolute constants $C, \xi > 0$ such that, for every isotropic convex body K in $\mathbb{R}^n$,

$$q_{-c}(K, \xi) \geqslant Cn.$$

We already know that there exists a parameter $q_* := q_*(K)$ (related to the L_q-centroid bodies of K) with the following properties:

1. $q_*(K) \geqslant c\sqrt{n}$,
2. $q_{-c}(K, \xi_0) \geqslant q_*(K)$ for some absolute constant $\xi_0 \geqslant 1$, and hence, $I_2(K) \leqslant \xi_0 I_{-q_*}(K)$.

What is not clear is the behavior of $I_{-p}(K)$ when p lies in the interval $[q_*, n]$.

The main idea of Dafnis and Paouris is to start with an "extremal" isotropic convex body K in $\mathbb{R}^n$ with maximal isotropic constant $L_K \simeq L_n$ which is at the same time in α-regular M-position. Their starting point, which has a rather technical proof, is the following precise statement.

Theorem 14.1 (Dafnis-Paouris)**.** *There exist absolute constants $\kappa, \tau > 1$ and $\delta > 0$ such that, for every $\alpha \in [1, 2)$, we can find an isotropic convex body K_α in $\mathbb{R}^n$ with the following properties:*

(i) $L_{K_\alpha} \geqslant \delta L_n$,

(ii) *for every* $t \geqslant \tau(2-\alpha)^{-3/2}$

$$\log N(K_\alpha, t\sqrt{n}B_2^n) \leqslant \frac{\kappa n}{(2-\alpha)^{2\alpha} t^\alpha}. \tag{14.2}$$

Then, they try taking advantage of the fact that small ball probability estimates are closely related to estimates for covering numbers. The key lemma is the following.

Lemma 14.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Assume that, for some $s > 0$,*

$$r_s := \log N(K, sB_2^n) < n. \tag{14.3}$$

Then,

$$I_{-r_s}(K) \leqslant 3es.$$

Proof. Let $z_0 \in \mathbb{R}^n$ be such that $\mathrm{Vol}_n(K \cap (-z_0 + sB_2^n)) \geqslant \mathrm{Vol}_n(K \cap (z + sB_2^n))$ for every $z \in \mathbb{R}^n$. It follows that

$$\mathrm{Vol}_n((K + z_0) \cap sB_2^n) \cdot N(K, sB_2^n) \geqslant \mathrm{Vol}_n(K) = 1. \tag{14.4}$$

Let $q = r_s$. Then, using Markov's inequality, the definition of $I_{-q}(K + z_0)$ and (14.3), we get

$$\mathrm{Vol}_n((K + z_0) \cap 3^{-1} I_{-q}(K + z_0) B_2^n) \leqslant 3^{-q} < e^{-q} = e^{-r_s} \leqslant \frac{1}{N(K, sB_2^n)}.$$

From (14.4) we obtain

$$\mathrm{Vol}_n((K+z_0)\cap 3^{-1}I_{-q}(K+z_0)B_2^n) < \mathrm{Vol}_n((K+z_0)\cap sB_2^n),$$

and this implies

$$3^{-1}I_{-q}(K+z_0)\leqslant s.$$

Since K is centered, we get that $I_{-k}(K+z)\geqslant \frac{1}{e}I_{-k}(K)$ for any $1\leqslant k<n$ and $z\in\mathbb{R}^n$. To see this, we write

$$\begin{aligned} I_{-k}(K+z) &= c_{n,k}\left(\int_{G_{n,k}}\mathrm{Vol}_{n-k}(K+z)\cap F^{\perp})\,d\nu_{n,k}(F)\right)^{-1/k}\\ &\geqslant \frac{c_{n,k}}{e}\left(\int_{G_{n,k}}\mathrm{Vol}_{n-k}(K\cap F^{\perp})\,d\nu_{n,k}(F)\right)^{-1/k}=\frac{1}{e}I_{-k}(K).\end{aligned}$$

This proves the lemma. $\square$

We can now prove the main theorem.

Theorem 14.3 (Dafnis-Paouris)**.** *Assume that $q_{-c}(K,\zeta)\geqslant\beta n$ for some $\zeta\geqslant 1$, some $\beta\in(0,1)$ and every isotropic convex body K in $\mathbb{R}^n$. Then,*

$$L_n\leqslant\frac{C\zeta}{\sqrt{\beta}}\log^2\Big(\frac{e}{\beta}\Big), \tag{14.5}$$

where $C>0$ is an absolute constant.

Proof. Set $\alpha:=2-\log(e/\beta)^{-1}$ and with this α apply Theorem 14.1 to find an isotropic convex body K_α which satisfies its conclusion: for some absolute constants $\kappa,\tau\geqslant 1$ and $\delta>0$ it holds that $L_{K_\alpha}\geqslant\delta L_n$ and

$$\log N(K_\alpha,t\sqrt{n}B_2^n)\leqslant\frac{\kappa n}{(2-\alpha)^{2\alpha}t^\alpha}\quad\text{for all } t\geqslant\tau\log^{3/2}\Big(\frac{e}{\beta}\Big).$$

We may clearly assume that $\tau^2\leqslant e\kappa$ as well. We choose

$$t_1=(e\kappa)^{1/\alpha}\frac{1}{\sqrt{\beta}}\log^2\Big(\frac{e}{\beta}\Big);$$

then $t_1^\alpha=e\kappa(2-\alpha)^{-2\alpha}(\sqrt{\beta})^{-\alpha}$ and, since $\tau\leqslant\sqrt{e\kappa}\leqslant(e\kappa)^{1/\alpha}$, we have that $t_1\geqslant\tau(2-\alpha)^{-3/2}=\tau\log^{3/2}(e/\beta)$. Therefore,

$$r_1:=\log N(K_\alpha,t_1\sqrt{n}B_2^n)\leqslant\frac{\kappa n}{(2-\alpha)^{2\alpha}t_1^\alpha}\leqslant\frac{1}{e}(\sqrt{\beta})^\alpha n\leqslant\beta n,$$

and hence by Lemma 14.2 we obtain that

$$I_{-r_1}(K_\alpha)\leqslant 3et_1\sqrt{n}.$$

On the other hand, since $r_1\leqslant\beta n$ and since $q_{-c}(K_\alpha,\zeta)\geqslant\beta n$, we have that

$$\sqrt{n}L_{K_\alpha}=I_2(K_\alpha)\leqslant\zeta I_{-r_1}(K_\alpha).$$

It follows that

$$L_{K_\alpha}\leqslant 3e\zeta t_1=3e\zeta(e\kappa)^{1/\alpha}\frac{1}{\sqrt{\beta}}\log^2\Big(\frac{e}{\beta}\Big)\leqslant\frac{3e^2\zeta\kappa}{\sqrt{\beta}}\log^2\Big(\frac{e}{\beta}\Big).$$

Since $L_{K_\alpha}\geqslant\delta L_n$, the result follows. $\square$

Remark 14.4. Since $q_{-c}(K,\xi_0) \geqslant q_*(K) \geqslant c\sqrt{n}$ for some absolute constants $\xi_0 \geqslant 1$ and $c > 0$, we may apply Theorem 14.3 with $\zeta = \xi_0$ and $\beta = c/\sqrt{n}$ to get

$$L_n \leqslant \frac{C\zeta}{\sqrt{c}} \sqrt[4]{n} \log^2\Big(\frac{e\sqrt{n}}{c}\Big) \leqslant C_1 \sqrt[4]{n} (\log n)^2, \tag{14.6}$$

where $C_1 > 0$ is an absolute constant.

In the opposite direction, one can show that if the hyperplane conjecture is correct then there are absolute constants $\sigma, \xi > 0$ such that, for every isotropic convex body K in $\mathbb{R}^n$, one has $q_{-c}(K,\xi) \geqslant \sigma n$. This is an immediate consequence of the next theorem.

Theorem 14.5 (Dafnis-Paouris)**.** *There exists an absolute constant $C > 0$ such that, for every n and for every isotropic convex body K in $\mathbb{R}^n$,*

$$q_{-c}(K, CL_n) \geqslant n - 1.$$

Proof. We start with the formula

$$I_{-s}(K) \simeq \sqrt{n} \left(\int_{G_{n,s}} \mathrm{Vol}_{n-s}(K \cap F^{\perp})\, d\nu_{n,s}(F) \right)^{-1/s}. \tag{14.7}$$

Recall from Proposition 9.5 that

$$\mathrm{Vol}_{n-s}(K \cap F^{\perp})^{1/s} \simeq \frac{L_{\overline{K_{s+1}}(\pi_F(\mu_K))}}{L_K}.$$

for every $F \in G_{n,s}$. Thus, we get

$$I_{-s}(K) \simeq \sqrt{n} \left(\int_{G_{n,s}} \left(\frac{L_{\overline{K_{s+1}}(\pi_F(f))}}{L_K} \right)^s d\nu_{n,s}(F) \right)^{-1/s}.$$

Now,

$$\int_{G_{n,s}} \left(\frac{L_{\overline{K_{s+1}}(\pi_F(f))}}{L_K} \right)^s d\nu_{n,s}(F) \leqslant \left(\frac{L_s}{L_K} \right)^s.$$

Therefore,

$$I_{-s}(K) \geqslant \frac{c_1\sqrt{n}L_K}{L_s} \geqslant \frac{c_2\sqrt{n}L_K}{L_n}$$

because it is known that $L_s \leqslant c_3 L_n$ for all integers $s \leqslant n-1$. Since $I_2(K) = \sqrt{n}L_K$, we get

$$q_{-c}(K,\delta) := \max\{p \geqslant 1 : I_2(K) \leqslant c_2^{-1} L_n I_{-p}(K)\} \geqslant n - 1.$$

This is the claim of the theorem. $\square$

15 A variant of Bourgain's argument and one more reduction

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. We consider the parameter

$$I_1(K, Z_q^{\circ}(K)) = \int_K \|\langle \cdot, x\rangle\|_{L_q(K)} dx.$$

Generally, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then for every symmetric convex body C in $\mathbb{R}^n$ and for every $q \in (-n, \infty)$, $q \neq 0$, we define

$$I_q(K, C) := \left(\int_K \|x\|_C^q \, dx \right)^{1/q}.$$

The notation $I_1(K, Z_q^\circ(K))$ is then justified by the fact that $\|\langle\cdot, x\rangle\|_{L_q(K)}$ is the norm induced on $\mathbb{R}^n$ by the polar body $Z_q^\circ(K)$ of the L_q-centroid body of K.

The purpose of this section is to describe a work of Giannopoulos, Paouris and Vritsiou which reduces the hyperplane conjecture to the study of the parameter $I_1(K, Z_q^\circ(K))$ when K belongs to the following subclass of isotropic convex bodies. We start with a definition which formalizes Theorem 14.1 from the previous section.

Definition 15.1. Let $\kappa, \tau > 0$. We say that an isotropic convex body K in $\mathbb{R}^n$ is (κ, τ)-*regular* if

$$\log N(K, tB_2^n) \leqslant \frac{\kappa n^2 \log^4 n}{t^2} \text{ for all } t \geqslant \tau\sqrt{n}\log^{3/2} n.$$

Applying Theorem 14.1 with $\alpha = 2 - (\log n)^{-1}$, we see that there are absolute constants $\kappa, \tau > 1$ and $\delta > 0$ such that, for every $n \in \mathbb{N}$, there exist (κ, τ)-regular isotropic convex bodies with maximal isotropic constant. More precisely, we start with the next fact.

Theorem 15.2. *For every $n \in \mathbb{N}$ we can find an isotropic convex body K in $\mathbb{R}^n$ with the following properties:*

(i) $L_K \geqslant \delta L_n$,

(ii) $\log N(K, tB_2^n) \leqslant \kappa n^2 \log^4 n / t^2$ *for all* $t \geqslant \tau\sqrt{n}\log^{3/2} n$,

where $\kappa \geqslant \tau^2 \geqslant 1$ and $\delta > 0$ are absolute constants.

The main result of this section is the next reduction of the slicing problem.

Theorem 15.3 (Giannopoulos-Paouris-Vritsiou)**.** *There exists an absolute constant $\rho \in (0, 1)$ with the following property. Given $\kappa \geqslant \tau^2 \geqslant 1$, for every $n \geqslant n_0(\tau)$ and every (κ, τ)-regular isotropic convex body K in $\mathbb{R}^n$ we have: if*

$$2 \leqslant q \leqslant \rho^2 n \text{ and } I_1(K, Z_q^\circ(K)) \leqslant \rho n L_K^2, \tag{15.1}$$

then

$$L_K^2 \leqslant C\kappa\sqrt{\frac{n}{q}}\log^4 n \max\left\{1, \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\},$$

where $C > 0$ is an absolute constant.

Observe that, for every isotropic convex body K in $\mathbb{R}^n$, we have that

$$I_1(K, Z_2^\circ(K)) \leqslant \sqrt{n}L_K^2 \leqslant \rho n L_K^2$$

if n is sufficiently large. From Theorem 15.2 we know that, for some absolute constants $\kappa \geqslant \tau^2 \geqslant 1$ and $\delta > 0$, there exists a (κ, τ)-regular isotropic convex body K in $\mathbb{R}^n$ with $L_K \geqslant \delta L_n$. Therefore, Theorem 15.3 gives

$$L_K^2 \leqslant C_1\sqrt{n}\log^4 n, \tag{15.2}$$

which already leads to the bound $L_n \leqslant C_2\sqrt[4]{n}\log^2 n$ for L_n.

