The SuperPAC: Geometry to Arithmetic of Sphere Packings

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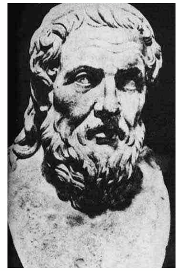
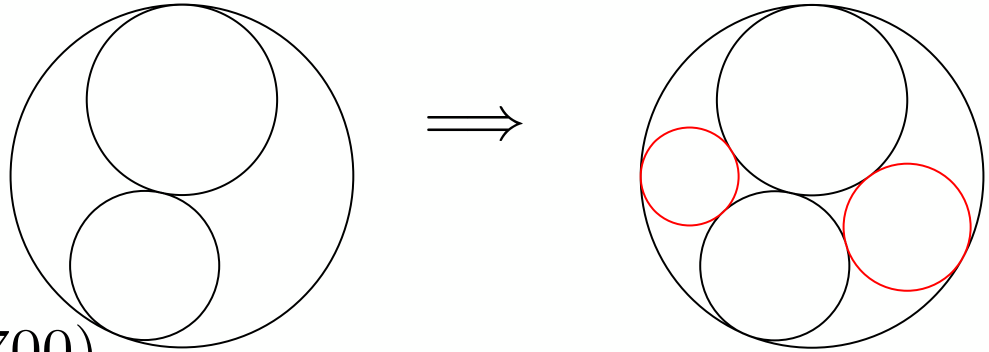
Apollonian Circle Packings



Geometry: (Apollonius, ∼200 BCE)

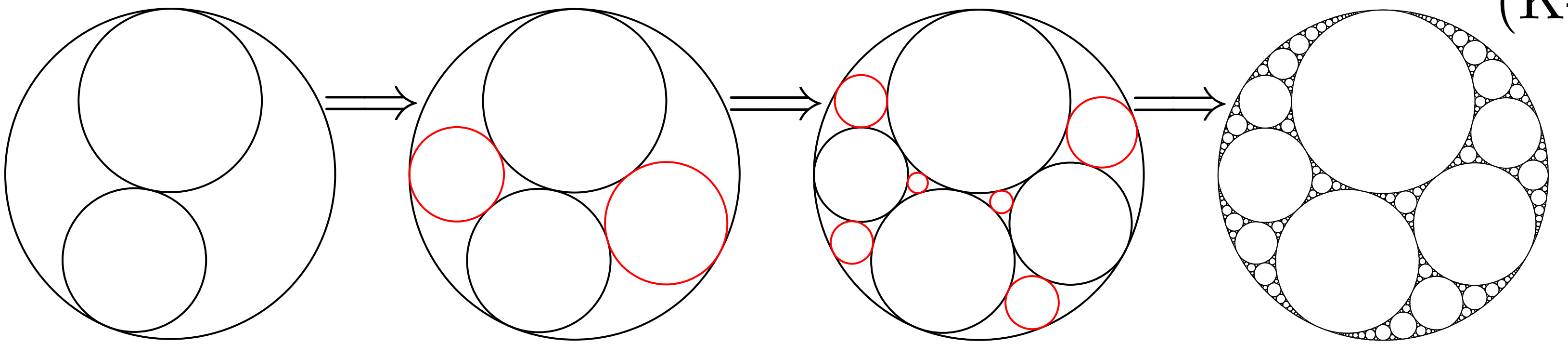
(Proof by Viète, ∼1600)

(Baragar-K 2017: Construction using 7 elementary moves)



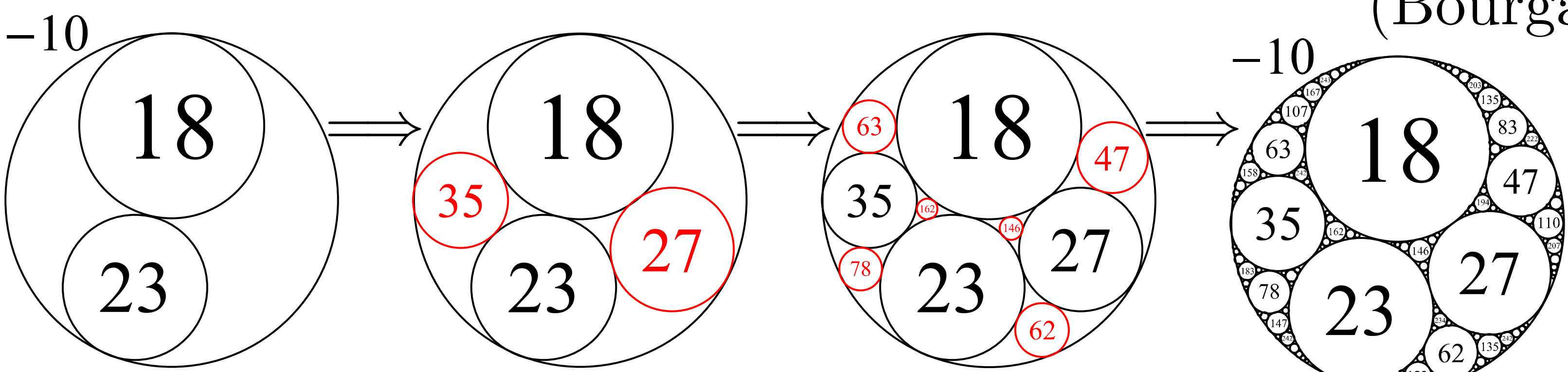
Dynamics: (Leibniz, ∼1700)

