5/1-5/5/17 MSRI Workshop 810:

Recent developments in Analytic Number Theory

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"Better than squareroot cancellation for multiplicative functions"

Focus on Dirichlet Characters

I. Intro (square-root cancellation + results)

II. Proof ideas in random case

III. Proof ideas in deterministic case

I. Let r be a larger prime, $1 \le x \le r$. Consider

$$\frac{1}{r-1}\sum_{\chi\,(\mathrm{mod}\,r)} \Big|\sum_{n\le x}\chi(n)\Big|^2 = \sum_{n_1,n_2\le x}\frac{1}{r-1}\sum_{\chi\,(r)}\chi(n_1)\overline{\chi(n_2)}$$

Vanishes unless $n_1 = n_2$

(*)
$$= \lfloor x \rfloor$$

Remarks • Provided $x \le 0.99\cdot(r-1)$, principal character doesn't dominate (i.e. $\frac{\lfloor x\rfloor^2}{r-1} \le 0.99\lfloor x\rfloor$)

• (*) is consistent with all/many sums satisfying

$$\Big|\sum_{n\le x}\chi(n)\Big| \sim \sqrt{x}$$

so-called "square-root cancellation"

L^2-moment of characters above. What about L^1?

$$\frac{1}{r-1}\sum_{\chi(r)}\Big|\sum_{n\le x}\chi(n)\Big|$$

Cauchy-Schwarz $\longrightarrow$ $\le \left(\frac{1}{r-1}\sum_{\chi(r)}\Big|\sum_{n\le x}\chi(n)\Big|^2\right)^{1/2} = \sqrt{\lfloor x\rfloor}$

<u>Qn</u>: Is this sharp?

Reasons to think $\sqrt{x}$ <u>is</u> sharp:

- This is sharp when $0.01r \le x \le 0.99r$, by Montgomery-Vaughan (1979): $\frac{1}{r-2}\sum_{\chi\ne\chi_0}\Big|\sum_{n\le x}\chi(n)\Big|^{2q} \ll_q r^q \;\forall q>0$

  Then w/ $q=2$, $0.01r\le x\le 0.99$,

  $$\frac{1}{r-2}\sum_{\chi\ne\chi_0}\Big|\sum_{n\le x}\chi(n)\Big|^4 \ll r^2 \ll x^2$$

  Heuristically, if 4th moment is small, then there can't be many large individual terms

  Hölder and (*) gives

  $$\frac{1}{r-2}\sum_{\chi\ne\chi_0}\Big|\sum_{n\le x}\chi(n)\Big| \gg \sqrt{x}$$

- Philosophically, people usually say squareroot cancellation is "like the Central Limit Theorem".

  i.e., for Gaussian, all L^{2q}-norms are $\asymp_q \sqrt{x}$ (Standard deviation)

Reasons to think $\sqrt{x}$ <u>isn't</u> sharp:

- Consider random model for problem: $(f(p))_{p \text{ prime}}$ indep. random variables, uniformly distributed on $\{|z|=1\}$ (Steinhaus distribution)

  Define $f(n) := \prod_{p^a \| n} f(p)^a$ ← "Steinhaus random multiplicative function"

Conj (Helson, 2010) L^1-norm

$$\mathbb{E}\Big[\Big|\sum_{n\le x} f(n)\Big|\Big] = o(\sqrt{x}) \quad \text{as} \quad x\to\infty$$

Lower bounds: Helson, Bondarenko-Seip, H.-Nikeghbali-Radziwiłł

$$\gg \sqrt{x}/(\log\log x)^3$$ Not much smaller than $\sqrt{x}$!

Possible analog: $\int_0^1 \Big|\underbrace{\sum_{n\le x} e(n\theta)}\Big|\, d\theta$ is very small, like $\log x$

"additive character"

- $\frac{1}{r-1}\sum_{\chi(r)} |L(\frac12,\chi)| \sim \frac{1}{r-1}\sum_{\chi}\Big|\sum_{n\le r}\frac{\chi(n)}{\sqrt{n}}\Big|$

With squareroot cancellation(?) $\sim \Big(\sum_{n\le r}\frac1n\Big)^{1/2} \sim \sqrt{\log r}$ False!

Correct order $\asymp (\log r)^{1/4}$ Rudnick-Sound, Heath-Brown

This happens because of Euler product (even though it diverges at $1/2$, terms still "collapse")

Thm 1 (H.) Let f be a Steinhaus random mult. fn., $0\le q<1$.

$$\mathbb{E}\Big|\sum_{n\le x} f(n)\Big|^{2q} \asymp_q \Big(\frac{x}{\sqrt{\log\log x}}\Big)^q$$

In particular, w/ $q=1/2$ get $\frac{\sqrt{x}}{(\log\log x)^{1/4}}$, confirming Helson's Conj.

Have seen exact $\sqrt{x}$ for character sums, so when/why does random model fail?