However, the behavior of $I_1(K, Z_q^\circ(K))$ may allow us to use much larger values of q. For every isotropic convex body K in $\mathbb{R}^n$ one can prove some simple general estimates:

(i) For every $2 \leqslant q \leqslant n$,

$$c_1 \max\left\{\sqrt{n}L_K^2, \sqrt{qn}, R(Z_q(K))L_K\right\} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c_2 q\sqrt{n}L_K^2.$$

(ii) If $2 \leqslant q \leqslant \sqrt{n}$, then

$$c_1 \max\left\{\sqrt{n}L_K^2, \sqrt{qn}L_K\right\} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c_2 q\sqrt{n}L_K^2.$$

Any improvement of the exponent of q in the upper bound $I_1(K, Z_q^\circ(K)) \leqslant cq\sqrt{n}L_K^2$ would lead to an estimate $L_n \leqslant Cn^\alpha$ with $\alpha < \frac{1}{4}$. It seems plausible that one could even have $I_1(K, Z_q^\circ(K)) \leqslant c\sqrt{qn}L_K^2$, at least when q is small, say $2 \leqslant q \ll \sqrt{n}$. Some evidence is given by the following facts:

(iii) If K is an unconditional isotropic convex body in $\mathbb{R}^n$, then

$$c_1\sqrt{qn} \leqslant I_1(K, Z_q^\circ(K)) \leqslant c_2\sqrt{qn}\,\log n$$

for all $2 \leqslant q \leqslant n$.

(iv) If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $2 \leqslant q \leqslant \sqrt{n}$, there exists a set $A_q \subseteq O(n)$ with $\nu(A_q) \geqslant 1 - e^{-q}$ such that $I_1(K, Z_q^\circ(U(K))) \leqslant c_3\sqrt{qn}\,L_K^2$ for all $U \in A_q$.

For the proof of Theorem 15.3 we need two auxiliary results. The first one provides an estimate for the L_q-norm of the maximum of N linear functionals on K.

Lemma 15.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$, and consider any points $z_1, z_2, \ldots, z_N \in \mathbb{R}^n$. If $q \geqslant 1$ and $p \geqslant \max\{\log N, q\}$, then*

$$\left(\int_K \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle|^q dx\right)^{1/q} \leqslant \overline{\beta}_1 \max_{1\leqslant i\leqslant N} h_{Z_p(K)}(z_i), \tag{15.3}$$

where $\overline{\beta}_1 > 0$ is an absolute constant.

Proof. Let $p \geqslant \max\{\log N, q\}$ and $\theta \in S^{n-1}$. Markov's inequality shows that

$$\mathrm{Vol}_n(\{x \in K : |\langle x, \theta\rangle) \geqslant e^3 h_{Z_p(K)}(\theta)\}) \leqslant e^{-3p}.$$

Since $x \mapsto |\langle x, \theta\rangle|$ is a seminorm, from Borell's lemma we get that

$$\mathrm{Vol}_n(\{x \in K : |\langle x, \theta\rangle) \geqslant e^3 t h_{Z_p(K)}(\theta)\}) \leqslant (1 - e^{-3p})\left(\frac{e^{-3p}}{1 - e^{-3p}}\right)^{\frac{t+1}{2}} \leqslant e^{-pt}$$

for every $t \geqslant 1$. We set $S := e^3 \max_{1\leqslant i\leqslant N} h_{Z_p(K)}(z_i)$. Then, for every $t \geqslant 1$ we have that

$$\mathrm{Vol}_n(\{x \in K : \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle| \geqslant St\}) \leqslant \sum_{i=1}^N \mathrm{Vol}_n(\{x \in K : |\langle x, z_i\rangle| \geqslant e^3 t h_{Z_p(K)}(z_i)\}) \leqslant Ne^{-pt}.$$

It follows that

$$\begin{aligned}
\int_K \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle|^q dx &= q\int_0^\infty s^{q-1}\mathrm{Vol}_n(\{x \in K : \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle| \geqslant s\})\,ds \\
&\leqslant S^q + q\int_S^\infty s^{q-1}\mathrm{Vol}_n(\{x \in K : \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle| \geqslant s\})\,ds \\
&= S^q\left(1 + q\int_1^\infty t^{q-1}\mathrm{Vol}_n(\{x \in K : \max_{1\leqslant i\leqslant N} |\langle x, z_i\rangle| \geqslant St\})\,dt\right) \\
&\leqslant S^q\left(1 + qN\int_1^\infty t^{q-1}e^{-pt}dt\right) \\
&= S^q\left(1 + \frac{qN}{p^q}\int_p^\infty t^{q-1}e^{-t}dt\right) \\
&\leqslant S^q\left(1 + \frac{qN}{p^q}e^{-p}p^q\right) \\
&\leqslant (3S)^q,
\end{aligned}$$

where we have also used the fact that, for every $p \geqslant q \geqslant 1$,

$$\int_p^\infty t^{q-1}e^{-t}dt \leqslant e^{-p}p^q.$$

This finishes the proof (with $\overline{\beta}_1 = 3e^3$). $\square$

The second lemma concerns the L_q-centroid bodies of subsets of K.

Lemma 15.5. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant q, r \leqslant n$. There exists an absolute constant $\overline{\beta}_2 > 0$ such that if A is a convex subset of K with $\mathrm{Vol}_n(A) \geqslant 1 - e^{-\overline{\beta}_2 q}$, then*

$$Z_p(K) \subseteq 2Z_p(\overline{A}) \tag{15.4}$$

for all $1 \leqslant p \leqslant q$. For the opposite inclusion, if $\mathrm{Vol}_n(A) \geqslant 2^{-\frac{r}{2}}$ then

$$Z_p(\overline{A}) \subseteq 2Z_p(K) \tag{15.5}$$

for all $r \leqslant p \leqslant n$.

Proof. Let $\theta \in S^{n-1}$. Note that

$$h_{Z_p(\overline{A})}(\theta) = \left(\int_{\overline{A}} |\langle x, \theta\rangle|^p dx\right)^{1/p} = \frac{1}{\mathrm{Vol}_n(A)^{\frac{1}{p}+\frac{1}{n}}} \left(\int_A |\langle x, \theta\rangle|^p dx\right)^{1/p}.$$

We first prove (15.5): since $A \subseteq K$ and assuming that $\mathrm{Vol}_n(A) \geqslant 2^{-\frac{r}{2}}$, we have

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta\rangle|^p dx\right)^{1/p} \geqslant \left(\int_A |\langle x, \theta\rangle|^p dx\right)^{1/p} \geqslant 2^{-\frac{r}{2p}-\frac{r}{2n}} \left(\int_{\overline{A}} |\langle x, \theta\rangle|^p dx\right)^{1/p} \geqslant \frac{1}{2} h_{Z_p(\overline{A})}(\theta)$$

for all $r \leqslant p \leqslant n$. On the other hand, assuming that $\mathrm{Vol}_n(A) \geqslant 1 - e^{-\overline{\beta}_2 q}$ and using the fact that $\|\langle \cdot, \theta\rangle\|_{2p} \leqslant c\|\langle \cdot, \theta\rangle\|_p$ for some absolute constant $c > 0$, we have

$$\begin{aligned}
\int_K |\langle x, \theta\rangle|^p dx &= \int_A |\langle x, \theta\rangle|^p dx + \int_{K\setminus A} |\langle x, \theta\rangle|^p dx \\
&\leqslant \mathrm{Vol}_n(A)^{1+\frac{p}{n}} \int_{\overline{A}} |\langle x, \theta\rangle|^p dx + \mathrm{Vol}_n(K \setminus A)^{1/2} \left(\int_K |\langle x, \theta\rangle|^{2p} dx\right)^{1/2} \\
&\leqslant \int_{\overline{A}} |\langle x, \theta\rangle|^p dx + e^{-\overline{\beta}_2 q/2} c^p \int_K |\langle x, \theta\rangle|^p dx \\
&\leqslant \int_{\overline{A}} |\langle x, \theta\rangle|^p dx + \frac{1}{2}\int_K |\langle x, \theta\rangle|^p dx
\end{aligned}$$

for every $1 \leqslant p \leqslant q$, if $\overline{\beta}_2 > 0$ is chosen large enough. This proves (15.4). $\square$

Proof of Theorem 15.3. Let $\kappa \geqslant \tau^2 \geqslant 1$ and consider a (κ, τ)-regular isotropic convex body K in $\mathbb{R}^n$. Assume that the conditions (15.1) are also satisfied. We define a convex body W in $\mathbb{R}^n$, setting

$$W := \{x \in K : h_{Z_q(K)}(x) \leqslant C_1 I_1(K, Z_q^\circ(K))\},$$

where $C_1 = e^{2\overline{\beta}_2}$ and $\overline{\beta}_2 > 0$ is the constant which was defined in Lemma 15.5. From Markov's inequality we have that $\mathrm{Vol}_n(W) \geqslant 1 - e^{-2\overline{\beta}_2}$ and also trivially that $\mathrm{Vol}_n(W) \geqslant \frac{1}{2} \geqslant 2^{-q/2}$ (as long as $\overline{\beta}_2 \geqslant 1$). Then we set

$$K_1 := \overline{W}.$$

Applying both cases of Lemma 15.5 to the set W with $p=2$, we see that

$$\frac{1}{2}Z_2(K_1)\subseteq Z_2(K)\subseteq 2Z_2(K_1).$$

This implies that

$$\frac{1}{4}L_K^2=\frac{1}{4}\int_K\langle x,\theta\rangle^2dx\leqslant\int_{K_1}\langle x,\theta\rangle^2dx\leqslant 4\int_K\langle x,\theta\rangle^2dx=4L_K^2$$

for every $\theta\in S^{n-1}$, and hence,

$$\frac{nL_K^2}{4}\leqslant\sum_{i=1}^n\int_{K_1}\langle x,e_i\rangle^2dx=\int_{K_1}|x|^2dx\leqslant 4nL_K^2.$$

We also have

$$K_1=\mathrm{Vol}_n(W)^{-1/n}W\subseteq 2W\subseteq 2K,$$

thus for every $x\in K_1$ we have $x/2\in W$ and, using (15.5) of Lemma 15.5 (with $p=q$) we write

$$h_{Z_q(K_1)}(x)\leqslant 2h_{Z_q(K)}(x)=4h_{Z_q(K)}(x/2)\leqslant 4C_1I_1(K,Z_q^\circ(K)). \tag{15.6}$$

Finally,

$$\log\overline{N}(K_1,tB_2^n)\leqslant\log N(2K_1,tB_2^n)\leqslant\log N(4K,tB_2^n)\leqslant\frac{16\kappa n^2\log^4 n}{t^2},$$

for all $t\geqslant 4\tau\sqrt{n}\log^{3/2}n$. We write

$$nL_K^2\leqslant 4\int_{K_1}|x|^2dx\leqslant 4\int_{K_1}\max_{z\in K_1}|\langle x,z\rangle|\,dx.$$

Observe now that for every $t\geqslant 4\tau\sqrt{n}\log^{3/2}n$ we can find $z_1,\dots,z_{N_t}\in K_1$, with $|N_t|\leqslant\exp(16\kappa n^2\log^4 n/t^2)$, such that $K_1\subseteq\bigcup_{i=1}^{N_t}(z_i+tB_2^n)$. It follows that

$$\max_{z\in K_1}|\langle x,z\rangle|\leqslant\max_{1\leqslant i\leqslant N_t}|\langle x,z_i\rangle|+\max_{w\in tB_2^n}|\langle x,w\rangle|=\max_{1\leqslant i\leqslant N_t}|\langle x,z_i\rangle|+t|x|,$$

and hence

$$nL_K^2\leqslant 4\int_{K_1}\max_{1\leqslant i\leqslant N_t}|\langle x,z_i\rangle|dx+4t\int_{K_1}|x|\,dx\leqslant 4\int_{K_1}\max_{1\leqslant i\leqslant N_t}|\langle x,z_i\rangle|dx+8t\sqrt{n}L_K. \tag{15.7}$$

Recall also that by Borell's lemma we can find absolute constants $\beta_1,\beta_2>0$ so that

$$Z_q(K)\subseteq\beta_1qZ_1(K)\quad\text{and}\quad Z_q(K)\subseteq\beta_2\frac{q}{p}Z_p(K) \tag{15.8}$$

for all $1\leqslant p<q$. We choose

$$t_0^2=64C_2\kappa\max\left\{1,\frac{I_1(K,Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}\frac{n^{3/2}}{\sqrt{q}}\log^4 n,$$

where $C_2=16C_1\beta_1\overline{\beta}_1$ with $\overline{\beta}_1$ the constant from Lemma 15.4. With this choice of t_0, we have

$$t_0^2\geqslant 64C_2\kappa\sqrt{\frac{n}{q}}\,n\log^4 n\geqslant\frac{64C_2\kappa}{\rho}\,n\log^4 n \tag{15.9}$$

and

$$t_0^2 \geqslant 64C_2\kappa\,\frac{I_1(K,Z_q^\circ(K))}{qL_K^2}\,n\log^4 n. \tag{15.10}$$

From (15.9) it is clear that

$$t_0^2 \geqslant 64C_2\kappa\frac{n\log^4 n}{\rho} \geqslant 16\tau^2 n\log^3 n,$$

provided that $n \geqslant n_0(\tau,\rho)$, so the above argument, leading up to (15.7), holds with $t=t_0$. We also set $p_0 := \frac{16\kappa n^2\log^4 n}{t_0^2}$. Observe that $p_0 \geqslant q$: since q is such that $I_1(K,Z_q^\circ(K)) \leqslant \rho n L_K^2$, we have $\max\left\{1,\frac{I_1(K,Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\} \leqslant \rho\sqrt{n/q}$, and hence

$$t_0^2 \leqslant 64C_2\kappa\rho\frac{n^2\log^4 n}{q}.$$

But then, if we choose $\rho < 1/(4C_2)$, we have

$$p_0 = \frac{16\kappa n^2\log^4 n}{t_0^2} \geqslant \frac{16\kappa n^2 q\log^4 n}{64C_2\kappa\rho n^2\log^4 n} = \frac{q}{4C_2\rho} \geqslant q$$

as claimed. Therefore, using Lemma 15.4 with $q'=1$, we can write

$$\int_{K_1}\max_{1\leqslant i\leqslant N_{t_0}}|\langle x,z_i\rangle|dx \leqslant \overline{\beta}_1\max_{1\leqslant i\leqslant N_{t_0}} h_{Z_{p_0}(K_1)}(z_i) \leqslant \overline{\beta}_1\beta_1\frac{p_0}{q}\max_{1\leqslant i\leqslant N_{t_0}} h_{Z_q(K_1)}(z_i).$$