(K-Oh)



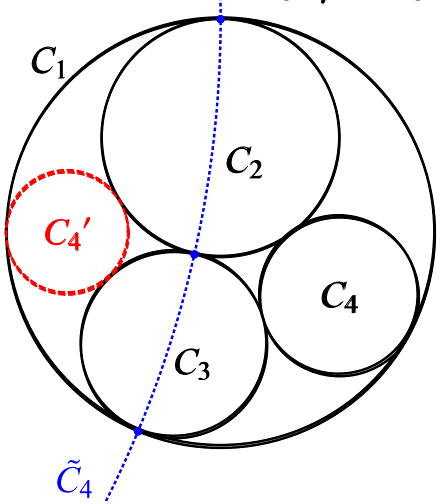
Arithmetic: Soddy (1936): Study the "bends" $b = 1/r$

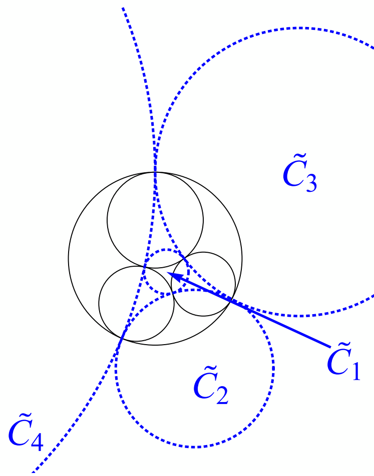
(Bourgain-K)



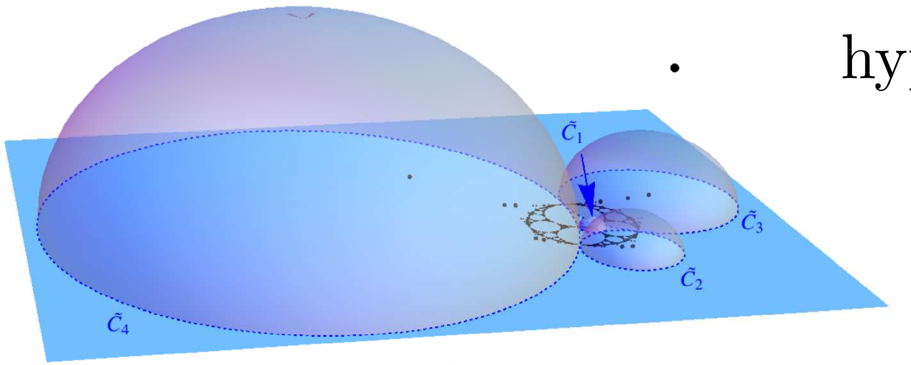
Main Question:
How does Geometry lead to Arithmetic?

Geometry/Dynamics: Packing is generated by "dual" circles:





$$\Gamma = \left\langle \tilde{C}_1, \ldots, \tilde{C}_4 \right\rangle$$
$$< \ \mathrm{Isom}(\mathbb{H}^3)$$



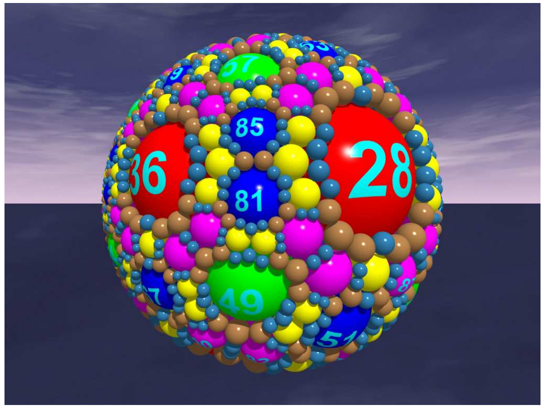
Apollonian Circle Packing = limit set of (geometrically finite) hyperbolic reflection group

Arithmetic: Descartes Theorem (∼1650)

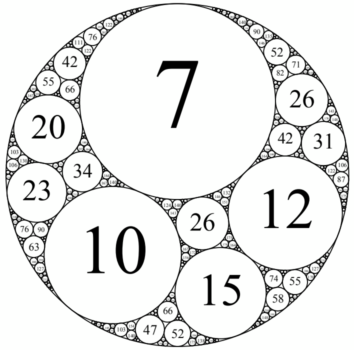
$$b_1^2 + b_2^2 + b_3^2 + b_4^2 \ = \ \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2$$

(Soddy:) $\Longrightarrow$ $b_4' = 2(b_1 + b_2 + b_3) - b_4$ ($\mathbb{Z}$-linear!)