<u>Thm 2</u> (H.) Let r be a large prime, $1 \le x \le r$, $L := \min\{x, r/x\}$. Then for $0 \le q < 1$,

$$\frac{1}{r-1} \sum_{\chi (r)} \Big| \sum_{n \le x} \chi(n) \Big|^{2q} \ll_q \left(\frac{x}{\min\left(\sqrt{\log\log L}, \sqrt{\log\log\log r}\right)} \right)^q$$

i.e., avoiding case where $x \asymp r$ so that $L \to \infty$

There is a symmetry in $x \leftrightarrow r/x$ (due to approx. func'l eqn.), so this is comparable to Thm. 1

Probably not needed

<u>II</u>. Only discuss upper bound in Thm 1.

<u>Theme 1</u>: Connect $\mathbb{E} \big| \sum_{n \le x} f(x) \big|^{2q}$ to Euler product

$$F(s) := \prod_{p \le x} \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

why? • Only clear hope of handling $2q \notin 2\mathbb{N}$

• In $F(s)$, the terms $\left(1 - \frac{f(p)}{p^s}\right)^{-1}$ are independent

• On $\frac{1}{2}$-line, Euler product was responsible for cancellation in L-fns.

Obvious approach: Use Perron

$$\sum_{n \le x} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} \, ds \qquad c > 0$$

This is inefficient, since $\int \frac{1}{|s|}$ diverges like $\log x$, and only have savings of $\sqrt{\log\log x}$

Also would need to see cancellation in $F(s) x^{it}$ — very hard!

Instead, let $\underline{P}(n) :=$ Largest prime $p | n$

$$\sum_{n \le x} f(n) = \sum_{k \ge 0} \sum_{\substack{n \le x \\ x^{e^{-k-1}} < \underline{P}(n) \le x^{e^{-k}}}} f(n) =: \sum_{k \ge 0} S_k$$

$$= \sum_{k \ge 0} \sum_{\substack{1 < m \le x \\ p|m \Rightarrow \underbrace{x^{e^{-k-1}} < p \le x^{e^{-k}}}}} f(m) \sum_{\substack{n \le x/m \\ x^{e^{-k-1}}\text{-smooth}}} f(n)$$

$\boxed{\text{Call interval } I_k}$

By multiplicativity, these $f(m)$, $f(n)$ are independent!

<u>Hölder</u>: if $c(m) \in \mathbb{C}$ are coeffs,

$$\mathbb{E} \Big| \sum_{\substack{1 \le m \le x \\ p|m \Rightarrow p \in I_k}} f(m) c(m) \Big|^{2q} \le \Big(\mathbb{E} \Big| \sum f(m) c(m) \Big|^2 \Big)^q$$

$$= \Big(\sum_{\substack{1 \le m \le x \\ p|m \Rightarrow p \in I_k}} |c(m)|^2 \Big)^q$$

By conditioning, $\mathbb{E} |S_k|^{2q} \le \mathbb{E} \Big(\sum_{\substack{1 < m \le x \\ p|m \Rightarrow p \in I_k}} \Big| \sum_{\substack{n \le x/m \\ n\ x^{e^{-k-1}}\text{-smooth}}} f(n) \Big|^2 \Big)^q$

- Now replace $\sum_m$ by an integral
- Apply Parseval, Physical side = $f(n)$ sum
  Frequency side = Dirichlet series

Result: $\mathbb{E}\left|\sum_{n\le x} f(n)\right|^{2q} \sim x^q\, \mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} \left|F(\tfrac12+it)\right|^2 dt\right)^q$

<u>Note</u>: Unlike Perron, Parseval avoids oscillations; rather compare two mean-squares.

Now we want to show

$$\mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} \left|F(\tfrac12+it)\right|^2 dt\right)^q \ll_q \left(\frac{1}{\sqrt{\log\log x}}\right)^q, \quad q<1$$

If $q=1$, have

$$\frac{1}{\log x}\int_{-1/2}^{1/2} \underbrace{\mathbb{E}\left|F(\tfrac12+it)\right|^2}_{\sim \log x} dt \asymp 1$$

$\sim \log x$ (e.g. second moments of L-fns.)

Is it true that terms on right are all "roughly independent"

$$\frac{1}{\log x}\int_{-1/2}^{1/2} \left|F(\tfrac12+it)\right|^2 dt \sim \frac{1}{\log^2 x}\sum_{|k|\le \frac{\log x}{2}} \left|F\left(\tfrac12+\tfrac{ik}{\log x}\right)\right|^2 \;?$$

No! This is the source of extra cancellation.

III. <u>Deterministic Case</u>

$L=\min(x, r/x)$ comes from approx. func'l eqn. / Poisson summation: Pólya Fourier expn.

$$\left|\sum_{n\le x}\chi(n)\right| \sim \frac{x}{\sqrt r}\left|\sum_{n\le r/x}\chi(n)\right|$$

- In Thm. 1, conditioned on $(f(p))_{p\le x^{e^{-k}}}$, lots of primes! (e.g. $p\le x^{1/e}$ for $k=1$)

For character sums, just "condition" on $\chi(p)$ for $p \le (\log r)^{3/4}$, much smaller set of primes.

Roughly $e^{(\log r)^{3/4}}$ "conditions", $< r-1$ characters, so enough room to see cancellation.