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"Subconvex equidistribution of cusp forms"

M = finite vol. Riemannian manifold

Take a sequence of eigenfunctions, square-integrable

$$\varphi \in L^2(M) \text{ s.t. } (\Delta + \lambda_\varphi)\varphi = 0.$$

Qn: How does φ behave as $\lambda \to \infty$?

(and interact w/ geodesic flow on M)

One description of behavior is L^2-mass μ_φ: prob. meas. on M

$\forall\ \Psi \in C_c(M)$,

$$\mu_\varphi(\Psi) := \frac{\langle \varphi, \Psi\varphi \rangle}{\langle \varphi, \varphi \rangle}$$

We know the most in arithmetic setting, $M = SL_2(\mathbb{Z}) \backslash \mathcal{H}$

Hecke eigenfns. $T_n \varphi = \sqrt{n}\, \lambda_\varphi(n) \cdot \varphi$

normalized Hecke eigenvals.

AQUE Thm: (Lindenstrauss '06, Soundararajan 2010)

For fixed $\Psi \in C_c(M)$ (M as above),

$$\mu_\varphi(\Psi) \longrightarrow \mu(\Psi) := \frac{\langle \Psi, 1 \rangle}{\langle 1, 1 \rangle}$$

- This answered part of a conj. of Rudnick-Sarnak (1994)
- refines QE thm. (Schnirelmann, Colin de Verdiere, Zelditch (1970s-80s))

<u>Qn</u>: At what rate does $\mu_\varphi \to \mu$?

<u>Defn</u>: Given $\mathcal{E} : \{\text{Eigenfns. } \varphi\} \to \mathbb{R}_+$, say $\mu_\varphi \to \mu$ at <u>rate $\mathcal{E}$</u> if $\forall\ \psi \in C_c^\infty(M)$, $\exists\ c_\psi \geq 0$ s.t.

$$|\mu_\varphi(\psi) - \mu(\psi)| \leq c_\psi \cdot \mathcal{E}(\varphi)$$

<u>Optimal AQUE Conj</u>: $\mu_\varphi \to \mu$ at rate $\lambda_\varphi^{-1/4+\varepsilon}$ $\forall \varepsilon > 0$

(Sarnak 93, Luo-Sarnak 94, Watson 02)

verified average version of statement, and showed true under GRH

<u>Strong AQUE Conj</u>: $\mathcal{E}(\varphi) = \lambda_\varphi^{-\delta}$, some $\delta > 0$

- AQUE proved via ergodic theory, provides "no rate"

<u>Aim</u>: Prove a prime level aspect analog of Strong AQUE

- <u>Sarnak 01, Liu-Ye '02, Lau-Liu-Ye 06</u>:
  Strong AQUE holds if φ is a <u>dihedral form</u> (theta lift from $\mathbb{Q}$)
- <u>Holowinsky-Sound '10</u>: Analog of AQUE in weight $k \to \infty$ aspect
  - with effective rate $\mathcal{E}(\varphi) = (\log k)^{-\delta}$, w/ $k = \mathrm{wt}(\varphi)$

N. 2011: Rate result in Square free level aspect

$\varphi =$ wt k holom. cuspidal newform on $\Gamma_0(N)\backslash\mathcal{H}$, N squarefree

$\mu_\varphi =$ measure on $SL_2(\mathbb{Z})\backslash\mathcal{H}$

$$\mu_\varphi(\Psi) = \frac{\langle \varphi, \Psi\varphi\rangle}{\langle \varphi, \varphi\rangle}$$

"associating Ψ w/ its pullback to modular curve" (see diagram to left)

$$\begin{tikzcd} \Gamma_0(N)\backslash\mathcal{H} \arrow[d] \arrow[dr, "{\varphi, \Psi}"] & \\ SL_2(\mathbb{Z})\backslash\mathcal{H} \arrow[r, "\Psi"] & \mathbb{C} \end{tikzcd}$$

Thm (N.) $\mu_\varphi \to \mu$ w/ rate $\log(kN)^{-\delta}$

"Thm" MN (Munshi 2015-17, N. 2017)

Fix weight k. Let φ traverse a sequence of newforms on $\Gamma_0(p)\backslash\mathcal{H}$, p prime $\to \infty$, with wt $\varphi = k$.

Then $\mu_\varphi \to \mu$ at rate $p^{-\delta}$, for some $\delta > 0$,

i.e., $\forall\ \Psi \in C_c^\infty(SL_2(\mathbb{Z})\backslash\mathcal{H})$,

$$\left| \frac{\langle \varphi, \Psi\varphi\rangle}{\langle \varphi, \varphi\rangle} - \frac{\langle 1, \Psi\rangle}{\langle 1, 1\rangle} \right| \leq \frac{C_\Psi}{p^\delta}$$

Recall Spectral Thm for $SL_2(\mathbb{Z})$:

$$L^2(SL_2(\mathbb{Z})\backslash\mathcal{H}) = \underbrace{L^2_{Eis}}_{} \oplus L^2_{Cusp}$$

$$\mathbb{C}\cdot 1 \oplus \left(\int_t \mathbb{C}\, E_{1/2+it} \frac{dt}{4\pi}\right)$$

Any $f \in C_c^\infty(M)$ has a rapidly convergent spectral decomposition:

$$f = \langle f, 1\rangle \cdot 1 + \sum_{\psi \in \mathcal{B}(L^2_{cusp})} \langle f, \psi\rangle + \left(\text{Eisenstein}\right)$$

orthonormal basis

Since $\mu_\varphi(1) = 1$, and $\mu(\psi) = 0$ for ψ cuspidal or Eisenstein,

it suffices to show that $|\mu_\varphi(\psi)| \ll_\psi p^{-\delta}$ for these cases

