Lillian Pierce: A Chebotarev density theorem for families of finite fields, with applications to class groups
(Duke)

① Counting primes:

$$\pi(x) = \sum_{p \le x} 1$$

Better to use weights $\quad \theta(x) = \sum_{p \le x} \log p$

von Mangoldt fn.

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n)$$

Since $\quad -\dfrac{\zeta'(s)}{\zeta(s)} = \sum_{n \ge 1} \Lambda(n) n^{-s}$

Asymptotically close: $\theta(x) = \psi(x) + \underbrace{O\Big(\sum_{\substack{p^m \le x \\ m \ge 2}} \log p\Big)}_{O(x^{1/2}\log x)}$

Also, by partial summation

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_{3/2}^{x} \frac{\theta(t)}{t(\log t)^2}\, dt$$

Landau's formula:

$$\psi(x) = x - \sum_{\substack{\rho = \beta + i\gamma \\ |\gamma| \le T}} \frac{x^\rho}{\rho} + O\big(xT^{-1}(\log x)^2\big) + O(\log x)$$

zeros of $\zeta(s)$

These are "acceptable" error terms

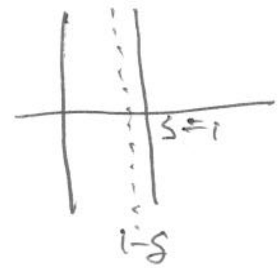
Turán: $\psi(x) = x + O(x \exp(-c(\log x)^{\alpha}))$ $\quad 0 < \alpha < 1,$

$\iff \zeta(s) \neq 0$ for $\sigma \geq 1 - c'(\log t)^{\frac{\alpha-1}{\alpha}}$ (Here $\sigma = s + it$)

To get errors $O(x^{1-\delta})$ for $\delta > 0$, need a better zero-free region: ("Box-Shape")

$$\psi(x) = x + O(x^{1-\delta} \log^2 x)$$

$$\iff \zeta(s) \neq 0 \text{ for } \sigma \geq 1 - \delta$$



(2) Restricted primes

$$\pi(x; a, q) := \sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} 1 = \frac{1}{\varphi(q)} \mathrm{Li}(x) + E$$

w/ $(a,q) = 1$

E depends on zero-free region for $L(s, \chi)$ w/ χ a char. mod q

This uses "finite Fourier inversion", since

$$\delta_{\equiv a(q)}(x) = \frac{1}{\varphi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \bar{\chi}(a)\chi(x)$$

$$\psi(x, \chi) = \sum_{n \leq x} \Lambda(n)\chi(n) = \delta_{\chi} x - \sum \frac{x^{\rho}}{\rho}$$

$\delta_\chi = 1 \iff \chi = \chi_0$ $\qquad$ ρ = zeroes of $L(s, \chi)$

(3) $$\pi(x; K) := \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p}) \leq x}} 1 = \mathrm{Li}(x) + E$$

K: alg. extn. $K/\mathbb{Q}$

E: zero-free region for $\zeta_K(s)$

④ $\pi_{\mathcal{C}}(x; L/k) := \sum_{\substack{\mathfrak{p} \in \mathcal{O}_k \\ \text{unram'd in } L \\ Nm_{k/\mathbb{Q}} \mathfrak{p} \leq x \\ \left[\frac{L/k}{\mathfrak{p}}\right] = \mathcal{C}}} 1 = \frac{|\mathcal{C}|}{|G|} Li(x) + \mathcal{E}$

"Prime # Thm"

Artin symbol

$\boxed{\begin{array}{l} L/k \text{ normal extn.} \\ G = Gal(L/k) \\ \mathcal{C} = \text{conj. class in } G \end{array}}$

Note that Case ② was $k = \mathbb{Q}$, $L = \mathbb{Q}(e^{2\pi i/q})$,

③ $L = k$, ① $L = k = \mathbb{Q}$.

Recall Artin symbol:

Let $|G| = n$. If $\mathfrak{p} \subseteq \mathcal{O}_k$ s.t. $\mathfrak{p} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$ in $\mathcal{O}_L$

(w/ all $e_i = e$, $n = efg$)

- Decomposition group:

$$D_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\} \twoheadrightarrow Gal\left(\mathcal{O}_L/\mathfrak{P} \big/ \mathcal{O}_k/\mathfrak{p}\right) \cong Gal\left(\mathbb{F}_{q^f}/\mathbb{F}_q\right)$$

where $q = Nm_{k/\mathbb{Q}}(\mathfrak{p})$

$\boxed{\text{Let } \tau_q : x \mapsto x^q}$

- Inertia group:
  Kernel of above map, cardinality is e

① If $\mathfrak{p}$ unramified, $e=1$, define $\left[\frac{L/k}{\mathfrak{p}}\right] := [\text{Preimage of } \tau_q]$

