Kaisa Matomäki: Correlations of von Mangoldt and higher order divisor functions 44

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Write von Mangoldt function $\Lambda(n) := \begin{cases} \log p & \text{if } n = p^r \\ 0 & \text{o.w.} \end{cases}$

Divisor fn $d_k(n) := \sum_{n = m_1 \cdots m_k} 1$

Twin primes $\Leftrightarrow$ $\sum_{n \le X} 1_{n \in \mathcal{P}} 1_{n+2 \in \mathcal{P}}$

$\langle \approx \rangle$ $\sum_{n \le X} \Lambda(n) \Lambda(n+2)$

as in Prime Number Thm., $\Lambda(n)$ is a good substitue for $1_{\mathcal{P}}$

Hardy-Littlewood Conj:

$$\sum_{n \le X} \Lambda(n)\Lambda(n+h) = \mathfrak{S}(h) X + O(X^{1/2 + o(1)})$$

Singular series defined by

$$\mathfrak{S}(h) := \begin{cases} 0 & h \text{ odd} \\ \underbrace{2 \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{(p-1)^2}\right)}_{\text{"Twin prime constant"}} \cdot \underbrace{\prod_{\substack{p \mid h \\ p > 2}} \frac{p-1}{p-2}}_{\text{mod } p \text{ obstructions}} \end{cases}$$

This is wide open.

Present Goal: Prove H-L averaging on h.

<u>Thm 1</u> (M-Radziwiłł-Tao) If $X \geq H \geq X^{8/33} \geq 2$,

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+h) = \mathfrak{S}(h) X (1 + O(\log^{-A} X))$$

for all but $O(H(\log X)^{-A})$ many h s.t. $|h| \leq H$

- Previously Mikawa (1991): True for H $\underbrace{X^{1/3+\varepsilon}}$

Here $\frac{8}{33} < 1/4$.

Here the $1/3$ comes from $2 \cdot 1/6$, as primes in short intervals results hold for $X^{1/6}$.

Taylor expansion of $\log \zeta(s)$:

$$\log(1-(1-\zeta(s))) = -\sum_{k \geq 1} \frac{(-1)^k}{k} (\zeta(s)-1)^k$$

"Linnik's identity"

Now compare Dirichlet series, coeffs

$$\frac{1_{n=p^\alpha}}{\alpha} = -\sum_{k \geq 1} \frac{(-1)^k}{k} \underbrace{d_k^*(n)}_{:= \sum_{\substack{n = m_1 \cdots m_k \\ m_j > 1 \, \forall j}} 1}$$

Morally, we need to understand

$$\sum_n d_k(n) d_k(n+h) \qquad \forall \; k \geq 1$$

<u>$k=1$</u> Trivial ($d_1 \equiv 1$)

<u>$k=2$</u> Known for fixed H

<u>$k \geq 3$</u> need variant of Thm 1.

<u>Add'l Motivation</u>: Related to $2k$-th moments of ζ

Thm 2 If $X \geq H \geq (\log X)^{C_k} \geq 2$, ($C_k \sim 10000 k \log k$ works)

$k \log k$ is essential due to divisor fn. oscillation in short intervals

Then $$\sum_{n \leq X} d_k(n) d_k(n+h) = \text{Main Term} \cdot (1 + o(1))$$

$\boxed{\text{order } X (\log X)^{2k-2}}$

for all but $o(H)$ values of $|h| \leq H$.

Remark: Compared to Thm 1, smaller H comes at the price of losing log-savings in error terms.

Proof Ideas

Circle Method: $$\sum_{n \leq X} a_n a_{n+h} = \int_0^1 \overbrace{\left|\sum a_n e(\alpha n)\right|^2}^{S(\alpha)} e(\alpha h)\, d\alpha$$

i.e. only $n_1 - n_2 = h$ survive

Split into Major Arcs + minor,

$[0,1] = \mathfrak{M} \cup \mathfrak{m}$

$\mathfrak{M}$ will give main term via standard arguments

- Challenging part is bounding minor arcs. Consider on average: Square magnitude

$$\sum_{|h| \leq H} \left| \int_{\mathfrak{m}} |S(\alpha)|^2 e(\alpha h)\, d\alpha \right|^2$$

$$\leq \cdots \leq H \sum_{i=1}^{H} \left| \int_{[\frac{i}{H}, \frac{i+1}{H}] \cap \mathfrak{m}} |S(\alpha)|^2\, d\alpha \right|^2$$

$$\leq H \underbrace{\left(\sup_{\alpha \in \mathfrak{m}} \int_{|\beta - \alpha| \leq \frac{1}{H}} |S(\beta)|^2\, d\beta \right)^{\delta}}_{S_1 \text{ - find pointwise savings}} \underbrace{\left(\sum_{i=1}^{H} \int_{[\frac{i}{H}, \frac{i+1}{H}] \cap \mathfrak{m}} |S(\alpha)|^2\, d\alpha \right)^{2-\delta}}_{S_2 \text{ - Use Parseval}} \qquad \delta \in [0,2]$$