Combining the above with (15.7), (15.6) and the definition of C_2, we get

$$nL_K^2 \leqslant C_2\frac{p_0}{q}I_1(K,Z_q^\circ(K)) + 8t_0\sqrt{n}L_K. \tag{15.11}$$

Also, from (15.10) and the definition of p_0, we have

$$C_2\frac{p_0}{q}I_1(K,Z_q^\circ(K)) = \frac{16C_2\kappa I_1(K,Z_q^\circ(K))}{qt_0^2}\,n^2\log^4 n \leqslant \frac{1}{4}\,nL_K^2.$$

Therefore, (15.11) gives

$$nL_K^2 \leqslant C_3 t_0\sqrt{n}L_K.$$

This shows that

$$L_K^2 \leqslant C_4\frac{t_0^2}{n} = C\kappa\max\left\{1,\frac{I_1(K,Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}\sqrt{\frac{n}{q}}\log^4 n,$$

as claimed. $\square$

16 Volume of the centroid bodies and the isotropic constant

Klartag and E. Milman further exploited the logarithmic Laplace transform to obtain additional information on the L_q-centroid bodies of an isotropic log-concave probability measure μ in $\mathbb{R}^n$ and an alternative proof of the bound $L_\mu = O(\sqrt[4]{n})$. Recall that the logarithmic Laplace transform of a Borel probability measure μ on $\mathbb{R}^n$ is defined by

$$\Lambda_\mu(\xi) = \log\Big(\int_{\mathbb{R}^n} e^{\langle\xi,x\rangle}d\mu(x)\Big).$$

It is easily checked that Λ_μ is convex and $\Lambda_\mu(0)=0$. If $\mathrm{bar}(\mu)=0$ then Jensen's inequality shows that

$$\Lambda_\mu(\xi) = \log\Big(\int_{\mathbb{R}^n} e^{\langle\xi,x\rangle}d\mu(x)\Big) \geqslant \int_{\mathbb{R}^n}\langle\xi,x\rangle d\mu(x) = 0$$

for all ξ; therefore, Λ_μ is a non-negative function. Further properties of Λ_μ in the log-concave case are described in the next proposition.

Proposition 16.1. *Let μ be an n-dimensional log-concave probability measure. The set $A(\mu) = \{\Lambda_\mu < \infty\}$ is open and Λ_μ is C^∞ and strictly convex on $A(\mu)$. Moreover, for every $t \geqslant 0$ and $\alpha \geqslant 1$,*

$$\frac{1}{\alpha}\{\Lambda_\mu \leqslant \alpha t\} \subseteq \{\Lambda_\mu \leqslant t\} \subseteq \{\Lambda_\mu \leqslant \alpha t\}. \tag{16.1}$$

Definition 16.2. For every $p > 0$ we define

$$\Lambda_p(\mu) = \{\Lambda_\mu \leqslant p\} \cap (-\{\Lambda_\mu \leqslant p\}).$$

The level sets $\Lambda_p(\mu)$ of Λ_μ can be expressed in terms of the L_q-centroid bodies of μ; it is not hard to check the following.

Proposition 16.3. *Let μ be a log-concave probability measure with* $\mathrm{bar}(\mu) = 0$. *For every $p \geqslant 1$,*

$$\Lambda_p(\mu) \simeq pZ_p(\mu)^\circ.$$

Lemma 16.4. *Let μ be a log-concave probability measure with* $\mathrm{bar}(\mu) = 0$. *For every $q, r > 0$,*

$$\nabla\Lambda_\mu\left(\frac{1}{2}\{\Lambda_\mu \leqslant q\}\right) \subseteq (q + r)\{\Lambda_\mu \leqslant r\}^\circ.$$

Proof. Let $x \in \frac{1}{2}\{\Lambda_\mu \leqslant q\}$. Then, $\Lambda_\mu(2x) \leqslant q$. For every $z \in \{\Lambda_\mu \leqslant r\}$ we may write

$$\left\langle \nabla\Lambda_\mu(x), \frac{z}{2}\right\rangle \leqslant \Lambda_\mu(x) + \left\langle \nabla\Lambda_\mu(x), \frac{z}{2}\right\rangle \leqslant \Lambda_\mu\left(x + \frac{z}{2}\right) \leqslant \frac{\Lambda_\mu(2x) + \Lambda_\mu(z)}{2} \leqslant \frac{q + r}{2},$$

using the fact that $\Lambda_\mu(x) \geqslant 0$ and the convexity of Λ_μ. Since

$$\langle \nabla\Lambda_\mu(x), z\rangle \leqslant q + r$$

for every $z \in \{\Lambda_\mu \leqslant r\}$ we see that $\nabla\Lambda_\mu(x) \in (q + r)\{\Lambda_\mu \leqslant r\}^\circ$. □

Corollary 16.5. *Let μ be a log-concave probability measure with* $\mathrm{bar}(\mu) = 0$. *For every $p > 0$,*

$$\nabla\Lambda_\mu\left(\frac{1}{2}\Lambda_p(\mu)\right) \subseteq 2p\,\Lambda_p(\mu)^\circ.$$

Proof. We apply Lemma 16.4 with $q = r = p$. We have

$$\nabla\Lambda_\mu\left(\frac{1}{2}\Lambda_p(\mu)\right) \subseteq \nabla\Lambda_\mu\left(\frac{1}{2}\{\Lambda_\mu \leqslant p\}\right) \subseteq 2p\,\{\Lambda_\mu \leqslant p\}^\circ \subseteq 2p\,\Lambda_p(\mu)^\circ,$$

because $\{\Lambda_\mu \leqslant p\} \supseteq \Lambda_p(\mu)$ implies that $\{\Lambda_\mu \leqslant p\}^\circ \subseteq \Lambda_p(\mu)^\circ$. □

Definition 16.6. For every $p > 0$ we define

$$\Psi_p = \left(\frac{1}{\mathrm{Vol}_n\left(\frac{1}{2}\Lambda_p(\mu)\right)}\int_{\frac{1}{2}\Lambda_p(\mu)} \det \mathrm{Hess}\,(\Lambda_\mu)(x)\,dx\right)^{1/n}.$$

Proposition 16.7. *For every $p > 0$,*

$$\mathrm{Vol}_n(\Lambda_p(\mu))^{1/n} \leqslant C\sqrt{\frac{p}{n}}\,\frac{1}{\sqrt{\Psi_p}}.$$

Proof. Using Corollary 16.5 and the change of variables $x = \nabla\Lambda_\mu(y)$, we write

$$\begin{aligned}\mathrm{Vol}_n(2p\Lambda_p(\mu)^\circ) \geqslant \mathrm{Vol}_n\left(\nabla\Lambda_\mu\left(\frac{1}{2}\Lambda_p(\mu)\right)\right) &= \int_{\frac{1}{2}\Lambda_p(\mu)} \det \mathrm{Hess}\,(\Lambda_\mu)(y)\,dy \\ &= \mathrm{Vol}_n\left(\frac{1}{2}\Lambda_p(\mu)\right)\,\Psi_p^n.\end{aligned}$$

In other words,

$$\mathrm{Vol}_n(\Lambda_p(\mu)^\circ)^{1/n} \geqslant \frac{\Psi_p}{4p}\,\mathrm{Vol}_n(\Lambda_p(\mu))^{1/n}.$$

From the Blaschke-Santaló inequality we have

$$\mathrm{Vol}_n(\Lambda_p(\mu)^\circ)^{1/n} \leqslant \frac{c}{n}\,\frac{1}{\mathrm{Vol}_n(\Lambda_p(\mu))^{1/n}},$$

and hence,

$$\mathrm{Vol}_n(\Lambda_p(\mu))^{2/n} \leqslant \frac{C^2 p}{n}\,\frac{1}{\Psi_p},$$

where $C^2 = 4c$. $\square$

Let μ be a log-concave probability measure on $\mathbb{R}^n$ with density ρ. For every $\xi \in A(\mu) = \{\Lambda_\mu < \infty\}$ we set

$$\rho_\xi(x) = \frac{1}{Z_\xi}\rho(x)e^{\langle \xi, x\rangle},$$

where $Z_\xi > 0$ is chosen so that ρ_ξ becomes a probability density. Next, we set

$$b_\xi = \frac{1}{Z_\xi}\int_{\mathbb{R}^n} x\rho(x)e^{\langle \xi, x\rangle}dx$$

and we define a probability measure μ_ξ with density

$$\frac{1}{Z_\xi}\rho(x+b_\xi)e^{\langle \xi, x+b_\xi\rangle}.$$

Lemma 16.8. *We have*

$$b_\xi = \nabla\Lambda_\mu(\xi) \quad \textit{and} \quad \mathrm{Cov}(\mu_\xi) = \mathrm{Hess}\,(\Lambda_\mu)(\xi).$$

Proof. Both equalities follow from simple calculations: just observe that since the log-concave density $\rho(x)e^{\langle \xi^0, x\rangle}$ decays exponentially for every $\xi^0 \in A(\mu)$, we can differentiate twice under the integral sign. $\square$

Theorem 16.9 (Klartag-E. Milman)**.** *Let μ be a log-concave probability measure on $\mathbb{R}^n$ with* $\mathrm{bar}(\mu) = 0$*. For every $1 \leqslant p \leqslant n$,*

$$\mathrm{Vol}_n(Z_p(\mu))^{1/n} \simeq \sqrt{\frac{p}{n}}\inf_{\xi\in\frac{1}{2}\Lambda_p(\mu)}[\det \mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}.$$

Proof of the lower bound. We combine Propositions 16.3 and 16.7. We have

$$\mathrm{Vol}_n(\Lambda_p(\mu))^{1/n} \leqslant C\sqrt{\frac{p}{n}}\,\frac{1}{\sqrt{\Psi_p}}$$

and

$$\Lambda_p(\mu) \simeq pZ_p(\mu)^\circ.$$

Therefore, by the reverse Santaló inequality,

$$\begin{aligned}\mathrm{Vol}_n(Z_p(\mu))^{1/n} &\geqslant \frac{c_1}{n\mathrm{Vol}_n(Z_p(\mu)^\circ)^{1/n}} \geqslant \frac{c_2 p}{n\mathrm{Vol}_n(\Lambda_p(\mu))^{1/n}} \geqslant c_3\frac{p}{n}\sqrt{\frac{n}{p}}\sqrt{\Psi_p} = c_3\sqrt{\frac{p}{n}}\sqrt{\Psi_p}\\ &= c_3\sqrt{\frac{p}{n}}\left(\frac{1}{\mathrm{Vol}_n\left(\frac{1}{2}\Lambda_p(\mu)\right)}\int_{\frac{1}{2}\Lambda_p(\mu)}\det\mathrm{Hess}\,(\Lambda_\mu)(\xi)\,d\xi\right)^{\frac{1}{2n}}\\ &\geqslant c_3\sqrt{\frac{p}{n}}\inf_{\xi\in\frac{1}{2}\Lambda_p(\mu)}[\det\mathrm{Hess}\,(\Lambda_\mu)(\xi)]^{\frac{1}{2n}}\\ &= c_3\sqrt{\frac{p}{n}}\inf_{\xi\in\frac{1}{2}\Lambda_p(\mu)}[\det\mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}.\end{aligned}$$

For the proof of the upper bound we need the following.

Proposition 16.10. *Let μ be a log-concave probability measure on $\mathbb{R}^n$ with $\mathrm{bar}(\mu)=0$. For every $\xi\in\frac{1}{2}\Lambda_p(\mu)$,*

$$Z_p(\mu_\xi)\simeq Z_p(\mu). \tag{16.2}$$

Proof. From Proposition 16.3, it is enough to show that, for every $\xi\in\frac{1}{2}\Lambda_p(\mu)$,

$$\Lambda_p(\mu_\xi)\simeq\Lambda_p(\mu).$$

We first observe that

$$\Lambda_{\mu_\xi}(z)=\Lambda_\mu(z+\xi)-\Lambda_\mu(\xi)-\langle z,\nabla\Lambda_\mu(\xi)\rangle. \tag{16.3}$$

To see this, first note that

$$\log Z_\xi=\log\int_{\mathbb{R}^n}\rho(x)e^{\langle\xi,x\rangle}dx=\Lambda_\mu(\xi)$$

and

$$\langle z,b_\xi\rangle=\langle z,\nabla\Lambda_\mu(\xi)\rangle.$$

Then, write

$$\begin{aligned}\Lambda_{\mu_\xi}(z)&=\log\Big(\int_{\mathbb{R}^n}\frac{1}{Z_\xi}e^{\langle z,y\rangle}\rho_\xi(y)dy\Big)=\log\Big(\int_{\mathbb{R}^n}e^{\langle z,y\rangle}e^{\langle\xi,y+b_\xi\rangle}\rho(y+b_\xi)dy\Big)-\log Z_\xi\\ &=\log\Big(\int_{\mathbb{R}^n}e^{-\langle z,b_\xi\rangle}e^{\langle z,y+b_\xi\rangle}e^{\langle\xi,y+b_\xi\rangle}\rho(y+b_\xi)dy\Big)-\Lambda_\mu(\xi)\\ &=-\langle z,b_\xi\rangle+\log\Big(\int_{\mathbb{R}^n}e^{\langle z+\xi,y+b_\xi\rangle}\rho(y+b_\xi)dy\Big)-\Lambda_\mu(\xi)\\ &=-\langle z,b_\xi\rangle+\Lambda_\mu(z+\xi)-\Lambda_\mu(\xi).\end{aligned}$$

Claim. Let $D,p>0$. If $\Lambda_\mu(2y)\leqslant Dp$ and $z\in\Lambda_p(\mu)$, then

$$\Lambda_\mu(z/2+y)-\Lambda_\mu(y)-\langle z/2,\nabla\Lambda_\mu(y)\rangle\leqslant(D+1)p.$$