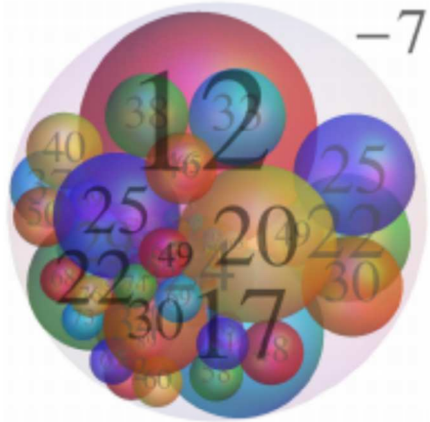
More General Setting



(Soddy 1936,
K 2012,
“4-simplex”)



(Boyd 1974,
Guettler-Mallows 2010,
Zhang 2014, “octahedron”)



(Dias/Nakamura 2014,
“4-orthoplex”)

…

Def: A packing $\mathcal{P}$ of $\widehat{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$, is an infinite collection of oriented $(n-1)$-spheres (or co-dim-1 planes) so that:

- the interiors of spheres are disjoint,
- the tangency graph is connected, and
- the packing fills up space; $\overline{\bigsqcup_{S\in\mathcal{P}} interior(S)} = \widehat{\mathbb{R}^n}$

Def: A Γ-packing $\mathcal{P}$ is one arising as the limit set of some geom finite, reflection group $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$.

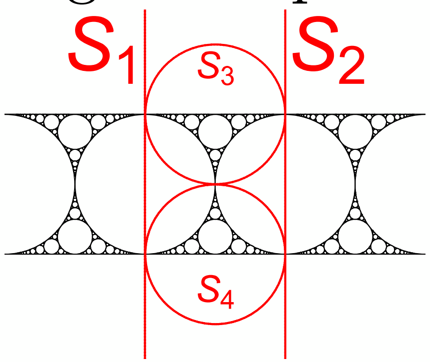
Problem: ~~Classify all integral Γ-packings.~~

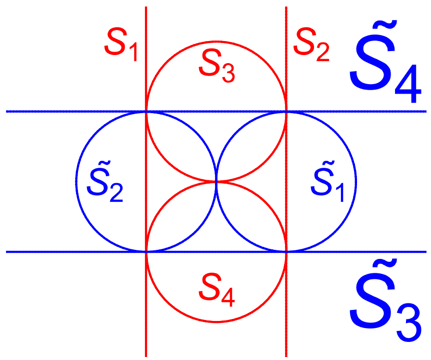
SuperIntegrality

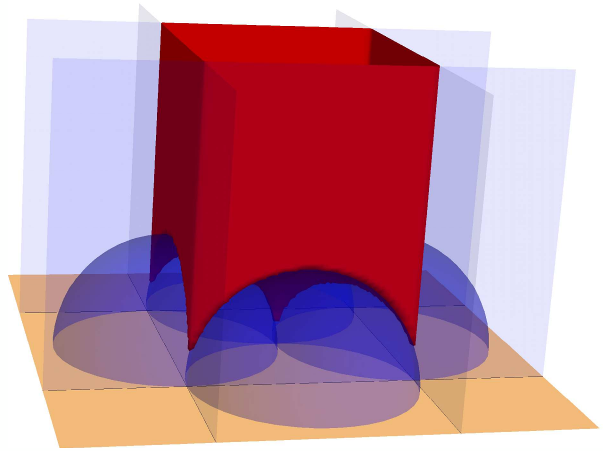
Def: The supergroup $\widetilde{\Gamma}$ of a Γ-packing $\mathcal{P}$ is:

$$\widetilde{\Gamma} \;:=\; \langle \Gamma, \mathcal{P} \rangle \,.$$

E.g. $\mathcal{P}$=Apollonian





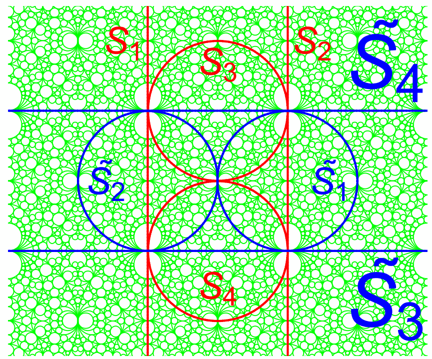


Apollonian supergroup has fundamental domain = ideal octahedron with all dihedral angles $\pi/2$. $\widetilde{\Gamma} \sim \mathrm{SL}_2(\mathbb{Z}[i])$

Def: The superpacking $\widetilde{\mathcal{P}}$ of a Γ-packing $\mathcal{P}$ is:

$$\widetilde{\mathcal{P}} \;:=\; \widetilde{\Gamma} \cdot \mathcal{P}.$$

E.g. $\mathcal{P}$=Apollonian



Def: If every $S \in \widetilde{\mathcal{P}}$ has integer bend, then $\mathcal{P}$ is called **superintegral**.