<u>Remark/Discussion:</u> For non-squarefree levels, newforms are not necessarily the natural vectors. The present results are the first example of a "subconvex system"

• <u>(Watson 2002, Harris-Kudla 1991, N. 2011)</u>:

$$|\mu_\varphi(\psi)|^2 = \frac{L(\varphi \times \bar{\varphi} \times \psi, \tfrac{1}{2})}{C(\varphi \times \bar{\varphi} \times \psi)^{\boxed{1/4 + o(1)}}}$$

as $\varphi \to \infty$, ψ fixed

<u>Remark</u>: Nonobvious the L-value is real!

Here $C \asymp_\psi p^4$, so $C^{1/4} \asymp p$

Thm MN reduces to subconvexity bound

$$(*) \quad \frac{L}{C^{1/4}} \ll C^{-\delta} \cdot \mathrm{cond}(\psi)^{A}$$

(for some $\delta > 0$, $A \geq 10^6$)

<u>Two cases</u>: (A) ψ = Eisenstein

(B) ψ = cusp form

<u>Thm M</u>: (A) holds <u>Thm N</u>: (A) $\Rightarrow$ (B)

Then (A) + (B) $\Rightarrow$ Thm MN.

<u>Pf of Thm N</u> Want to bound $\frac{L}{C^{1/4+o(1)}} \doteq |\langle \varphi, \psi\varphi \rangle|^2$

Shimizu Theta-lift: <u>Linearizes</u> role of φ

$$\varphi(z_1)\bar{\varphi}(z_2) = \int \varphi(w)\,\Theta(z_1, z_2, w)$$

$$\doteq |\langle \psi\Theta^{\#}, h^{\#} \rangle|^2$$

Constants depending on normalization of φ

Y. Qiu + "local Calculations" (N.) connects this to $\frac{L}{C^{1/4+o(1)}}$

Here $\Theta(z) := \frac{1}{y^4} \sum_{n \in \mathbb{Z}} e(n^2 z)$

$\Theta^{\#}(z) := \Theta(pz)$

$h(z) := \sum \frac{b(n)}{|n|^{1/2}} W(ny) e(nx)$ "inverse Shimura lift of ψ"

$h^{\#}(z) := \sum \frac{b(pn)}{|n|^{1/2}} W(ny) e(nx)$ ~~weight~~ on $\Gamma_0(4p)$

$1/2$-integral wt.

Have $\|h\| = \|h^\#\| \asymp 1$

Want to show: $\langle \varphi\Theta^\#, h^\# \rangle \ll p^{-\delta}$ (✻✻)

By "amplification" method (Iwaniec, Michel, Venkatesh, ...),

$$\|\varphi\Theta^\#\|^2 = \|\varphi\|^2 \cdot \|\Theta^\#\|^2 + O(p^{-\delta})$$ with "some room" left over

(✻✻✻)

Really need: For small $\ell \asymp p^\alpha$, α small, ℓ square

$$\langle T_\ell(\varphi\Theta^\#), \varphi\Theta^\# \rangle = \frac{1}{\deg T_\ell} \langle T_\ell \varphi, \varphi \rangle \langle T_\ell \Theta^\#, \Theta^\# \rangle + O(p^{-\delta}\ell^A) \quad (✻✻✻)_\ell$$

Idea: If (✻) fails, Cauchy-Schwarz $\Rightarrow$

$|\langle \varphi\Theta^\#, h^\# \rangle|^2 \ll \|\varphi\Theta^\#\|^2$ is nearly sharp, so

$\varphi\Theta^\#$ is nearly in $\mathbb{C}h^\#$, contradicting $(✻✻✻)_\ell$,

Since $T_\ell h^\# \approx \underbrace{\lambda_\varphi(h^\#)}_{\text{Typically} \sim 1} \sqrt{\ell}\, h^\#$, but

$$\langle T_\ell \varphi, \varphi \rangle \langle T_\ell \Theta^\#, \Theta^\# \rangle \ll \ell^{1/2 - \beta} \cdot \deg(T_\ell)$$

$\ll \ell^{1/2 + 7/64}$ $\quad \ll \ell^{3/4}$

$\ll \ell^{3/4 - \beta}$

To show (✻✻✻), $\|\varphi\Theta^\#\|^2 = \langle |\varphi|^2, |\Theta^\#|^2 \rangle$

$$= \langle |\varphi_\#|^2, |\Theta| \rangle = \langle |\varphi_\#|^2, 1 \rangle \cdot \langle 1, |\Theta|^2 \rangle + \dots$$

$= \|\varphi^\#\|^2 \cdot \|\Theta^\#\|^2$

where ...

$$= \sum_{\varphi \in \mathcal{B}(L^2_{cusp})} \langle |\varphi_\#|^2, \varphi \rangle \underbrace{\langle \varphi, |\Theta|^2 \rangle}_{=0,} + \int_t \underbrace{\langle |\varphi_\#|^2, E_{1/2+it} \rangle}_{\ll p^{-\delta}} \underbrace{\langle E_{1/2+it}, |\Theta|^2 \rangle}_{\ll t^{-1000}}$$

Noticed by N. recently,

by (M)

$|\Theta|^2$ is an Eisenstein series

(though note subtlety that level is 4)