Proof of the Claim. We apply Lemma 16.4 with $q=Dp$ and $r=p$. We have $\Lambda_\mu(2y)\leqslant Dp$ and $\Lambda_\mu(-z)\leqslant p$. Therefore,

$$-\langle\nabla\Lambda_\mu(y),z\rangle\leqslant(D+1)p.$$

Then,

$$\Lambda_\mu(z/2+y)-\Lambda_\mu(y)-\langle z/2,\nabla\Lambda_\mu(y)\rangle \leqslant \Lambda_\mu(z/2+y)+\frac{(D+1)p}{2}$$
$$\leqslant \frac{\Lambda_\mu(z)+\Lambda_\mu(2y)}{2}+\frac{(D+1)p}{2}\leqslant (D+1)p.$$

We can now continue the proof of Proposition 16.10:
(i) Assume that $z\in\Lambda_p(\mu)$. Note that $\Lambda_\mu(2\xi)\leqslant p$ because $\xi\in\frac{1}{2}\Lambda_p(\mu)$. The claim (with $D=1$ and $y=\xi$) combined with (16.3) shows that

$$\Lambda_{\mu_\xi}(z/2)=\Lambda_\mu(z/2+\xi)-\Lambda_\mu(\xi)-\langle z/2,\nabla\Lambda_\mu(\xi)\rangle\leqslant 2p.$$

From Proposition 16.1 it follows that $\Lambda_{\mu_\xi}(z/4)\leqslant p$. By symmetry, the same argument applies to $-z$, and hence $z\in 4\Lambda_p(\mu_\xi)$. In other words,

$$\Lambda_p(\mu)\subseteq 4\Lambda_p(\mu_\xi).$$

(ii) Assume that $z\in\Lambda_p(\mu_\xi)$. From (16.3) we know that

$$\Lambda_{\mu_\xi}(-2\xi)=\Lambda_\mu(-\xi)-\Lambda_\mu(\xi)+2\langle\xi,\nabla\Lambda_\mu(\xi)\rangle.$$

Note that

$$\Lambda_\mu(-\xi)\leqslant\frac{\Lambda_\mu(-2\xi)+\Lambda_\mu(0)}{2}\leqslant\frac{p+0}{2}=\frac{p}{2},$$

and similarly $\Lambda_\mu(\xi)\leqslant\frac{p}{2}$. From Lemma 16.4 we have

$$\langle\xi,\nabla\Lambda_\mu(\xi)\rangle\leqslant\frac{3p}{2}.$$

Since $\Lambda_\mu(\xi)\geqslant 0$ we conclude that

$$\Lambda_{\mu_\xi}(-2\xi)\leqslant\frac{7p}{2}.$$

Since $(\mu_\xi)_{-\xi}=\mu$, we may apply the argument from (i), using that $\Lambda_{\mu_\xi}(-2\xi)\leqslant Dp$ for $D=\frac{7}{2}$. We write

$$\Lambda_\mu(z/2)=\Lambda_{\mu_\xi}(z/2-\xi)-\Lambda_{\mu_\xi}(-\xi)+\langle -z/2,\nabla\Lambda_{\mu_\xi}(-\xi)\rangle.$$

Using the facts that $\Lambda_{\mu_\xi}(z/2-\xi)\leqslant\frac{1}{2}(\Lambda_{\mu_\xi}(z)+\Lambda_{\mu_\xi}(-2\xi))\leqslant\frac{9p}{4}$, $\Lambda_{\mu_\xi}(-\xi)\geqslant 0$ and $\langle -z/2,\nabla\Lambda_{\mu_\xi}(-\xi)\rangle\leqslant\frac{9p}{4}$ (by a last application of Lemma 16.4 for the pair $-z,-\xi$) we see that $\Lambda_\mu(z/2)\leqslant 9p/2$, which shows that $\Lambda_\mu(z/9)\leqslant p$. Using the same argument for $-z$ we finally conclude that

$$\Lambda_p(\mu_\xi)\subseteq 9\Lambda_p(\mu),$$

and the result follows. $\square$

We will also use the known upper bound for $\mathrm{Vol}_n(Z_p(\mu))^{1/n}$:

Fact 16.11. *Let ν be a log-concave probability measure on $\mathbb{R}^n$ with $\mathrm{bar}(\nu)=0$. For every $2\leqslant p\leqslant n$,*

$$\mathrm{Vol}_n(Z_p(\nu))^{1/n}\leqslant C\sqrt{p}\,\mathrm{Vol}_n(Z_2(\nu))^{1/n}.$$

Proof of the upper bound. Since

$$[\det\mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}\simeq\sqrt{n}\,\mathrm{Vol}_n(Z_2(\mu_\xi))^{1/n},$$

applying Fact 16.11 we get

$$\inf_{\xi\in\frac{1}{2}\Lambda_p(\mu)}\mathrm{Vol}_n(Z_p(\mu_\xi))^{1/n}\leqslant C\sqrt{\frac{p}{n}}\inf_{\xi\in\frac{1}{2}\Lambda_p(\mu)}[\det\mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}.$$

From Proposition 16.10 we know that $Z_p(\mu_\xi) \simeq Z_p(\mu)$ for all $\xi \in \frac{1}{2}\Lambda_p(\mu)$. It follows that

$$\mathrm{Vol}_n(Z_p(\mu))^{1/n} \leqslant C\sqrt{\frac{p}{n}} \inf_{\xi \in \frac{1}{2}\Lambda_p(\mu)} [\det \mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}.$$

This completes the proof of Theorem 16.9. $\square$

An immediate consequence of Theorem 16.9 is the following.

Theorem 16.12 (Klartag-E. Milman)**.** *Let μ be a log-concave probability on $\mathbb{R}^n$ with $\mathrm{bar}(\mu) = 0$. For every $1 \leqslant p \leqslant q \leqslant n$,*

$$\frac{\mathrm{Vol}_n(Z_p(\mu))^{1/n}}{\sqrt{p}} \geqslant c \frac{\mathrm{Vol}_n(Z_q(\mu))^{1/n}}{\sqrt{q}},$$

where $c > 0$ is an absolute constant.

Proof. Since $\Lambda_p(\mu) \subseteq \Lambda_q(\mu)$, we have

$$\inf_{\xi \in \frac{1}{2}\Lambda_p(\mu)} [\det \mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}} \geqslant \inf_{\xi \in \frac{1}{2}\Lambda_q(\mu)} [\det \mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}.$$

Then, we apply the formula of Theorem 16.9. $\square$

Remark 16.13. Another consequence of Theorem 16.9 is that if $x_0 \in \frac{1}{2}\Lambda_p(\mu)$ is such that

$$[\det \mathrm{Cov}(\mu_{x_0})]^{\frac{1}{2n}} \simeq \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} [\det \mathrm{Cov}(\mu_x)]^{\frac{1}{2n}},$$

then, using (16.2) as well, we get that

$$\mathrm{Vol}_n(Z_p(\mu_{x_0}))^{1/n} \simeq \sqrt{\frac{p}{n}}\, [\det \mathrm{Cov}(\mu_{x_0})]^{\frac{1}{2n}}.$$

Naturally, the aim is to show a similar equivalence for the corresponding quantities of the measure μ instead of those of μ_{x_0}. To accomplish this, we need to be able to prove that

$$\inf_{x \in \frac{1}{2}\Lambda_p(\mu)} [\det \mathrm{Cov}(\mu_x)]^{\frac{1}{2n}} \geqslant \frac{1}{\gamma} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}} \tag{16.4}$$

for as small a constant $\gamma \geqslant 1$ and for as large an interval of $p \in [1, n]$ as possible. Observe that if we establish (16.4) for some p and $\gamma \geqslant 1$, then we have by Theorem 16.9 that

$$\mathrm{Vol}_n(Z_p(\mu))^{1/n} \geqslant \frac{c}{\gamma}\sqrt{\frac{p}{n}}\, [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

and hence, by the definition of L_μ and by Theorem 9.1, we can conclude that

$$L_\mu = \|\mu\|_\infty^{1/n} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}} \leqslant c' \frac{[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}}{\mathrm{Vol}_n(Z_n(\mu))^{1/n}} \leqslant c' \frac{[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}}{\mathrm{Vol}_n(Z_p(\mu))^{1/n}} \leqslant c''\gamma\sqrt{\frac{n}{p}}, \tag{16.5}$$

where $c > 0$, c' and c'' are absolute constants (independent of the measure μ, the dimension n, or p and γ).

Klartag and E. Milman defined a hereditary parameter $q_\sharp^H(\mu)$ for isotropic measures μ, and gave a lower bound of the correct order for the volume radius of $Z_p(\mu)$ for every p up to that parameter. We shall work with a different parameter, introduced afterwards by Vritsiou.

Definition 16.14. Let μ be an isotropic log-concave measure in $\mathbb{R}^n$. For any $\gamma \geqslant 1$, we define

$$r_\sharp(\mu,\gamma) := \max\{1 \leqslant k \leqslant n-1 : \exists E \in G_{n,k} \text{ such that } L_{\pi_E\mu} \leqslant \gamma\} \tag{16.6}$$

In other words, $r_\sharp(\mu,\gamma)$ is the largest dimension $\leqslant n-1$ in which we can find at least one marginal of μ that has isotropic constant bounded above by γ; as a convention, when μ is an 1-dimensional measure, we set $r_\sharp(\mu,\gamma)=1$ for every γ. Next, we define a "hereditary" variant of $r_\sharp(\mu,\gamma)$ which controls the behavior of all marginals of μ with respect to $r_\sharp(\cdot,\gamma)$: set

$$r_\sharp^H(\mu,\gamma) := n \inf_k \inf_{E\in G_{n,k}} \frac{r_\sharp(\pi_E\mu,\gamma)}{k}. \tag{16.7}$$

A modification of the method of Klartag and E. Milman establishes the lower bound $\mathrm{Vol}_n(Z_p(\mu))^{1/n} \geqslant c\gamma^{-1}\sqrt{p/n}$ for all $p \leqslant r_\sharp(\mu,\gamma)$. Also, both $q_\sharp(\mu)$ and $q_*(\mu)$ are dominated by $r_\sharp(\mu,\gamma_0)$ for some absolute constant $\gamma_0 \geqslant 1$. Thus, the next theorem extends the result of Klartag and E. Milman.

Theorem 16.15 (Klartag-E. Milman, Vritsiou)**.** *Let μ be an isotropic log-concave measure in $\mathbb{R}^n$ and let $\gamma \geqslant 1$. Then, for every $p \in [1, r_\sharp^H(\mu,\gamma)]$, we have that*

$$|Z_p(\mu)|^{1/n} \geqslant \frac{c}{\gamma}\sqrt{\frac{p}{n}},$$

where $c>0$ is an absolute constant.

In Remark 16.13 we explained that what we have to show is that

$$[\det \mathrm{Cov}(\mu_x)]^{\frac{1}{2n}} \geqslant \frac{c'}{\gamma}$$

for every $x \in \frac{1}{2}\Lambda_p(\mu)$. The reason for introducing the hereditary parameter $r_\sharp^H(\mu,\gamma)$ is that in order to compare $\det\mathrm{Cov}(\mu_x)$ with $\det\mathrm{Cov}(\mu)$ we need to compare the corresponding eigenvalues of each covariance matrix, taken in increasing order, one pair at a time; this requires that we have control over $r_\sharp(\pi_E\mu,\gamma)$ of several marginals of μ of different dimensions.

In what follows, we denote the eigenvalues of $\mathrm{Cov}(\mu_x)$ by $\lambda_1^x \leqslant \lambda_2^x \leqslant \cdots \leqslant \lambda_n^x$, and we write E_k for the k-dimensional subspace which is spanned by eigenvectors corresponding to the first k eigenvalues of $\mathrm{Cov}(\mu_x)$.

Lemma 16.16. *For every two integers $1 \leqslant s \leqslant k \leqslant n$ we have that*

$$\sqrt{\lambda_k^x} \geqslant c_1 \sup_{F\in G_{E_k,s}} \mathrm{Vol}_n(Z_s(\pi_F\mu_x))^{1/s}, \tag{16.8}$$

where $c_1>0$ is an absolute constant.

Proof. Note that

$$\lambda_k^x = \max_{\theta\in S_{E_k}} \int_{E_k} \langle z,\theta\rangle^2\, d\pi_{E_k}\mu_x(z) = \sup_{F\in G_{E_k,s}} \max_{\theta\in S_F} \int_F \langle z,\theta\rangle^2\, d\pi_F\mu_x(z). \tag{16.9}$$

This is because, for every subspace F of E_k and every $\theta \in S_F \subseteq S_{E_k}$, we have that

$$\int_F \langle z,\theta\rangle^2\, d\pi_F\mu_x(z) = \int_{\mathbb{R}^n} \langle z,\theta\rangle^2\, d\mu_x(z) = \int_{E_k} \langle z,\theta\rangle^2\, d\pi_{E_k}\mu_x(z),$$

while λ_k^x is the largest eigenvalue of $\mathrm{Cov}(\pi_{E_k}\mu_x)$.

On the other hand, since μ_x is a centered, log-concave probability measure, which means that so are its s-dimensional marginals $\pi_F\mu_x$, we get from Theorem 9.1 that

$$\mathrm{Vol}_n(Z_s(\pi_F\mu_x))^{1/s} \simeq \frac{1}{\|f_{\pi_F\mu_x}\|_\infty^{1/s}} = \frac{[\det\mathrm{Cov}(\pi_F\mu_x)]^{\frac{1}{2s}}}{L_{\pi_F\mu_x}}. \tag{16.10}$$

Since $L_\nu \geqslant c$ for any isotropic measure ν, for some absolute constant $c>0$, it follows that

$$\mathrm{Vol}_n(Z_s(\pi_F\mu_x))^{1/s} \leqslant c'[\det\mathrm{Cov}(\pi_F\mu_x)]^{\frac{1}{2s}} \leqslant c' \max_{\theta\in S_F}\sqrt{\int_F \langle z,\theta\rangle^2\, d\pi_F\mu_x(z)}$$

for every $F\in G_{E_k,s}$, which combined with (16.9) gives us (16.8). □

To bound the right-hand side of (16.8) by an expression that involves $\det\mathrm{Cov}(\mu)$, we have to compare the volume of $Z_s(\pi_F\mu_x)$ to that of $Z_s(\pi_F\mu)$ (we are able to do that because of Proposition 16.10). Recall that for some fixed $x\in\frac{1}{2}\Lambda_p(\mu)$ and every integer $k\leqslant n$, we denote by E_k the k-dimensional subspace which is spanned by eigenvectors corresponding to the first k eigenvalues of $\mathrm{Cov}(\mu_x)$. For convenience, we also set $s_k^x := r_\sharp(\pi_{E_k}\mu,\gamma)$. The right choice of s is prompted by the following lemma.