The SuperPAC (K-N = K-Nakamura 2017)

SuperPAC (SuperIntegral Packing Arithmeticity Conjecture):
If a Γ-packing $\mathcal{P}$ is superintegral, then its supergroup $\widetilde{\Gamma}$ (a lattice generated by reflections) is arithmetic!

Remark: Superintegrality is necessary. Lemma: (K-N) There exist integral Γ-packings for which $\widetilde{\Gamma}$ is **not** arithmetic.

If true, SuperPAC would be very useful: **Thm:** (Vinberg, Nikulin, Long-Maclahlan-Reid, Agol, Agol-Belilopetsky-Storm-Whyte) There are only finitely many maximal arithmetic hyperbolic reflection groups! None once $n \geq 30$.

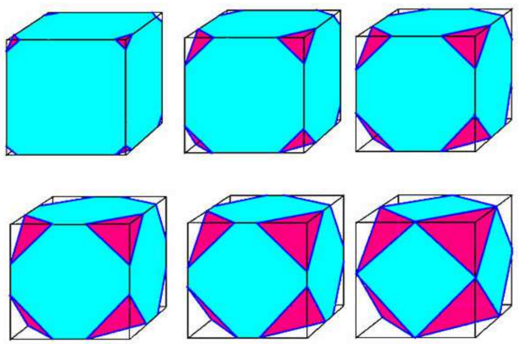
Corollary: SuperPAC $\Longrightarrow$ essentially only finitely many superintegral Γ-packings.

Main Theorem: (K-N) SuperPAC holds for Γ-packings "modeled on" **uniform polyhedra**.

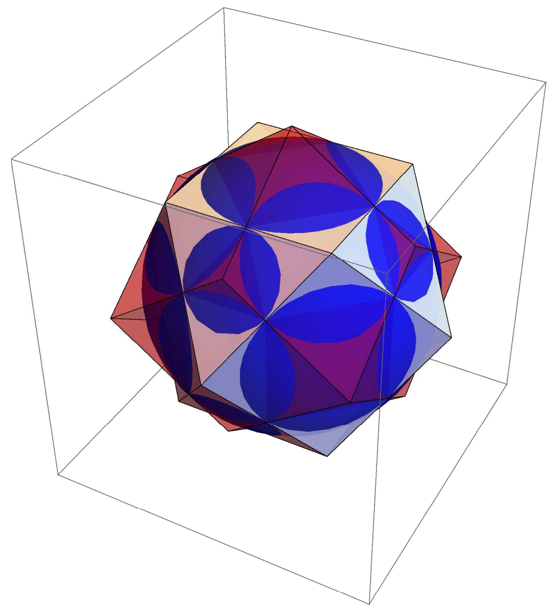
Polyhedral (Circle) Packings

Thm: (Koebe-Andreev-Thurston/Schramm) Every convex polyhedron admits a combinatorially equivalent geometrization with a midsphere. (Tangent to all edges.)

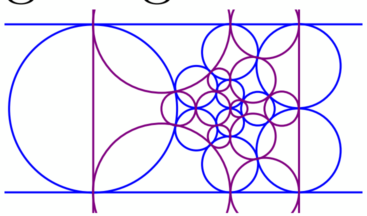
E.g. Cuboctahedron (Archimedean):



(KAT)
$\Longrightarrow$

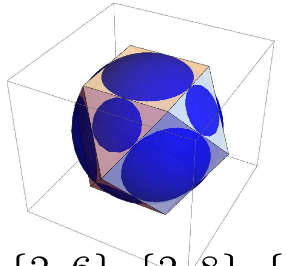


KAT allows one to attach a Γ-packing to a polyhedron Π:
Once geometrized, the midsphere is also that of the dual, $\widehat{\Pi}$, giving two sets of clusters, with tangency graphs $\cong \Pi$ and $\widehat{\Pi}$:



Then $\Gamma := \left\langle \text{reflections through } \widehat{\Pi} \text{ cluster} \right\rangle$
acts on Π cluster giving packing $\mathcal{P}$ *modeled* on Π.

