



If  $f: V \rightarrow M_d$  satisfies

$\text{rk}(f(g+h) - f(g) - f(h)) \leq r$ , then

$\exists \chi: V \rightarrow M_d$  homom. s.t.  $\text{rk}(f(g) - \chi(g)) \leq R(r)$

No quantitative information known

## Approximate Cohomology

$M$  = abelian group w/ filtration

$M_i \hookrightarrow M$  s.t.  $M_i \subseteq M_j$ ,  $M_i + M_j \subseteq M_{\min(i,j)}$

Ex:  $M_d$  above,  $M =$  <sup>homog.</sup> polys. on  $V$

Suppose  $G$  acts on  $M$ ,  $f: G^n \rightarrow M$

$f$  is an approximate  $n$ -cocycle if  $\text{Im}(df) \subseteq M_i$  for some  $i \in \mathbb{Z}_+$

$n=1$ :  $df(g,h) = f(g) + g \cdot f(h) - f(g+h)$

$f$  is an  $n$ -approximate coboundary if  $\exists t: G^{n-1} \rightarrow M$  s.t.  $\text{Im}(f - dt) \subseteq M_i$

Set of approximate cocycles form a group  $\tilde{Z}_n(G, M)$   
" " " " coboundaries  $\tilde{B}_n(G, M)$

$\exists a: H_n(G, M) \rightarrow \tilde{H}_n(G, M) = \tilde{Z}_n / \tilde{B}_n$

Thm: " $a_1$  is onto."

Qn: what about  $a_n$  for  $n \geq 2$ ?

Thm (Proper statement)

$G = V$   $\infty$ -dim'l /  $\mathbb{F}_p$

$M_d =$  homog. polys. as above, with rank filtration

$a_i$  is onto  $p > d-1$

Ingredients of Proof

1. See rank in an analytic way

Bias  $\Rightarrow$  low rank

All bounds independent of dim.  $V$

Thm (Green-Tao, Lovett-Kaufman, Lovett-Bhounnich)

$G = V =$  fin. dim. vector sp. /  $\mathbb{F}_p$ , "large" dim.

If  $|\sum_{x \in V} e(\frac{p(x)}{p})| \geq \frac{1}{p^s}$ , then  $p$  is of rank  $\leq C(d, s)$

Fix  $d$  in advance, work in  $M_d$

$e_p(y) := e^{\frac{2\pi i y}{p}}$

2. If  $p$  is rank  $r$ , consider finite differences

$\Delta_h(p(x)) := p(x+h) - p(x)$

Then  $\Delta_{h_d} \dots \Delta_{h_1} p$  is multilinear in  $h_1, \dots, h_d$

Lemma:  $|\sum_{h_1, \dots, h_d} e(\Delta_{h_d} \dots \Delta_{h_1} p)| \geq C(r)$

Note: Algebraic proof of poly. result could lead to new proof of Inverse Thm

3. Inverse theorem for Gowers norms over finite fields

(Bergelson-Tao-Z., Tao-Z.)

If  $f: V \rightarrow \mathbb{F}_p$ ,  $|\sum \Delta_{h_d} \dots \Delta_{h_1} (e(f(x)))| \geq \delta$ , then

$e(f)$  correlates with a fn.  $g$  s.t.  $\Delta_{h_d} \dots \Delta_{h_1} g = 1$

i.e.  $\exists \tilde{c}(\delta)$  s.t.

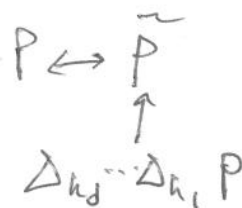
$$\left| \sum_{x \in V} e(f(x)) e(g(x)) \right| \geq \tilde{c}(\delta)$$

If  $p > d$ ,  $g$  is a poly.

(11)

1.  $g: V \rightarrow M_d$ ,  $\text{rk}(p(g+h) - p(g) - p(h)) \leq r$

Consider  $f(g, h_1, \dots, h_d) := \tilde{f}_g(h_1, \dots, h_d)$



Show that  $\Delta_{(g_{d+2}, r_{d+2})} \dots \Delta_{(g_1, r_1)} e(f) =$   
 $\|e(f)\| u^{d+d}(V^{d+d}) \geq \delta$

2. Use Inverse Thm. for Gowers Norm  
 $f$  correlates w/ a poly  $g$  on  $V^{d+1}$

3. Get  $\text{rk}(p(g) - p(g))$  is bounded for pos. density of  $g$

4. Additive combinatorics. Propagate using approx. cocycle condn.  
to all of  $G=V$ .