Lemma 16.17. *We have*

$$\sup_{F\in G_{E_k,s_k^x}} \mathrm{Vol}_{s_k^x}(Z_{s_k^x}(\pi_F\mu))^{1/s_k^x} \geqslant \frac{c_2}{\gamma}\,[\det\mathrm{Cov}(\mu)]^{\frac{1}{2n}} = \frac{c_2}{\gamma},$$

where $c_2>0$ is an absolute constant.

Proof. As in (16.10), we can write

$$\mathrm{Vol}_{s_k^x}(Z_{s_k^x}(\pi_F\mu))^{1/s_k^x} \geqslant \frac{c_2}{\|f_{\pi_F\mu}\|_\infty^{1/s_k^x}} = \frac{c_2\,[\det\mathrm{Cov}(\pi_F\mu)]^{\frac{1}{2s_k^x}}}{L_{\pi_F\mu}}$$

for some absolute constant $c_2>0$ and for every $F\in G_{E_k,s_k^x}$. Since μ is isotropic, we have

$$[\det\mathrm{Cov}(\pi_F\mu)]^{1/(2s_k^x)} = [\det\mathrm{Cov}(\mu)]^{1/(2n)} = 1.$$

Moreover, by the definition of $s_k^x = r_\sharp(\pi_{E_k}\mu,\gamma)$, there is at least one s_k^x-dimensional subspace of E_k, say F_0, such that the marginal $\pi_{F_0}(\pi_{E_k}\mu)\equiv\pi_{F_0}\mu$ has isotropic constant bounded above by γ. Combining all of these, we get

$$\sup_{F\in G_{E_k,s_k^x}} \mathrm{Vol}_{s_k^x}(Z_{s_k^x}(\pi_F\mu))^{1/s_k^x} \geqslant \mathrm{Vol}_{s_k^x}(Z_{s_k^x}(\pi_{F_0}\mu))^{1/s_k^x} \geqslant \frac{c_2}{\gamma}$$

as required. □

In order to compare $Z_{s_k^x}(\pi_F\mu_x)$ and $Z_{s_k^x}(\pi_F\mu)$ for every $F\in G_{E_k,s_k^x}$, we have two cases to consider:

(i) If $p\leqslant s_k^x = r_\sharp(\pi_{E_k}\mu,\gamma)$, then by Proposition 16.10 (and the fact that $x\in\frac{1}{2}\Lambda_p(\mu)\subseteq\frac{1}{2}\Lambda_{s_k^x}(\mu)$) we have that $Z_{s_k^x}(\mu_x)\simeq Z_{s_k^x}(\mu)$, and therefore for every $F\in G_{E_k,s_k^x}$,

$$Z_{s_k^x}(\pi_F\mu_x) = P_F\big(Z_{s_k^x}(\mu_x)\big) \simeq P_F\big(Z_{s_k^x}(\mu)\big) = Z_{s_k^x}(\pi_F\mu)$$

as well.

(ii) If $s_k^x<p$, then we can write

$$Z_{s_k^x}(\pi_F\mu_x) \supseteq c_0\frac{s_k^x}{p}Z_p(\pi_F\mu_x) \supseteq c_0'\frac{s_k^x}{p}Z_p(\pi_F\mu) \supseteq c_0'\frac{s_k^x}{p}Z_{s_k^x}(\pi_F\mu)$$

for some absolute constants $c_0, c_0' > 0$. We also recall that since

$$p \leqslant r_\sharp^H(\mu,\gamma) = n \inf_k \inf_{E\in G_{n,k}} \frac{r_\sharp(\pi_E\mu,\gamma)}{k} \leqslant \frac{n}{k} r_\sharp(\pi_{E_k}\mu,\gamma),$$

it holds that $s_k^x/p = r_\sharp(\pi_{E_k}\mu,\gamma)/p \geqslant k/n$.

To summarize the above, we see that in any case and for every $F \in G_{E_k, s_k^x}$,

$$Z_{s_k^x}(\pi_F\mu_x) \supseteq c_0'' \min\Big\{1, \frac{s_k^x}{p}\Big\} Z_{s_k^x}(\pi_F\mu) \supseteq c_0'' \frac{k}{n} Z_{s_k^x}(\pi_F\mu), \tag{16.11}$$

where $c_0'' > 0$ is a small enough absolute constant. We now have everything we need to bound $\mathrm{Vol}_n(Z_p(\mu))^{1/n}$ from below.

Proof of Theorem 16.15. Combining Lemmas 16.16 and 16.17 with (16.11), we see that for every $p \in [1, r_\sharp^H(\mu,\gamma)]$ and for every $x \in \frac{1}{2}\Lambda_p(\mu)$,

$$[\det \mathrm{Cov}(\mu_x)]^{1/2} = \prod_{k=1}^n \sqrt{\lambda_k^x} \geqslant \prod_{k=1}^n \frac{c}{\gamma}\frac{k}{n} = \frac{c^n}{\gamma^n}\frac{n!}{n^n}.$$

If we take n-th roots, the theorem then follows from Theorem 16.9. $\square$

We will now show that $r_\sharp^H$ is equivalent to a hereditary variant of the parameter $q_{-c}(\mu,\delta)$ of Dafnis and Paouris. Recall that for every $\delta \geqslant 1$ and every isotropic log-concave measure μ, we write

$$q_{-c}(\mu,\delta) := \max\{1 \leqslant p \leqslant n-1 : I_{-p}(\mu) \geqslant \delta^{-1} I_2(\mu) = \delta^{-1}\sqrt{n}\}.$$

Now, set

$$q_{-c}^H(\mu,\delta) := n \inf_k \inf_{E\in G_{n,k}} \frac{\lfloor q_{-c}(\pi_E\mu,\delta)\rfloor}{k}.$$

Then, the following theorem comparing $r_\sharp^H$ and q_{-c}^H holds.

Theorem 16.18. *There exist absolute constants $C_1, C_2 > 0$ such that for every isotropic measure μ on $\mathbb{R}^n$ and every $\gamma \geqslant 1$,*

$$r_\sharp^H(\mu,\gamma) \leqslant q_{-c}^H(\mu, C_1\gamma) \leqslant r_\sharp^H(\mu, C_2\gamma). \tag{16.12}$$

Note that by Theorem 16.15 and Remark 16.13 we get:

Theorem 16.19. *Let μ be an isotropic log-concave measure in $\mathbb{R}^n$. Then,*

$$L_\mu \leqslant C\gamma\sqrt{\frac{n}{r_\sharp^H(\mu,\gamma)}} \leqslant C\gamma\sqrt{\frac{n}{q_{-c}^H\Big(\mu, \frac{C_1}{C_2}\gamma\Big)}} \tag{16.13}$$

for every $\gamma \geqslant C_2/C_1$.

Since $q_{-c}^H(\mu,\xi_0) \geqslant c\sqrt{n}$ for an absolute constant ξ_0, from Theorem 16.19 we get (at least) once again the bound $L_\mu \leqslant C\sqrt[4]{n}$.

For the proof of Theorem 16.18 we need the following consequence of Theorem 16.15.

Lemma 16.20. *There exists a positive absolute constant C_1 such that, for every n-dimensional isotropic measure μ and every $\gamma \geqslant 1$,*

$$r_\sharp^H(\mu,\gamma) \leqslant \lfloor q_{-c}(\mu, C_1\gamma)\rfloor.$$

In other words, for every $p \leqslant \lceil r_\sharp^H(\mu,\gamma)\rceil$ we have that

$$I_{-p}(\mu) \geqslant \frac{1}{C_1\gamma} I_2(\mu) = \frac{1}{C_1\gamma}\sqrt{n}.$$

Proof. Set $p_\gamma := r_\sharp^H(\mu,\gamma)$ and observe that

$$\mathrm{Vol}_n(Z_{\lceil p_\gamma\rceil}(\mu))^{1/n} \geqslant \mathrm{Vol}_n(Z_{p_\gamma}(\mu))^{1/n} \geqslant \frac{c'}{\gamma}\sqrt{\frac{\lceil p_\gamma\rceil}{n}}.$$

By Hölder's and Santaló's inequalities, this gives us that

$$w_{-\lceil p_\gamma\rceil}\big(Z_{\lceil p_\gamma\rceil}(\mu)\big) \geqslant w_{-n}\big(Z_{\lceil p_\gamma\rceil}(\mu)\big) \geqslant \frac{\mathrm{Vol}_n(Z_{\lceil p_\gamma\rceil}(\mu))^{1/n}}{\omega_n^{1/n}} \geqslant \frac{c''}{\gamma}\sqrt{\lceil p_\gamma\rceil}.$$

Since $r_\sharp^H(\mu,\gamma) \leqslant r_\sharp(\mu,\gamma) \leqslant n-1$ by definition, we have $\lceil p_\gamma\rceil \leqslant n-1$, and thus we can use (13.6) to conclude that

$$I_{-\lceil p_\gamma\rceil}(\mu) \geqslant \frac{1}{C_1\gamma}\sqrt{n}$$

for some absolute constant $C_1>0$. This completes the proof. □

Proof of Theorem 16.18. For the left-hand side inequality of (16.12) we apply Lemma 16.20 for every marginal $\pi_E\mu$ of μ; we get that

$$r_\sharp^H(\pi_E\mu,\gamma) \leqslant \lfloor q_{-c}(\pi_E\mu, C_1\gamma)\rfloor.$$

In addition, we observe that

$$r_\sharp^H(\mu,\gamma) = n\inf_k \inf_{F\in G_{n,k}} \frac{r_\sharp(\pi_F\mu,\gamma)}{k} \leqslant n \inf_{s\leqslant \dim E}\ \inf_{F\in G_{E,s}} \frac{r_\sharp(\pi_F\mu,\gamma)}{s} = \frac{n}{\dim E}\, r_\sharp^H(\pi_E\mu,\gamma),$$

which means that for every integer k, for every subspace $E\in G_{n,k}$,

$$r_\sharp^H(\mu,\gamma) \leqslant \frac{n}{k}\, r_\sharp^H(\pi_E\mu,\gamma) \leqslant \frac{n}{k}\lfloor q_{-c}(\pi_E\mu, C_1\gamma)\rfloor,$$

or equivalently that $r_\sharp^H(\mu,\gamma) \leqslant q_{-c}^H(\mu, C_1\gamma)$.

For the other inequality of (16.12) we recall the formula

$$I_{-k}(\mu) \simeq \sqrt{n}\left(\int_{G_{n,k}} f_{\pi_E\mu}(0)\, d\nu_{n,k}(E)\right)^{-1/k} \geqslant \frac{1}{C_1\gamma} I_2(\mu) = \frac{1}{C_1\gamma}\sqrt{n},$$

namely if $k \leqslant \lfloor q_{-c}(\mu, C_1\gamma)\rfloor$, then there must exist at least one $E\in G_{n,k}$ such that $f_{\pi_E\mu}(0) \leqslant (C_1'\gamma)^k$ for some absolute constant C_1' (depending only on C_1). Since $\pi_E\mu$ is isotropic, we have

$$L_{\pi_E\mu} = \|f_{\pi_E\mu}\|_\infty^{1/k} \leqslant e(f_{\pi_E\mu}(0))^{1/k} \leqslant C_2\gamma.$$

This means that

$$r_\sharp(\mu, C_2\gamma) \geqslant \lfloor q_{-c}(\mu, C_1\gamma)\rfloor,$$

and the same will hold for every marginal $\pi_F\mu$ of μ. The inequality now follows from the definitions of $r_\sharp^H(\mu, C_2\gamma)$ and $q_{-c}^H(\mu, C_1\gamma)$. □

17 E. Milman's bound for the mean width

In this section we describe E. Milman's almost sharp estimate for the mean width of an isotropic convex body in $\mathbb{R}^n$. In fact, the next theorem gives sharp bounds for the mean width $w(Z_q(K))$ of the L_q-centroid bodies $Z_q(K)$ of K.

Theorem 17.1 (E. Milman)**.** *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$w(K) \leqslant C\sqrt{n}(\log n)^2 L_K, \tag{17.1}$$

and for every $q \geqslant 1$ we have

$$w(Z_q(K)) \leqslant C\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\} L_K, \tag{17.2}$$

where $C > 0$ is an absolute constant.

The starting point of E. Milman is the idea to use Dudley-type estimates. Recall that the covering number $N(K,T)$ of K by T is the least number of translates of T whose union covers K. For every $k \geqslant 1$ we set

$$e_k(K,T) := \inf\{s > 0 : N(K,sT) \leqslant 2^k\}. \tag{17.3}$$

In particular, the k-th entropy number of K is $e_k(K) := e_k(K,B_2^n)$. Dudley's bound for the mean width takes the following form: if K is a symmetric convex body in $\mathbb{R}^n$, then

$$\sqrt{n}w(K) \leqslant c_1 \sum_{k\geqslant 1} \frac{1}{\sqrt{k}} e_k(K,B_2^n), \tag{17.4}$$

where $c_1 > 0$ is an absolute constant. If K is an isotropic convex body in $\mathbb{R}^n$ then one can check in an "elementary" way that

$$\log N(K, sB_2^n) \leqslant C_1 \frac{n^{3/2}L_K}{s} \tag{17.5}$$

for every $s > 0$. Therefore,

$$e_k(K,B_2^n) = \inf\{s > 0 : N(K,sB_2^n) \leqslant 2^k\} \leqslant C_2\sqrt{n}L_K\frac{n}{k}. \tag{17.6}$$

Combining this estimate with (17.4) we get

$$w(K) \leqslant C_3 \sum_{k\geqslant 1} \frac{1}{\sqrt{k}}\frac{n}{k}\, L_K, \tag{17.7}$$

which finally gives the bound $w(K) \leqslant Cn^{3/4}L_K$.