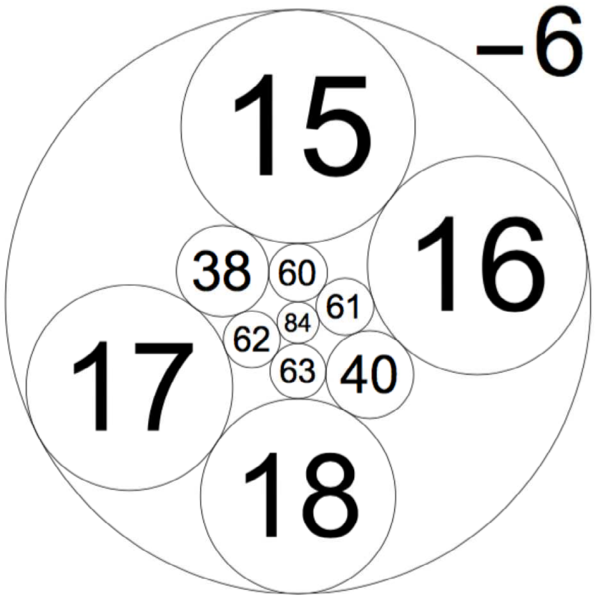
E.g.: Π=Cuboctahedron



Combinatorics/Topology: $G = (V, E)$,

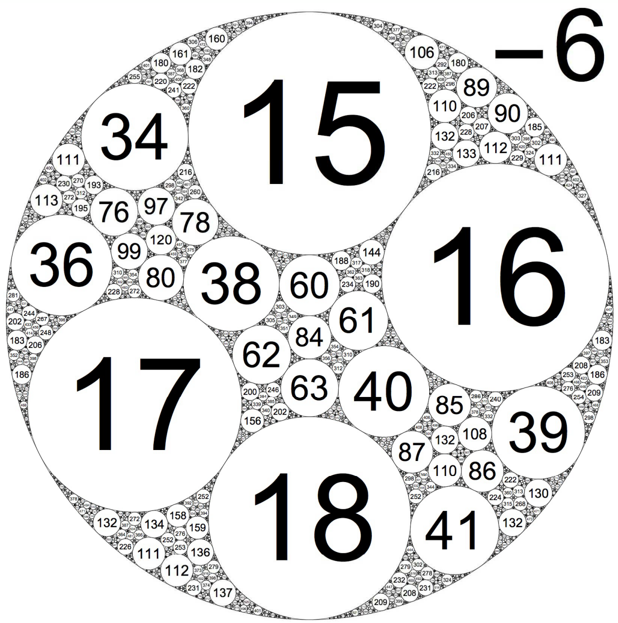
$V = (1, \ldots, 12)$, $E = (\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,6\}, \{2,8\}, \{3,5\}, \{3,6\}, \{3,9\}, \{4,7\}, \{4,10\}, \{5,7\}, \{5,11\}, \{6,8\}, \{6,9\}, \{7,10\}, \{7,11\}, \{8,10\}, \{8,12\}, \{9,11\}, \{9,12\}, \{10,12\}, \{11,12\})$, $F = (\{4,10,8,2\}, \{3,9,11,5\}, \{9,6,8,12\}, \{3,1,2,6\}, \{5,7,4,1\}, \{11,12,10,7\}, \{12,11,9\}, \{3,5,1\}, \{6,9,3\}, \{5,11,7\}, \{8,10,12\}, \{1,4,2\}, \{2,8,6\}, \{7,10,4\})$

Geometry: Π-cluster



$\Longrightarrow$

Dynamics: $\mathcal{P}(\Pi)$:



Def: Π is (super)integral if *some* packing $\mathcal{P}$ modeled on Π is.

Lemma: (K-N) Π = cuboctahedron is superintegral.

Problem: Classify these!

SuperIntegral Polyhedra

Determining whether a given Π is (super)integral is non-trivial:
- KAT is an existence proof; actual geometrization is achieved through infinite limiting process (see Stephenson).
- To the rescue: Selberg/Mostow/Prasad rigidity: can make all bends and centers algebraic! After enough decimal places, *guess* the algebraic values, then verify tangencies rigorously.

Even then there are difficulties:

Thm (K-N):

(i) Infinitely many polyhedra are integral!

This is an immediate corollary of:

(ii) Infinitely many distinct polyhedra give rise to the *same* circle packing $\mathcal{P}$!

Moreover,

(iii) There are infinitely many non-isomorphic integral circle packings!

Proof: Certain constructions ("growths") gluing along vertices/faces. (Non-maximal reflection groups)

Main Theorem

Recall: Π is uniform if its faces are regular polygons and it is vertex-transitive. These are: Platonic, Archim, (Anti-)Prisms.

Main Theorem: (K-N) SuperPAC holds for Γ-packings modeled on **uniform polyhedra** and their growths.

The classification is:

Platonic:

- tetrahedron,
- octahedron (Guettler-Mallows, Zhang),
- cube (Stange).

▶ Dodec- and icosahedra are "golden": $\mathbb{Z}[\varphi]$-integral bends, $\varphi = \frac{1+\sqrt{5}}{2}$.

Archimedean:

- cuboctahedron,
- truncated tetrahedron,
- truncated octahedron.

