Tamar Ziegler : "Approximate Cohomology"

(Hebrew University) (w/ Kazhdan)

G = abelian group, countable

V = vector space / K, countable
field

$$\rho : G \to End_K(V)$$

$\forall\ g, h \in G$, $rank(\rho(g+h) - \rho(g) - \rho(h)) \le r$

"almost-homom.", note if $r=0$ it's exact

Qn: Is there a homom. $\chi : G \to End_K(V)$ s.t. $rk(\rho - \chi) \le R(r)$

Ex: $G = \mathbb{Z}$, $\rho : G \to End_K(V)$

$$rk(\rho(g+h) - \rho(g) - \rho(h)) \le 1$$

Can show there is χ s.t. $R(r) \le 2$.

Don't know answer for $r=2$!

Thm: $G = V$ = ∞-dim'l vector spaces / K, $char(K) = p \ge 5$,

Then $R(r) < \infty$.

Thm: Fix $d \ge 2$, M_d = homog. polys. deg. d on V = ∞-dim'l vector space / $K = \mathbb{F}_p$

If $p \in M_d$, it is of rank r if This is Schmidt's h-invariant (Audience comment)

$$p = \sum_{i=1}^{r} l_i R_i, \quad w/\ l_i, R_i \text{ homog. deg} \ge 1$$

If $\rho : V \to M_d$ satisfies

$$\mathrm{rk}(\rho(g+h) - \rho(g) - \rho(h)) \leq r, \text{ then}$$

$$\exists\ \chi : V \to M_d \text{ homom. s.t. } \mathrm{rk}(\rho(g) - \chi(g)) \leq R(r)$$

No quantitative information known

Approximate Cohomology

M = abelian group w/ filtration

$M_i \nearrow M$ s.t. $M_i \subseteq M$, $M_i + M_j \subseteq M_{c(i,j)}$

<u>Ex</u>: M_d above, M = homog. polys. on V

Suppose G acts on M, $\rho : G^n \to M$

- ρ is an approximate n-cocycle if $\mathrm{Im}(\partial\rho) \subseteq M_i$ for some $i \in \mathbb{Z}_+$

$n=1$: $\partial\rho(g,h) = \rho(g) + g\cdot\rho(h) - \rho(g+h)$

- ρ is an n-approximate coboundary if $\exists\ t : G^{n-1} \to M$ s.t. $\mathrm{Im}(\rho - \partial t) \subset M_i$

Set of approximate cocycles form a group $\widetilde{Z}_n(G,M)$

" " coboundaries $\widetilde{B}_n(G,M)$

$$\exists\ a : H_n(G,M) \to \widetilde{H}_n(G,M) = \widetilde{Z}_n / \widetilde{B}_n$$

<u>Thm</u>: "a_1 is onto."

<u>Qn</u>: What about a_n for $n \geq 2$?

Thm (Proper statement)

$G = V$ ∞-dim'l $/ \mathbb{F}_p$

M_d = homog. polys. as above, with rank filtration

a_1 is onto $p > d-1$

Ingredients of Proof

1. See rank in an analytic way

Bias $\Rightarrow$ low rank

All bounds independent of dim. V

Thm (Green-Tao, Lovett-Kaufman, Lovett-Bhoumick)

$G = V$ = fin. dim. vector sp. $/ \mathbb{F}_p$, "large" dim.

If $\left|\mathbb{E}_{x \in V} e_p(p(x))\right| \geq 1/p^s$, then p is of rank $\leq C(d,s)$

$e_p(y) := e^{\frac{2\pi i y}{p}}$

Fix d in advance, work in M_d

2. If p is rank r, consider finite differences
$\Delta_h(p(x)) := p(x+h) - p(x)$

Then $\Delta_{h_d} \cdots \Delta_{h_1} p$ is multilinear in $h_1, \ldots, \cancel{\ } h_d$

Lemma: $\mathbb{E}_{h_1, \ldots, h_d} e(\Delta_{h_d} \cdots \Delta_{h_1} p) \geq c(r)$

Note: Algebraic proof of poly. result could lead to new proof of Inverse Thm

3. Inverse theorem for Gowers norms over finite fields
(Bergelson-Tao-Z., Tao-Z.)

If $f: V \to \mathbb{F}_p$. $\left|\mathbb{E}\, \Delta_{h_d} \cdots \Delta_{h_1} (e(f(x))\right| \geq \delta$, then

$e(f)$ correlates with a fn. $\cancel{\ }$ q s.t. $\Delta_{h_d} \cdots \Delta_{h_1} \cancel{\ }$ $q = 1$

i.e. $\exists\ \tilde{c}(\delta)$ s.t.

$$\left| \mathbb{E}_{x \in V} e(f(x))\, e(g(x)) \right| \geq \tilde{c}(\delta)$$

If $p > d$, g is a poly.

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1. $\rho : \cancel{}\, V \to M_d$. $\quad rk(\rho(g+h) - \rho(g) - \rho(h)) \leq r$

$$P \leftrightarrow \tilde{P}$$
$$\uparrow$$
$$\Delta_{h_d} \cdots \Delta_{h_1} P$$

Consider $f(g, h_1, \ldots, h_d) := \tilde{\rho}_g (h_1, \ldots, h_d)$

Show that $\Delta_{(g_{d+2}, r_{d+2})} \cdots \Delta_{(g_1, r_1)} e(f) =$

$$\| e(f) \|_{U^{d+2}(V^{d+1})} \geq \delta$$

2. Use Inverse Thm. for Gowers Norm
   f correlates w/ a poly g on V^{d+1}

3. Get $rk(\rho(g) - p(g))$ is bounded for pos. density of g

4. Additive combinatorics. Propagate using approx. cocycle condn.
   to all of $G = V$.