E. Milman uses a stronger version of Dudley's bound, which had been proved by V. Milman and Pisier. For every $k \geqslant 1$ they introduced the parameter

$$v_k(K) := \sup\{\mathrm{vrad}(P_F(K)) : F \in G_{n,k}\}. \tag{17.8}$$

Note that, for every $F \in G_{n,k}$,

$$\mathrm{Vol}_k(P_F(K)) \leqslant N(P_F(K), e_kP_F(B_2^n))\mathrm{Vol}_k(e_kB_F) \leqslant N(K, e_k(K)B_2^n)e_k^k\mathrm{Vol}_k(B_F) \leqslant (2e_k)^k\mathrm{Vol}_k(B_F), \tag{17.9}$$

and hence,

$$v_k(K) \leqslant 2e_k(K). \tag{17.10}$$

From (17.10) it is clear that the next theorem gives an estimate which is stronger than (17.4).

Theorem 17.2 (V. Milman-Pisier)**.** *For every symmetric convex body K in $\mathbb{R}^n$ one has*

$$\sqrt{n}w(K) \leqslant c_2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \mathrm{Rad}_k(K) v_k(K), \tag{17.11}$$

where $\mathrm{Rad}_k(K) := \sup\{\mathrm{Rad}(X_{P_F(K)}) : F \in G_{n,k}\}$*, and* $\mathrm{Rad}(Y) \leqslant c_3 \log(d(Y, \ell_2^{\dim(Y)})+1)$ *is the Rademacher constant of* Y*.*

A direct consequence of Theorem 17.2 is the inequality

$$\sqrt{n}w(K) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(K), \tag{17.12}$$

where $\lesssim$ denotes that (17.12) holds true up to a fixed power of $\log n$. We shall apply (17.12) for an isotropic convex body K in $\mathbb{R}^n$. In this case we know that

$$Z_n(K) \simeq Z_\infty(K) \supseteq K, \tag{17.13}$$

and hence, in order to prove (17.1), it suffices to obtain an upper bound fror $w(Z_n(K))$, and more generally for $w(Z_q(K))$, $1 \leqslant q \leqslant n$. From (17.12) we have

$$\sqrt{n}w(Z_q(K)) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(Z_q(K)). \tag{17.14}$$

It is necessary to estimate the parameter $v_k(Z_q(K))$; we consider any $F \in G_{n,k}$ and try to give an upper bound for

$$\mathrm{vrad}(P_F(Z_q(K))) = \left(\frac{\mathrm{Vol}_k(P_F(Z_q(K)))}{\mathrm{Vol}_k(B_2^k)}\right)^{1/k}. \tag{17.15}$$

At this point we use the fact that if μ is an isotropic log-concave probability measure in $\mathbb{R}^n$ then for every $F \in G_{n,k}$ we have $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$. Since $\pi_F(\mu)$ is an isotropic log-concave probability measure on F we may use the next estimates:

Lemma 17.3. *If ν is an isotropic log-concave probability measure in $\mathbb{R}^k$ then*

$$\mathrm{vrad}(Z_q(\nu)) \leqslant c_4\sqrt{q} \quad q \leqslant k, \tag{17.16}$$

and

$$\mathrm{vrad}(Z_q(\nu)) \leqslant c_5(q/k)\sqrt{k} \quad q \geqslant k. \tag{17.17}$$

Proof. We have already seen (17.16). Since $\mathrm{vrad}(Z_k(\nu)) \leqslant c_4\sqrt{k}$ and $Z_q(\nu) \subseteq c\frac{q}{k} Z_k(\nu)$ for all $q \geqslant k$, we easily get (17.17). □

Proof of Theorem 17.1. Applying Lemma 17.3 for $\nu = \pi_F(\mu_K)$, we get

$$v_k(Z_q(K)) \leqslant c_6 \sqrt{\frac{q}{k}} \max(\sqrt{q}, \sqrt{k}) L_K. \tag{17.18}$$

It follows that

$$\begin{aligned} L_K^{-1}\sqrt{n}w(Z_q(K)) &\lesssim L_K^{-1} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(z_q(K)) \simeq \sum_{k=1}^{n} \max\left(\sqrt{\frac{q}{k}}, \frac{q}{k}\right) \\ &\simeq q \sum_{k=1}^{q} \frac{1}{k} + \sqrt{q} \sum_{k=q}^{n} \frac{1}{\sqrt{q}} \simeq q \log q + \sqrt{q}\sqrt{n} \leqslant \sqrt{n}\sqrt{q} \log q. \end{aligned} \tag{17.19}$$

This concludes the proof of (17.2) and of the theorem. □

18 Sub-Gaussian directions

We have seen that every $\theta \in S^{n-1}$ is a ψ_1-direction for any convex body K in $\mathbb{R}^n$ with an absolute constant C. An open question, asked by V. Milman, is if there exists an absolute constant $C > 0$ such that every K has at least one sub-Gaussian direction (ψ_2-direction) with constant C. It was first proved by Klartag in that for every centered convex body K of volume 1 in $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$\mathrm{Vol}_n(\{x \in K : |\langle x, \theta\rangle| \geqslant ct\|\langle \cdot, \theta\rangle\|_2\}) \leqslant e^{-\frac{t^2}{[\log(t+1)]^{2a}}} \tag{18.1}$$

for all $t \geqslant 1$, where $a = 3$ (equivalently, $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^a \|\langle \cdot, \theta\rangle\|_2$). The best known estimate is due to Giannopoulos, Paouris and Valettas who showed that the body $\Psi_2(K)$ with support function $y \mapsto \|\langle \cdot, y\rangle\|_{L_{\psi_2}(K)}$ has volume

$$\left(\frac{\mathrm{Vol}_n(\Psi_2(K))}{\mathrm{Vol}_n(Z_2(K))|}\right)^{1/n} \leqslant c\sqrt{\log n}. \tag{18.2}$$

From (18.2) it follows immediately that there exists at least one sub-Gaussian direction for K with constant $b \leqslant C\sqrt{\log n}$.

Using Theorem 17.1, Brazitikos and Hioni proved that if K is isotropic then logarithmic bounds for $\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)}$ hold true with probability polynomially close to 1: For any $a > 1$ one has

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^{3/2} \max\left\{\sqrt{\log n}, \sqrt{a}\right\} L_K$$

for all θ in a subset Θ_a of S^{n-1} with $\sigma(\Theta_a) \geqslant 1 - n^{-a}$, where $C > 0$ is an absolute constant.

Here, we consider the question if one can have an estimate of this type *for all* directions θ of a subspace $F \in G_{n,k}$ of dimension k increasing to infinity with n. We say that $F \in G_{n,k}$ is a *sub-Gaussian subspace* for K with constant $b > 0$ if

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_\alpha}(K)} \leqslant b\|\langle \cdot, \theta\rangle\|_2 \tag{18.3}$$

for all $\theta \in S_F := S^{n-1} \cap F$. We will show that if K is isotropic then a random subspace of dimension $(\log n)^4$ is sub-Gaussian with constant $b \simeq (\log n)^2$.

Theorem 18.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $k \simeq (\log n)^4$ then there exists a subset Γ of $G_{n,k}$ with $\nu_{n,k}(\Gamma) \geqslant 1 - n^{-(\log n)^3}$ such that*

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K \tag{18.4}$$

for all $F \in \Gamma$ and all $\theta \in S_F$, where $C > 0$ is an absolute constant.

We need the next fact on the diameter of k-dimensional projections of symmetric convex bodies.

Proposition 18.2. *Let D be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leqslant k < n$ and $\alpha > 1$. Then there exists a subset $\Gamma_{n,k} \subset G_{n,k}$ with measure $\nu_{n,k}(\Gamma_{n,k}) \geqslant 1 - e^{-c_2\alpha^2 k}$ such that the orthogonal projection of D onto any subspace $F \in \Gamma_{n,k}$ satisfies*

$$R(P_F(D)) \leqslant c_3\alpha \max\{w(D), R(D)\sqrt{k/n}\}, \tag{18.5}$$

where $c_2 > 0, c_3 > 1$ are absolute constants.

Combining Proposition 18.2 with Theorem 17.1 and the fact that $R(Z_q(K)) \leqslant cqL_K$, we get:

Lemma 18.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Given $1 \leqslant q \leqslant n$ define $k_0(q)$ by the equation*

$$k_0(q) = \log^2(1+q)\max\{\log^2(1+q), n/q\}. \tag{18.6}$$

Then, for every $1 \leqslant k \leqslant k_0(q)$, a random $F \in G_{n,k}$ satisfies

$$R(P_F(Z_q(K))) \leqslant c_1\alpha\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\}L_K \tag{18.7}$$

with probability greater than $1 - e^{-c_2\alpha^2 k_0(q)}$, where $c_1, c_2 > 0$ are absolute constants.

Proof. Since $R(Z_q(K)) \leqslant cqL_K$ we see that

$$\frac{R(Z_q(K))\sqrt{k_0(q)}}{\sqrt{n}} \leqslant \frac{cq}{\sqrt{n}}\log(1+q)\max\left\{\log(1+q), \frac{\sqrt{n}}{\sqrt{q}}\right\}L_K = c\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\}L_K. \tag{18.8}$$

From Theorem 17.1 we have an upper bound of the same order for $w(Z_q(K))$. Then, we apply Proposition 18.2 for $Z_q(K)$. $\square$

Remark 18.4. Note that if $1 \leqslant s \leqslant k$ then the conclusion of Proposition 18.2 continues to hold for a random $F \in G_{n,s}$ with the same probability on $G_{n,s}$; this is an immediate consequence of Fubini's theorem and of the fact that $R(P_H(D)) \leqslant R(P_F(D))$ for every s-dimensional subspace H of a k-dimensional subspace F of $\mathbb{R}^n$.

Proof of Theorem 18.1. We define q_0 by the equation

$$q_0\log^2(1+q_0) = n. \tag{18.9}$$

Note that $q_0 \simeq n/(\log n)^2$ and $\log(1+q_0) \simeq \log n$. For every $2 \leqslant q \leqslant q_0$ we have $q\log^2(1+q) \leqslant n$, therefore

$$k_0(q) = \frac{n\log^2(1+q)}{q} \geqslant \frac{c_1 n\log^2(1+q_0)}{q_0} \tag{18.10}$$

for some absolute constant $c_1 > 0$, because $q \mapsto \log^2(1+q)/q$ is decreasing for $q \geqslant 4$. It follows that

$$k_0(q) \geqslant c_1\log^4(1+q_0) \geqslant c_2(\log n)^4 \tag{18.11}$$

for all $2 \leqslant q \leqslant q_0$.

Now, we fix $\alpha > 1$ and define

$$k_0 = c_1\log^4(1+q_0). \tag{18.12}$$

Using Lemma 18.3 and Remark 18.4, for every $q \leqslant q_0$ we can find a set $\Gamma_q \subseteq G_{n,k_0}$ with $\nu_{n,k_0}(\Gamma_q) \geqslant 1 - e^{-c\alpha^2 k_0}$ such that

$$R(P_F(Z_q(K))) \leqslant c_3\alpha\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\}L_K \leqslant c_3\alpha\sqrt{q}\log(1+q)L_K \tag{18.13}$$

for all $F \in G_{n,k_0}$. If $\Gamma := \bigcap_{s=1}^{\lfloor\log_2 q_0\rfloor}\Gamma_{2^s}$, then

$$\nu_{n,k_0}\big(G_{n,k_0}\setminus\Gamma\big) \leqslant \nu_{n,k_0}\Big(G_{n,k_0}\setminus\bigcap_{s=1}^{\lfloor\log_2 n\rfloor}\Gamma_{2^s}\Big) \leqslant c(\log n)e^{-c\alpha^2 k_0} \leqslant \frac{1}{n^{\log^3 n}} \tag{18.14}$$

if $\alpha \simeq 1$ is chosen large enough. Then for every $F \in \Gamma$, for all $\theta \in S_F$ and for every $1 \leqslant s \leqslant \lfloor \log_2 q_0 \rfloor$ we have

$$\frac{h_{Z_{2^s}(K)}(\theta)}{\sqrt{2^s}} = \frac{h_{P_F(Z_{2^s}(K))}(\theta)}{\sqrt{2^s}} \leqslant c_3\alpha \log(1+2^s)L_K \leqslant c_4\alpha(\log n)L_K. \tag{18.15}$$

Taking into account the fact that if $2^s \leqslant q < 2^{s+1}$ then

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant \frac{h_{Z_{2^{s+1}}(K)}(y)}{2^{s/2}} = \sqrt{2}\frac{h_{Z_{2^{s+1}}(K)}(y)}{2^{(s+1)/2}}, \tag{18.16}$$

we see that

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant c_5\alpha(\log n)L_K \tag{18.17}$$

for every $F \in \Gamma$, for all $\theta \in S_F$ and for every $2 \leqslant q \leqslant q_0$.

Next, observe that if $q_0 \leqslant q \leqslant n$ then we may write

$$\begin{aligned}\frac{h_{Z_q(K)}(y)}{\sqrt{q}} &\leqslant \frac{c_6 q}{q_0}\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q}} = \frac{c_6\sqrt{q}}{\sqrt{q_0}}\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}} \leqslant \frac{c_6\sqrt{n}}{\sqrt{q_0}}\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}} \\ &= c_6\log(1+q_0)\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}} \leqslant c_7(\log n)\frac{h_{Z_{q_0}(K)}(y)}{\sqrt{q_0}},\end{aligned} \tag{18.18}$$

and hence

$$\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \leqslant c_7\alpha(\log n)^2 L_K \tag{18.19}$$

for every $F \in \Gamma$, for all $\theta \in S_F$ and for every $q_0 \leqslant q \leqslant n$.