Conj. Class

② If $\mathfrak{p}$ splits completely in L: $g = n$, $e = f = 1$
decomp'n gp trivial, so $\mathcal{C} = \{id\}$

③ p tamely ramified: $p \nmid e$, ~~inte~~ inertia gp. cyclic

④ p wildly ramified: Controllable up to a constant $C(G)$

Chebotarev Density Thm: (Fourier inversion)

$$\delta_{\mathcal{C}}(x) = \sum_{\substack{\rho \text{ irred.} \\ \text{representations} \\ \text{of } G}} \langle \delta_\rho, \operatorname{tr}\rho \rangle \operatorname{tr}\rho$$

indicate whether $x \in \mathcal{C}$

Control error via zero-free regions $L(s, \rho, L/K)$ (Artin L-fn)

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\rho \text{ irred.} \\ \text{nontriv. repns.} \\ \text{of } G}} L(s, \rho, L/K)^{\dim \rho}$$

assume Artin conjecture (this guarantees the nontrivial ρ contribute only L-fns., no poles)

Lagarias-Odlyzko (1975): Unconditionally, if L/K normal, $\deg. (L/\mathbb{Q}) = n_L > 1$, $D_L := |\operatorname{Disc} L/\mathbb{Q}|$, then

$$\left| \pi_{\mathcal{C}}(x; L/K) - \frac{|\mathcal{C}|}{|G|} Li(x) \right| \le \frac{|\mathcal{C}|}{|G|} Li(x^{\beta_0}) + c_1 x \left(\exp(-c_2 n_L^{1/2}) |\log x|\right)$$

for $x \ge \exp(10 n_L (\log D_L)^2) \ge D_L^{10 n_L}$

and where β_0 is a (real simple) exceptional zero

$$\sigma \ge \frac{1}{4 \log D_L}, \quad |t| \le \frac{1}{4 \log D_L}$$

Conditional on GRH:

$$|E| \le c_3 \frac{|\mathcal{C}|}{|G|} x^{1/2} \log(D_L x^{n_L}) \quad \text{for } x \ge 2$$

Current Goals:

1. Remove β_0 term
2. Reduce threshold x to D_L^{δ}, $\delta > 0$

Do this without GRH, at the expense of limiting to almost every field (in various families)

(w/ C. Turnage-Butterbaugh and M. Wood)

<u>Remark</u>: Can already remove β_0 term if ζ_k has no real zeroes, $\deg(L/k) \geq 3$, and no quad. subextns. $k \subseteq F \subseteq L$.

This follows from a result of Stark

<u>(Skeletal) Theorem</u>: For a family of # fields $\mathcal{L}(X)$

- of deg. $n / \mathbb{Q}$
- Fixed $G = \mathrm{Gal}(\tilde{K}/\mathbb{Q})$ for $K \in \mathcal{L}(X)$
- Disc $K \leq X$
- Possible restrictions on generators of inertia gp. of all tamely ramid primes

Known that $|\mathcal{L}(X)| \gg X^a$ for some $a > 0$. For at most $O(X^b)$ ~~extens~~ exceptions w/ $b < a$, Chebotarev density is true w/

$$\left| \pi_{\mathcal{C}}(x; \tilde{K}/\mathbb{Q}) - \frac{|\mathcal{C}|}{|G|} \mathrm{Li}(x) \right| \leq \frac{|\mathcal{C}|}{|G|} \frac{x}{(\log x)^2},$$

if $x \geq K_1 \exp(K_2 (\log \log D_L^{K_3})^2)$

K_i depend on degrees, $|G|$, a, b

Note this range includes $x \leq D_L^{\varepsilon}$

<u>Cor</u>: For $\mathcal{F}(X)$ s.t. theorem holds, for non-exceptional K we have $\forall\ \ell \geq 1$

$$|Cl_K[\ell]| \ll D_K^{\frac{1}{2} - \frac{1}{2\ell(n-1)} + \varepsilon}$$

if ~~primes~~ a positive proportion of primes $\leq D_K^{\frac{1}{2\ell(n-1)}}$ split completely

- Uses result of Ellenberg - Venkatesh

Table of example families:

$\deg K/\mathbb{Q}$	$\mathrm{Gal}\ \tilde{K}/\mathbb{Q}$	Generators of inertia group for tame primes	Exceptional family	Total family
n	C_n cyclic	totally ramified	$\ll X^{\varepsilon}\ \forall \varepsilon>0$	$\sim c \cdot X^{\frac{1}{n-1}}$
3	$S_3 = D_3$	D_K sq free (inertia gp. gen'd by (12))	$\ll X^{1/3}$ Ellen.-Venk. (2007)	$\sim c \cdot X$ (Bhargava '14)
4	S_4	" "	$\ll X^{1/2+\varepsilon}$ (Klüners)	cX
5	S_5	" "	~~$X^{0.27}$~~ $X^{39/40}$ ~~(Bhargava et al 2017)~~ (Bhargava)	cX
4	A_4	$\cong K_4$	$X^{0.27}$ (Bhargava et al 2017)	$\gg X^{1/2}$
$p \geq 5$	D_p (order $2p$)	a reflection	$\ll X^{6/(p-1)^2}$	$\gg X^{2/p-1}$