▶ Icosidodecahedron, Great/Small Rhombicosidodecahedra, and Truncated Dodec- and Icosahedra, are golden.

▶ Truncated Cube, and Great/Small Rhombicuboctahedra are "silver": $\mathbb{Z}[\rho]$-integral bends, $\rho = 1 + \sqrt{2} = [\overline{2}]$.

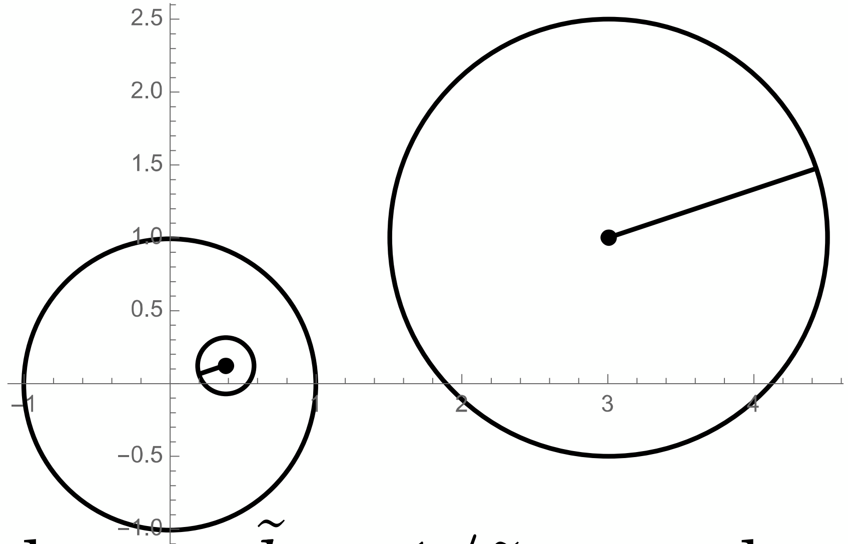
▶ Snub Cube has cubic bends, Snub Dodecahedron has sextic bends.

Prisms/Antiprisms: • 3-/4-/6-prisms, and 3-anti-prism.

thm: For all known superintegral $\mathcal{P}(\Pi)$, $\frac{\#\{\text{bends} < X\}}{\#\{\text{admissible} < X\}} \to 1$

Tools

Inversive Coordinates:



(Clifford/Darboux/...)

Let sphere S have center $z \in \mathbb{R}^n$, radius $r \Longrightarrow$ co-radius

$$\tilde{r} = \frac{r}{|z|^2 - r^2}.$$

$\Longrightarrow$

$$b\tilde{b} - (b|z|)^2 = -1, \quad \text{where} \quad \tilde{b} = 1/\tilde{r} = \text{co-bend}.$$

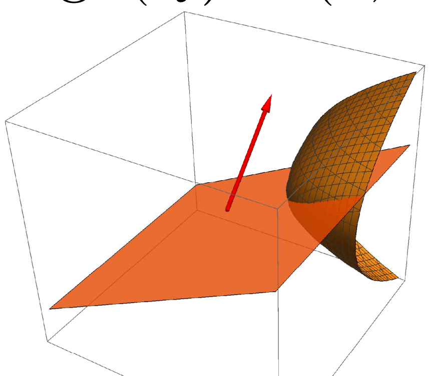
Attach $\quad S \leftrightsquigarrow v_S := (b, \tilde{b}, bz) \in \mathbb{R}^{n+2}$.

Then

$$v_S \in \; Q = -1, \text{ where } Q = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -I_n \end{pmatrix}.$$

Set $\langle v_1, v_2 \rangle \; := \; v_1.Q.v_2^{\dagger}$.

$\text{Sign}(Q) = (1, n+1)$. Thus one sheet of $Q = 1$ is $\cong \mathbb{H}^{n+1}$.



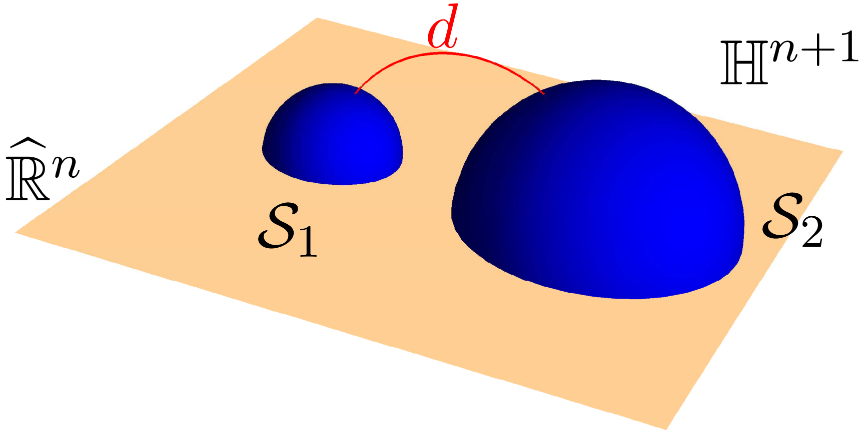
Given S, v_S is normal to plane $P_S : \langle v_S, \cdot \rangle = 0$