Recall that $\Psi_2(K)$ is the convex body with support function $h_{\Psi_2(K)}(y) = \|\langle \cdot, y\rangle\|_{L_{\psi_2}(K)}$. One also has

$$h_{\Psi_2(K)}(y) \simeq \sup_{q\geqslant 2}\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \simeq \sup_{2\leqslant q\leqslant n}\frac{h_{Z_q(K)}(y)}{\sqrt{q}} \tag{18.20}$$

because $h_{Z_q(K)}(y) \simeq h_{Z_n(K)}(y)$ for all $q \geqslant n$. Then, (18.17) and (18.19) and the fact that $\alpha \simeq 1$ show that

$$\|\langle \cdot, \theta\rangle\|_{L_{\psi_2}(K)} \leqslant C(\log n)^2 L_K \tag{18.21}$$

for every $F \in \Gamma$ and for all $\theta \in S_F$, where $C > 0$ is an absolute constant. □

19 Notes and References

1. The hyperplane conjecture appears for the first time in the work of Bourgain [13] on high-dimensional maximal functions associated with arbitrary convex bodies. The conjecture was stated in this form in the article of V. Milman and Pajor [50] and in the PhD Thesis of K. Ball [8].

Bourgain's article [13] concerned high-dimensional maximal functions associated with arbitrary convex bodies. He was interested in bounds for the L_p-norm of the maximal function

$$M_K f(x) = \sup\left\{\frac{1}{\mathrm{Vol}_n(tK)}\int_{tK}|f(x+y)|\,dy \mid t>0\right\}$$

of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where K is a centrally symmetric convex body in $\mathbb{R}^n$. Let $C_p(K)$ denote the best constant such that $\|M_K f\|_p \leqslant C_p(K)\|f\|_p$ is satisfied. Bourgain showed that there exists an absolute constant $C > 0$ (independent of n and K) such that

$$\|M_K\|_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} \leqslant C.$$

Earlier, Stein had proved in [65] that if $K = B_2^n$ is the Euclidean unit ball then $C_p(B_2^n)$ is bounded independently of the dimension for all $p > 1$. By the definition of M_K it is clear that in order to obtain a uniform bound on $\|M_K\|_{2\to 2}$ one can start with a suitable position $T(K)$ (where $T \in GL_n$) of K. Bourgain used the isotropic position; the property that played an important role in his argument was that when K is isotropic then $L_K \mathrm{Vol}_{n-1}(K\cap\theta^{\perp}) \simeq 1$ for all $\theta \in S^{n-1}$. Bourgain mentioned the fact that $L_K \geqslant c$ and asked whether a reverse inequality holds true.

The result for $\|M_K\|_{2\to 2}$ was generalized to all $p > 3/2$ by Bourgain [14] and, independently, by Carbery [22]. Afterwards, Müller [52] obtained dimension free maximal bounds for all $p > 1$, which however depend on L_K and on the maximal volume of hyperplane projections of K. In the case of the cube, Bourgain [17] showed that for every $p > 1$ there exists a constant $C_p > 0$ such that $C_p(B_\infty^n) \leqslant C_p$ for all n.

2. The bodies $K_p(\mu)$ were introduced by K. Ball (see [8] and [9]) who established their convexity. One should also mention Busemann's paper [20], where the case of a density which is the indicator of a convex body (and, say, $p = 1$) is proved. Ball [9] showed that if μ is an even isotropic log-concave measure then the body $T = K_{n+2}(\mu)$ is an isotropic symmetric convex body with $L_\mu \simeq L_T$. The observation that one can reduce the study of the behavior of the isotropic constants of all log-concave measures to the class of centrally symmetric convex bodies is due to Klartag [38].

3. Bourgain's bound $L_K = O(\sqrt[4]{n}\log n)$ appeared in [15]. We present a modification of his argument, which is due to Dar [24]. Bourgain showed in [16] that if K is a symmetric convex body and K is a ψ_2-body with constant b then one can improve this estimate to $L_K \leqslant cb\log(1+b)$. It was later proved by Klartag and E. Milman [42] that in this case one has $L_K \leqslant cb$.

4. There are several results confirming the hyperplane conjecture for important classes of convex bodies. To be more precise, let us say that a class $\mathcal{C}$ of symmetric convex bodies satisfies the hyperplane conjecture uniformly if there exists a positive constant C such that $L_K \leqslant C$ for all $K \in \mathcal{C}$.

The fact that the isotropic constants of unconditional convex bodies are bounded by an absolute constant is due to Bourgain, see [50]; a different proof using the Loomis-Whitney inequality is given by Schmuckenschläger in [63]. One more proof, leading to the bound $L_K \leqslant 1/\sqrt{2}$, can be found in the article [11] of Bobkov and Nazarov. A more general result, with a different proof, can be found in Milman and Pajor [50]. Uniform bounds are known for the isotropic constants of some other classes of convex bodies: convex bodies whose polar bodies contain large affine cubes (see again [50]), the unit balls of 2-convex spaces with a given constant α (see Klartag and E. Milman [41]), bodies with small diameter (in particular, the class of zonoids) etc.

Uniform boundedness of the isotropic constants of the unit balls of the Schatten classes was established by König, Meyer and Pajor in [44]. One of the main ingredients of the proof is a formula of Saint-Raymond from [59]. Before the work of König, Meyer and Pajor, Dar had obtained the estimate $L_{B(S_p^n)} \leqslant C\sqrt{\log n}$ in [25] (see [24] for the case $p = 1$).

Upper bounds for the isotropic constant of polytopes, which depend on the number of their vertices or facets, follow from results of Ball [10], Junge [36] and [37] and E. Milman [48]. A more geometric approach, that covers the case of not necessarily symmetric polytopes too, was given of Alonso-Gutiérrez in [7].

5. L_q-centroid bodies were introduced by Lutwak and Zhang [46] who used a different normalization. If K is a convex body in $\mathbb{R}^n$ then, for every $1 \leqslant q < \infty$, the body $\Gamma_q(K)$ was defined in [46] through its support function

$$h_{\Gamma_q(K)}(y) = \left(\frac{1}{c_{n,q}\mathrm{Vol}_n(K)}\int_K |\langle x, y\rangle|^q dx\right)^{1/q},$$

where

$$c_{n,q} = \frac{\omega_{n+q}}{\omega_2\omega_n\omega_{q-1}}.$$

In other words, $Z_q(K) = c_{n,q}^{1/q}\Gamma_q(K)$ if $\mathrm{Vol}_n(K) = 1$. The normalization of $\Gamma_q(K)$ is chosen so that $\Gamma_q(B_2^n) = B_2^n$ for every q. Lutwak, Yang and Zhang [47] have established the following L_q affine isoperimetric inequality (see Campi and Gronchi [21] for an alternative proof): For every $q \geqslant 1$,

$$\mathrm{Vol}_n(\Gamma_q(K)) \geqslant 1,$$

with equality if and only if K is a centered ellipsoid of volume 1.

Alesker's theorem is from [6]; it is the starting point of the work of Paouris. His study of the L_q-centroid bodies from an asymptotic point of view started with [53] and [54], where the parameter $q_*(\mu)$ is introduced. The deviation inequality was proved in [55] and the extension to negative moments in [56].

In the particular case of unconditional isotropic convex bodies, the inequality of Paouris had been previously proved by Bobkov and Nazarov (see [11] and [12]). The origin of the work of Bobkov and Nazarov is in the work of

Schechtman, Zinn and Schmuckenschläger on the volume of the intersection of two L_p^n-balls (see [60], [61], [62] and [64]). Before Paouris' theorem, Guédon and Paouris had studied in [34] the case of the unit balls of the Schatten classes.

6. Klartag's solution to the isomorphic slicing problem and his $O(\sqrt[4]{n})$ bound for the isotropic constant are from [39]. A second proof of the same estimate was given by Klartag and E. Milman in [42].

Klartag and E. Milman in [42] defined the "hereditary" variant

$$q_*^H(\mu) := n \inf_k \inf_{E\in G_{n,k}} \frac{q_*(\pi_E\mu)}{k} \tag{19.1}$$

of $q_*(\mu)$ and then, for every $q \leqslant q_*^H(\mu)$, they showed that $\mathrm{Vol}_n(Z_q(\mu))^{1/n} \geqslant c_3\sqrt{q/n}$ where $c_3 > 0$ is an absolute constant. An immediate consequence of this inequality and of the fact that $q_*^H(\mu) \geqslant c\sqrt{n}$ is an alternative proof of the bound $L_\mu = O(\sqrt[4]{n})$ for the isotropic constant. Vritsiou [66] introduced a new parameter $r_\sharp(\mu, A)$ that dominates $q_*^H(\mu)$ and modified the argument of Klartag and E. Milman to show that the lower bound $\mathrm{Vol}_n(Z_p(\mu))^{1/n} \geqslant cA^{-1}\sqrt{p/n}$ continues to hold for all $p \leqslant r_\sharp(\mu, A)$.

Giannopoulos, Paouris and Vritsiou observed in [32] that one can use Klartag's approach in order to give a purely convex geometric proof of the reverse Santaló inequality is due to

7. Bourgain, Klartag and V. Milman proved in [18] a reduction of the hyperplane conjecture to the case of convex bodies whose volume ratio is bounded by some absolute constant. The same fact follows from Klartag's approach in [38].

Dafnis and Paouris introduced in [23] the parameter $q_{-c}(\mu,\zeta) := \max\{p \geqslant 1 : I_2(\mu) \leqslant \zeta I_{-p}(\mu)\}$; among other things they proved that a positive answer to the hyperplane conjecture is equivalent to the existence of two absolute constants $C, \xi > 0$ such that $q_{-c}(K,\xi) \geqslant Cn$ for every isotropic convex body K in $\mathbb{R}^n$.

Giannopoulos, Paouris and Vritsiou proposed in [31] a reduction of the hyperplane conjecture to the study of the parameter $I_1(K, Z_q^\circ(K)) = \int_K \|\langle\cdot,x\rangle\|_q dx$ in the sense that it immediately recovers a bound that is slightly worse than Bourgain's and Klartag's bounds and opens the possibility for improvements: an upper bound of the form $I_1(K, Z_q^\circ(K)) \leqslant C_1 q^s \sqrt{n} L_K^2$ *for some* $q \geqslant 2$ and $\frac{1}{2} \leqslant s \leqslant 1$ and *for all* isotropic convex bodies K in $\mathbb{R}^n$ leads to the estimate

$$L_n \leqslant \frac{C_2\sqrt[4]{n}\log n}{q^{\frac{1-s}{2}}}.$$

8. A question, originally posed by V. Milman in the framework of convex bodies, asks if there exists an absolute constant $C > 0$ such that every centered convex body K of volume 1 has at least one ψ_2 direction with constant C. Klartag, using again properties of the logarithmic Laplace transform, proved in [40] that for every log-concave probability measure μ on $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$\mu\left(\{x : |\langle x,\theta\rangle| \geqslant ct\|\langle\cdot,\theta\rangle\|_2\}\right) \leqslant e^{-\frac{t^2}{(\log(t+1))^{2\alpha}}},$$

for all $1 \leqslant t \leqslant \sqrt{n}\log^\alpha n$, where $\alpha = 3$ (see also [28] for a first improvement). The best known estimate, is due to Giannopoulos, Paouris and Valettas who proved in [29] and [30] that one can always have $\alpha = 1/2$.

The main idea in all these works is to define the symmetric convex set $\Psi_2(\mu)$ whose support function is $h_{\Psi_2(\mu)}(\theta) = \|\langle\cdot,\theta\rangle\|_{\psi_2}$ and to estimate its volume. A logarithmic in the dimension bound on the volume radius of $\Psi_2(\mu)$ was first obtained by Klartag in [40] and then by Giannopoulos, Pajor and Paouris in [28]. The best known estimate $\mathrm{v.rad}(\Psi_2(\mu)) \leqslant c\sqrt{\log n}$ is proved in [29]. The main tool in the proof of this result is estimates for the covering numbers $N(Z_q(K), sB_2^n)$.

9. The question to obtain an upper bound for the mean width of an isotropic convex body

$$w(K) := \int_{S^{n-1}} h_K(x)\, d\sigma(x),$$

that is, the L_1-norm of the support function of K with respect to the Haar measure on the sphere, was open for a number of years. The upper bound $w(K) \leqslant cn^{3/4}L_K$ appeared in the Ph.D. Thesis of Hartzoulaki [35]. Other approaches leading to the same bound can be found in Pivovarov [58] and in Giannopoulos Paouris and Valettas [30]. E. Milman showed in [49] that if K is an isotropic convex body in $\mathbb{R}^n$ then, for all $q \geqslant 1$ one has

$$w(Z_q(K)) \leqslant C\log(1+q)\max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\} L_K$$

where $C>0$ is an absolute constant. In particular,

$$w(K)\leqslant C\sqrt{n}(\log n)^2 L_K.$$

The dependence on n is optimal up to the logarithmic term.

An interesting related question is to determine the distribution of the function $\theta\mapsto\|\langle\cdot,\theta\rangle\|_{\psi_2}$ on the unit sphere; that is, to understand whether most of the directions have ψ_2-norm that is, say, logarithmic in the dimension. For a discussion and partial results see [30]. As a consequence of E. Milman's theorem, Brazitikos and Hioni showed in [19] that the answer is affirmative. More precisely, they showed that for any $a>1$ one has

$$\|\langle\cdot,\theta\rangle\|_{L_{\psi_2}(K)}\leqslant C(\log n)^{3/2}\max\left\{\sqrt{\log n},\sqrt{a}\right\}L_K$$

for all θ in a subset Θ of S^{n-1} with $\sigma(\Theta)\geqslant 1-n^{-a}$, where $C>0$ is an absolute constant. Theorem 18.1 is from [26].