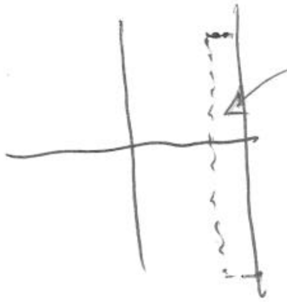
Conditional results, continue table

$n \geq 6$	S_n	D_K sq. free	$\ll X^{\Delta}$ if $\exists \ll D^{\Delta}$ deg. n fields s.t. $D_K = D$	$\gg X^{1/2 + 1/n}$
n	Simple A_n	none	On strong Artin Conj $\ll X^2$	$\gg$ unknown?

Proof Ideas

Let $L = \widetilde{K}$ (Galois clos.)

First Step If ζ_L / ζ_K were zero-free in box



$[1-\delta, 1] \times [-(\log D_L)^{2/\delta}, (\log D_L)^{2/\delta}]$,

then theorem holds. (see Cho-Kim prime ideal thm.)

Second Step Show most $K \in \mathcal{L}(X)$ satisfy this zero-free region.

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\rho \text{ irred. nontriv.} \\ \text{rep. of } G}} L(s, \rho, L/K)^{\dim \rho}$$

• Let $k = \mathbb{Q}$ shortly

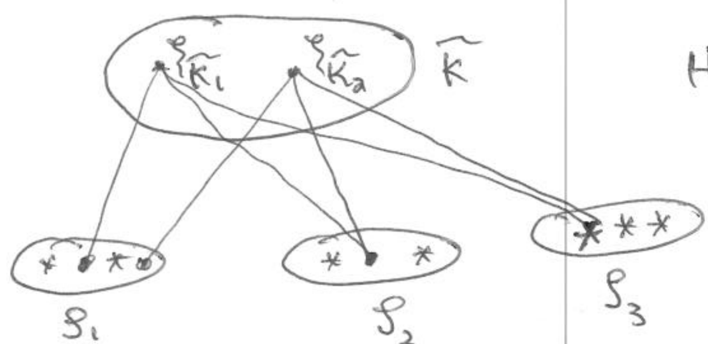
If strong Artin Conj. holds, for each i

$\{L(s, \rho_i, L/\mathbb{Q})\}$ collection of cuspidal automorphic L-fns,

$\uparrow$ L varies over $\widetilde{K}$ for $K \in \mathcal{L}(X)$

- Kowalski–Michel (2002): In each such collection, most are zero-free in a box.

Thus ζ_L is zero free in $\left\{\begin{matrix}\text{zero-free} \\ \text{region } \zeta_K\end{matrix}\right\} \cap \{\text{boxes}\}$

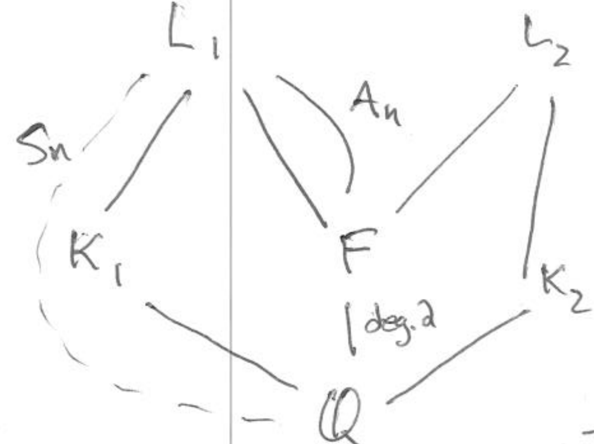


$*$ = bad factors

Hard to rule out bad factors in ρ_i appearing in many $\zeta_{\widetilde{K}}$.

But $\zeta(s, \rho, L_1/\mathbb{Q}) = \zeta(s, \rho, L_2/\mathbb{Q})$

$\iff L_1^{\ker(\rho)} = L_2^{\ker(\rho)}$



Control by fact that

$p \mid D_K \Rightarrow p \mid D_{\widetilde{K}}$

$\Rightarrow p \mid D_F$

By restriction on ramification

- This idea is source of $\ll X^{\Delta}$ bound for $n \geq 6$

Comment:

- Best hope for more unconditional results might be for case of $\mathrm{Gal}(\widetilde{K}/\mathbb{Q})$ simple, since then there is no room for propagation of bad factors as in diagrams above.