Then $P_S \cap \{Q = 1\} \cong \mathbb{H}^n$

And $\partial(P_S \cap \{Q = 1\}) = S$

Möbius: reflection S: $x \mapsto x - 2\frac{\langle x, v_S \rangle}{\langle v_S, v_S \rangle} v_S$

Koebe-Andreev-Thurston, Again



$$\langle v_1, v_2\rangle := v_1.Q.v_2^\dagger = \cosh d(\mathcal{S}_1,\mathcal{S}_2) = \begin{cases} 1 & \text{tangent} \\ 0 & \text{orthogonal} \\ -1 & \mathcal{S}_1 = \mathcal{S}_2 \end{cases}$$

So what is **KAT**? Given tangency data of Π (say, n vertices) and $\widehat{\Pi}$ (m faces), one solves:

$$\begin{pmatrix} v_{\text{vertex}_1} \\ v_{\text{vertex}_n} \\ \hline v_{\text{face}_1} \\ v_{\text{face}_m} \end{pmatrix} \cdot Q \cdot \begin{pmatrix} \\ \hline \\ \end{pmatrix}^\dagger = \left(\begin{array}{c|c} \text{vertex tangency} & \text{intersection data} \\ \hline \text{intersection data} & \text{face tangency} \end{array}\right) = \text{superGramian} = \widetilde{G}$$

E.g., Π = square pyramid, $\widetilde{G} =$

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 1 & -1 & 1 & 3 & 1 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 \\ 1 & 1 & -1 & 1 & 3 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 \\ 1 & 3 & 1 & -1 & 1 & 2\sqrt{2} & 0 & 0 & 2\sqrt{2} & 0 \\ 1 & 1 & 3 & 1 & -1 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & -1 & 1 & 3 & 1 & 1 \\ 0 & 2\sqrt{2} & 0 & 0 & 2\sqrt{2} & 1 & -1 & 1 & 3 & 1 \\ 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 3 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 1 & 3 & 1 & -1 & 1 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

$\widetilde{\Gamma} = \langle v_{\text{vertices}}, v_{\text{faces}}\rangle$

KAT: $\exists!$ solution,
up to Möbius/orientation

E.g.:

Arithmeticity: Vinberg

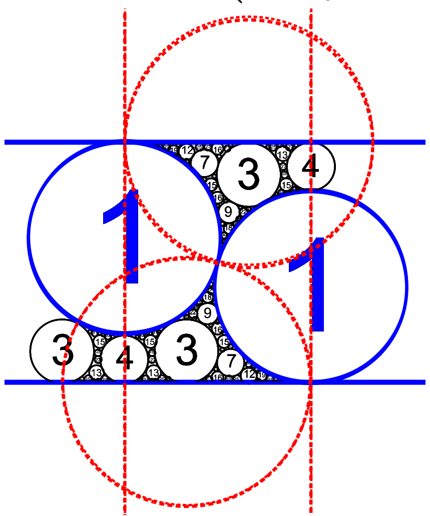
Supergroup $\widetilde{\Gamma} = \langle v_{\text{vertices}}, v_{\text{faces}} \rangle$ is a hyperbolic reflection group with Gramian $\widetilde{G}$.

E.g. For Π = 4-Pyramid, $\widetilde{G} = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 1 & -1 & 1 & 3 & 1 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 \\ 1 & 1 & -1 & 1 & 3 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 \\ 1 & 3 & 1 & -1 & 1 & 2\sqrt{2} & 0 & 0 & 2\sqrt{2} & 0 \\ 1 & 1 & 3 & 1 & -1 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & -1 & 1 & 3 & 1 & 1 \\ 0 & 2\sqrt{2} & 0 & 0 & 2\sqrt{2} & 1 & -1 & 1 & 3 & 1 \\ 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 3 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 1 & 3 & 1 & -1 & 1 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$

Thm: (Vinberg 1967)
$\widetilde{\Gamma}$ is arithmetic iff
all cyclic products of $2\widetilde{G}$
are integers.

$$\widetilde{\Gamma} \sim \mathrm{SL}_2(\mathcal{O}_2)$$

E.g. 2 (Boyd 1974): $\widetilde{G} = \begin{pmatrix} -1 & 1 & 1 & \frac{3}{2} & 0 & 0 & 0 & \frac{\sqrt{15}}{2} \\ 1 & -1 & \frac{3}{2} & 1 & 0 & 0 & \frac{\sqrt{15}}{2} & 0 \\ 1 & \frac{3}{2} & -1 & 1 & 0 & \frac{\sqrt{15}}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 1 & -1 & \frac{\sqrt{15}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{15}}{2} & -1 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{15}}{2} & 0 & 1 & -1 & \frac{1}{2} & 1 \\ 0 & \frac{\sqrt{15}}{2} & 0 & 0 & 1 & \frac{1}{2} & -1 & 1 \\ \frac{\sqrt{15}}{2} & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 & -1 \end{pmatrix}$



$$\widetilde{\Gamma} \sim \mathrm{SL}_2(\mathcal{O}_{15})$$

Also superintegral!