The dual problem to estimate as the respective L_1-norm of the Minkowski functional of K,

$$M(K):=\int_{S^{n-1}}\|x\|_K\,d\sigma(x),$$

when K is a centrally symmetric isotropic convex body, had not been studied until recently; partial non-trivial results can be found in [33]. The currently best known estimate estimate $M(K)\leqslant\frac{C\log^{2/5}(e+n)}{\sqrt[10]{n}L_K}$ is due to Giannopoulos and E. Milman (see [27]).

10. There are several other challenging conjectures and important results about isotropic log-concave measures. The first one is the *central limit problem*, that asks if the 1-dimensional marginals of high-dimensional isotropic log-concave measures are approximately Gaussian with high probability. It is generally known through results of Sudakov that, if μ is an isotropic probability measure in $\mathbb{R}^n$ that satisfies the *thin shell condition*,

$$\mu\left(\left|\frac{|x|}{\sqrt{n}}-1\right|\geqslant\varepsilon\right)\leqslant\varepsilon$$

for some $\varepsilon\in(0,1)$, then, for all directions θ in a subset A of S^{n-1} with $\sigma(A)\geqslant 1-\exp(-c_1\sqrt{n})$, one has

$$|\mu(\{x:\langle x,\theta\rangle\leqslant t\})-\Phi(t)|\leqslant c_2(\varepsilon+n^{-\alpha})\qquad\text{for all } t\in\mathbb{R},$$

where $\Phi(t)$ is the standard Gaussian distribution function and $c_1,c_2,\alpha>0$ are absolute constants. Thus, the central limit problem is reduced to the question whether every isotropic log-concave measure μ in $\mathbb{R}^n$ satisfies such a thin shell condition with $\varepsilon=\varepsilon_n$ tending to 0 as n tends to infinity. An affirmative answer to the problem was given by Klartag who obtained power-type estimates verifying the thin-shell condition; he showed that if μ is an isotropic log-concave measure in $\mathbb{R}^n$ then

$$\mathbb{E}\Big(\frac{|x|^2}{n}-1\Big)^2\leqslant\frac{C}{n^{\alpha}}$$

with some $\alpha\simeq 1/5$, and, as a consequence, that the density f_θ of $x\mapsto\langle x,\theta\rangle$ with respect to μ satisfies

$$\int_{-\infty}^{\infty}|f_\theta(t)-\gamma(t)|\,dt\leqslant\frac{1}{n^{\kappa}}$$

and

$$\sup_{|t|\leqslant n^{\kappa}}\left|\frac{f_\theta(t)}{\gamma(t)}-1\right|\leqslant\frac{1}{n^{\kappa}},$$

for all θ in a subset A of S^{n-1} with measure $\sigma(A)\geqslant 1-c_1\exp(-c_2\sqrt{n})$, where γ is the density of a standard Gaussian random variable, and c_1,c_2,κ are absolute constants. Although some sharper estimates were obtained afterwards, the following quantitative conjecture remains open:

> *There exists an absolute constant $C>0$ such that, for any $n\geqslant 1$ and any isotropic log-concave measure μ in $\mathbb{R}^n$, one has*
>
> $$\sigma_\mu^2:=\int_{\mathbb{R}^n}\big(|x|-\sqrt{n}\big)^2d\mu(x)\leqslant C^2.$$

Another conjecture concerns the *Cheeger constant* Is_μ of an isotropic log-concave measure μ: this is defined as the best constant $\kappa\geqslant 0$ such that

$$\mu^+(A)\geqslant\kappa\min\{\mu(A),1-\mu(A)\}$$

for every Borel subset A of $\mathbb{R}^n$, where $\mu^+(A)$ is the Minkowski content of A. The *Kannan-Lovász-Simonovits conjecture* asks if there exists an absolute constant $c > 0$ such that

$$\mathrm{Is}_n := \inf\{\mathrm{Is}_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\} \geqslant c.$$

Another way to formulate this conjecture is to ask for a *Poincaré inequality* to be satisfied by every isotropic log-concave measure μ in $\mathbb{R}^n$ with a constant $c > 0$ that is independent of the measure or the dimension n; more precisely, the KLS-conjecture is equivalent to asking if there exists an absolute constant $c > 0$ such that

$$c \int_{\mathbb{R}^n} \varphi^2 d\mu \leqslant \int_{\mathbb{R}^n} |\nabla\varphi|^2 d\mu$$

for every isotropic log-concave measure μ in $\mathbb{R}^n$ and for every smooth function φ with $\int_{\mathbb{R}^n} \varphi \, d\mu = 0$. For a detailed discussion of this area we refer to the book [2].

References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Part I*, Amer. Math. Soc., Mathematical Surveys and Monographs **202** (2015).

[2] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Amer. Math. Society, Mathematical Surveys and Monographs **196** (2014).

[3] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. **1200** (1986), Springer, Berlin.

[4] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics **94** (1989).

[5] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Second expanded edition. Encyclopedia of Mathematics and Its Applications 151, Cambridge University Press, Cambridge, 2014.

[6] S. Alesker, *ψ_2-estimate for the Euclidean norm on a convex body in isotropic position*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Oper. Theory Adv. Appl. **77** (1995), 1-4.

[7] D. Alonso-Gutiérrez, *A remark on the isotropy constant of polytopes*, Proc. Amer. Math. Soc. **139** (2011), 2565-2569.

[8] K. M. Ball, *Isometric problems in ℓ_p and sections of convex sets*, Ph.D. Dissertation, Trinity College, Cambridge (1986).

[9] K. M. Ball, *Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$*, Studia Math. **88** (1988), 69–84.

[10] K. M. Ball, *Normed spaces with a weak Gordon-Lewis property*, Lecture Notes in Mathematics **1470**, Springer, Berlin (1991), 36-47.

[11] S. G. Bobkov and F. L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 53-69.

[12] S. G. Bobkov and F. L. Nazarov, *Large deviations of typical linear functionals on a convex body with unconditional basis*, Stochastic Inequalities and Applications, Progr. Probab. 56, Birkhauser, Basel, 2003, 3–13.

[13] J. Bourgain, *On high dimensional maximal functions associated to convex bodies*, Amer. J. Math. **108** (1986), 1467-1476.

[14] J. Bourgain, *On the L^p-bounds for maximal functions associated to convex bodies*, Israel J. Math. **54** (1986), 257-265.

[15] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1469** (1991), 127–137.

[16] J. Bourgain, *On the isotropy constant problem for ψ_2-bodies*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 114-121.

[17] J. Bourgain, *On the Hardy-Littlewood maximal function for the cube*, Israel J. Math. **203** (2014), 275-294.

[18] J. Bourgain, B. Klartag and V. D. Milman, *Symmetrization and isotropic constants of convex bodies*, Geom. Aspects of Funct. Analysis, Lecture Notes in Math. **1850** (2004), 101-115.

[19] S. Brazitikos and L. Hioni, *Sub-Gaussian directions of isotropic convex bodies*, Journal of Mathematical Analysis and Applications **425** (2015), 919-927.

[20] H. Busemann, *A theorem on convex bodies of the Brunn-Minkowski type*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949), 27-31.

[21] S. Campi and P. Gronchi, *The L_p-Busemann-Petty centroid inequality*, Adv. in Math. **167** (2002), 128-141.

[22] A. Carbery, *Radial Fourier multipliers and associated maximal functions*, Recent Progress in Fourier Analysis, North-Holland Math. Studies **111** (1985), 49-56.

[23] N. Dafnis and G. Paouris, *Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture*, J. Funct. Anal. **258** (2010), 1933-1964.

[24] S. Dar, *Remarks on Bourgain's problem on slicing of convex bodies*, in Geometric Aspects of Functional Analysis, Operator Theory: Advances and Applications **77** (1995), 61-66.

[25] S. Dar, *Isotropic constants of Schatten class spaces*, Convex Geometric Analysis, MSRI Publications **34** (1998), 77-80.

[26] A. Giannopoulos, L. Hioni and A. Tsolomitis, *Geometry of random sections of isotropic convex bodies*, Bulletin of the Hellenic Mathematical Society **60** (2016), 20-40.

[27] A. Giannopoulos and E. Milman, *M-estimates for isotropic convex bodies and their L_q-centroid bodies*, in Geom. Aspects of Funct. Analysis, Lecture Notes in Mathematics **2116** (2014), 159-182.

[28] A. Giannopoulos, A. Pajor and G. Paouris, *A note on subgaussian estimates for linear functionals on convex bodies*, Proc. Amer. Math. Soc. **135** (2007), 2599–2606.

[29] A. Giannopoulos, G. Paouris and P. Valettas, *On the existence of subgaussian directions for log-concave measures*, Contemporary Mathematics **545** (2011), 103–122.

[30] A. Giannopoulos, G. Paouris and P. Valettas, *On the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies*, in Geom. Aspects of Funct. Analysis, Lecture Notes in Mathematics **2050** (2012), 227–254.

[31] A. Giannopoulos, G. Paouris and B-H. Vritsiou, *A remark on the slicing problem*, Journal of Functional Analysis **262** (2012), 1062-1086.

[32] A. Giannopoulos, G. Paouris and B-H. Vritsiou, *The isotropic position and the reverse Santaló inequality*, Israel J. Math. **203** (2014), 1-22.

[33] A. Giannopoulos, P. Stavrakakis, A. Tsolomitis and B-H. Vritsiou, *Geometry of the L_q-centroid bodies of an isotropic log-concave measure*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4569-4593.

[34] O. Guédon and G. Paouris, *Concentration of mass on the Schatten classes*, Ann. Inst. H. Poincare Probab. Statist. **43** (2007), 87-99.

[35] M. Hartzoulaki, *Probabilistic methods in the theory of convex bodies*, Ph.D. Thesis (March 2003), University of Crete.

[36] M. Junge, *On the hyperplane conjecture for quotient spaces of L_p*, Forum Math. **6** (1994), 617-635.

[37] M. Junge, *Proportional subspaces of spaces with unconditional basis have good volume properties*, in Geometric Aspects of Functional Analysis, Operator Theory: Advances and Applications **77** (1995), 121-129.

[38] B. Klartag, *An isomorphic version of the slicing problem*, J. Funct. Anal. **218** (2005), 372-394.

[39] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274-1290.

[40] B. Klartag, *Uniform almost sub-Gaussian estimates for linear functionals on convex sets*, Algebra i Analiz **19** (2007), 109–148.

[41] B. Klartag and E. Milman, *On volume distribution in 2-convex bodies*, Israel J. Math. **164** (2008), 221-249.

[42] B. Klartag and E. Milman, *Centroid Bodies and the Logarithmic Laplace Transform - A Unified Approach*, J. Funct. Anal. **262** (2012), 10-34.

[43] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky theorem*, Israel J. Math. **157** (2007), 193-207.

[44] H. König, M. Meyer and A. Pajor, *The isotropy constants of the Schatten classes are bounded*, Math. Ann. **312** (1998), 773-783.

[45] A. Litvak, V. D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), 95–124.

[46] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1-16.

[47] E. Lutwak, D. Yang and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111-132.

[48] E. Milman, *Dual mixed volumes and the slicing problem*, Adv. Math. **207** (2006), 566-598.

[49] E. Milman, *On the mean width of isotropic convex bodies and their associated L_p-centroid bodies*, Int. Math. Res. Not. IMRN (2015), no. 11, 3408-3423.

[50] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1376** (1989), 64–104.

[51] V. D. Milman and G. Pisier, *Gaussian processes and mixed volumes*, Annals of Probability **15** (1987), 292–304.

[52] D. Müller, A geometric bound for maximal functions associated to convex bodies, Pacific J. Math. **142** (1990), 297-312.

[53] G. Paouris, *On the ψ_2-behavior of linear functionals on isotropic convex bodies*, Studia Math. **168** (2005), 285-299.

[54] G. Paouris, *Concentration of mass and central limit properties of isotropic convex bodies*, Proc. Amer. Math. Soc. **133** (2005), 565-575.

[55] G. Paouris, *Concentration of mass in convex bodies*, Geom. Funct. Analysis **16** (2006), 1021-1049.

[56] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. **364** (2012), 287-308.

[57] G. Pisier, *A new approach to several results of V. Milman*, J. Reine Angew. Math. **393** (1989), 115–131.

[58] P. Pivovarov, *On the volume of caps and bounding the mean-width of an isotropic convex body*, Math. Proc. Cambridge Philos. Soc. **149** (2010), 317-331.

[59] J. Saint-Raymond, *Le volume des ideaux d'operateurs classiques*, Studia Math. **80** (1984), 63-75.

[60] G. Schechtman and M. Schmuckenschläger, *Another remark on the volume of the intersection of two L_p^n balls*, Geom. Aspects of Funct. Analysis (1989-90), Lecture Notes in Math., **1469**, Springer, Berlin (1991), 174-178.

[61] G. Schechtman and J. Zinn, *On the volume of the intersection of two L_p^n balls*, Proc. Am. Math. Soc. **110** (1990), 217-224.

[62] G. Schechtman and J. Zinn, *Concentration on the ℓ_p^n ball*, Geom. Aspects of Funct. Analysis, Lecture Notes in Math., **1745**, Springer, Berlin (2000), 245-256.

[63] M. Schmuckenschläger, *Die Isotropiekonstante der Einheitskugel eines n-dimensionalen Banachraumes mit einer 1-unbedingten Basis*, Arch. Math. (Basel) **58** (1992), 376-383.

[64] M. Schmuckenschläger, *Volume of intersections and sections of the unit ball of ℓ_p^n*, Proc. Amer. Math. Soc. **126** (1998), 1527-1530.

[65] E. Stein, *The development of square functions in the work of A. Zygmund*, Bull. Amer. Math. Soc. **7** (1982), 359-376.

[66] B-H. Vritsiou, *Further unifying two approaches to the hyperplane conjecture*, International Math. Research Notices (2014), no. 6, 1493-1514.

Apostolos Giannopoulos: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis 157-84, Athens, Greece.

E-mail: `apgiannop@math.uoa.gr`