Standard decomposition:

• Taking $\delta = 1$, second term is bounded by

$$S_2 \le \int_0^1 |S(\alpha)|^2 \, d\alpha \underset{\uparrow}{=} \sum_{n \le X} |a_n|^2$$

Parseval

<u>Issue</u>: $\sum_{n \le X} d_k(n)^2 \asymp X (\log X)^{k^2-1}$,

much larger than Main Term!
$\left(X (\log X)^{2k-2} \right)$

Thus when $H = (\log X)^{O(1)}$, S_2 must be handled differently,
e.g. $\delta = 1/2$

Need bounds like:

$$\sum_{j=1}^{J} \int_{\alpha - 1/H}^{\alpha + 1/H} |S(\alpha_j)|^2 \, d\alpha \ll J^{1/4} (\log X)^{2(k-1) + o(1)} X$$

Now the $\log X$ terms are no worse than Main Term, so savings from S_1 will suffice

• For S_1, need

$$\int_{\alpha - 1/H}^{\alpha + 1/H} |S(\beta)|^2 \, d\beta = O\left(\frac{1}{X} \left(\sum |a_n| \right)^2 \right) \quad \forall \, \alpha \in \mathfrak{m}$$

Write $\alpha = \frac{a}{q} + \theta$; on $\mathfrak{m}$, not too close to r/s for small s.
$|\theta| \le \frac{1}{qQ}$

Reduces to bounding (via Fourier inversion)

$$\int_X^{2X} \left| \frac{1}{H} \sum_{x \le n \le x+H} a_n e(\alpha n) \right|^2 dx$$

Note: a is not important; a bounded Gauss sum term

which becomes (using short interval sum techniques, stationary phase)

$$(*1) \quad \frac{1}{q} \int_{t \asymp \theta X} \left(\sum_{\chi \pmod q} \frac{1}{|\theta| \cdot H} \int_{t - \theta H}^{t + \theta H} \left| \sum_{n \le X} \frac{a_n \chi(n)}{n^{1/2 + it'}} \right|^2 dt' \right)^2 dt$$

$$(\ast 2) \le \sum_{\chi(q)} \int_{t \asymp \theta X} \left| \sum_{n \le X} \frac{a_n \chi(n)}{n^{1/2+it}} \right|^2 dt$$

Bound this by method of M.–Radziwiłł.

- For Thm 1, decompose $\Lambda(n)$ into convolutions via Heath-Brown's identity.

$\Rightarrow$ For $X^{1/3} \ge H \ge X^{1/4}$, most difficult term is

$$a_n = 1_{m \sim M} \ast \beta \qquad H \le M \le X^{1/3}$$

This is "roughly d_3" — 2-fold divisor fn. — Source of $X^{1/3}$ in Mikawa

For simplicity, let $q=1$, $\theta = \frac{1}{X^{\varepsilon}}$, $H = X^{2/7 + 2\varepsilon}$ in $(\ast 1)$:

$H_1 = \theta H$

$$\int_{t \asymp X^{1-\varepsilon}} \left(\frac{1}{H_1} \int_{t-H_1}^{t+H_1} \left| \sum_{n \le X} \frac{1 \ast \beta(n)}{n^{1/2+it}} \right|^2 dt' \right) dt$$

Use Cauchy-Schwarz to separate β.

$$\text{Need} \quad \int_T^{T+H_1} \left| \sum_{m \sim M} \frac{1}{m^{1/2+it}} \right|^2 dt = O\left(H_1^{1+\varepsilon}\right)$$

$$\forall\ |T| \le X^{1-\varepsilon}$$

- If second moments of zeta were known, would be finished, but they aren't.

$$\text{Smoothing} \Rightarrow \ll \frac{H}{M} \sum_{|n| \le M/H} \sum_{m \sim M} \left(1 + \frac{n}{m}\right)^{iT} \ll H^{1+\varepsilon}$$

by exponent pair $(1/6, 1/6)$

- Discussion: Under RH, Thm 1 improves to $H \ge X^{\varepsilon}$, or (maybe $\log X^c$)