Superintegrality: Bends Group

First Möbius group: Recall reflection through S given by:

$$x \mapsto x - 2\frac{\langle x, v_S\rangle}{\langle v_S, v_S\rangle} v_S = x.R_S, \qquad \text{where}$$

$$R_S = I + 2Q \cdot (v_S^\dagger \cdot v_S) \in O_Q.$$

Then symmetry group $\Gamma = \langle R_{\text{faces}} \rangle$ acts on the right as Möbius.

If Π-cluster is given by $V = \begin{pmatrix} v_{\text{vertex}_1} \\ \cdots \\ v_{\text{vertex}_n} \end{pmatrix}$, then bends are in the first column. (Recall $v = (b, \widetilde{b}, bz)$.)

So to study bends, we need a left-action. Define A_S $(n \times n)$ by:

$$A_S \cdot V = V \cdot R_S.$$

Note: this is rigid! Unaffected by choice of Möbius.

But: in general, very underdetermined. Only up to coker(V).

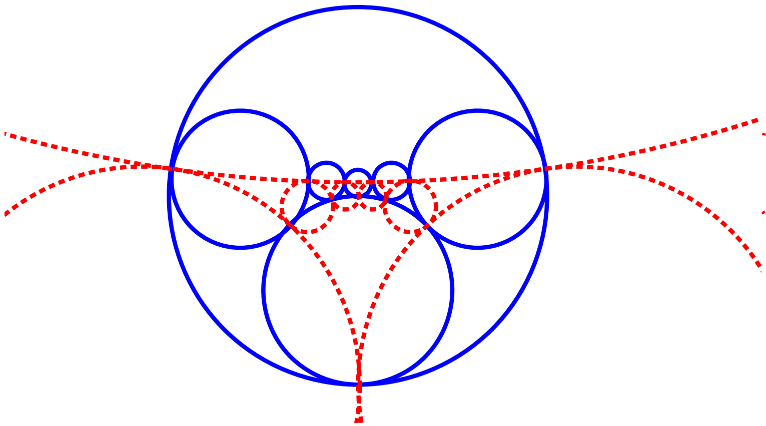
E.g. Π = 4-Pyramid.

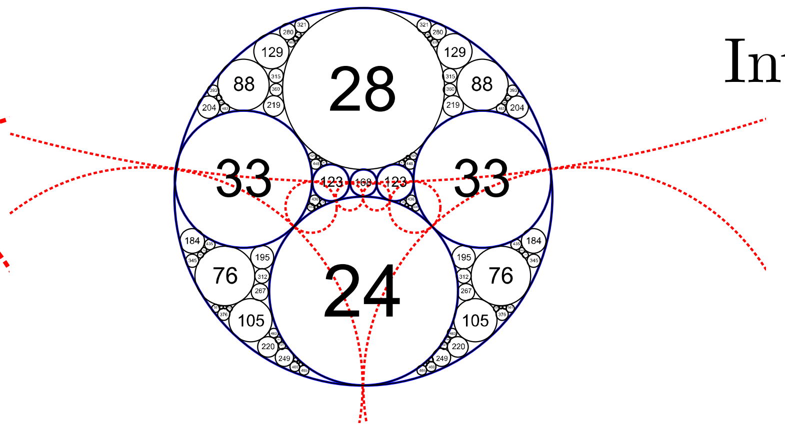
$$A_S \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 4 & 2 & -2\sqrt{2} & 1 \\ 8 & 1 & -2\sqrt{2} & 1 \end{pmatrix} = V \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 32 & 1 & -4\sqrt{2} & 0 \\ 8\sqrt{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & 1 \\ 32 & 1 & -4\sqrt{2} & 1 \\ 36 & 2 & -6\sqrt{2} & 1 \\ 8 & 1 & -2\sqrt{2} & 1 \end{pmatrix}$$

Linear Diophantine equation to solve for $A_S \overset{?}{\in} M_{n\times n}(\mathbb{Z})$

So: if A_S integral for *all* $S \in \Pi \cup \widehat{\Pi}$, then $\mathcal{P}(\Pi)$ is superintegral

Non-arithmetic: Π = Hexagonal pyramid





Integral!

$$\text{For } S \in \Pi,\ A_S = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ 6 & 4 & 4 & & -1 \\ 12 & 9 & 6 & & -2 \\ 12 & 10 & 5 & & -2 \\ 6 & 6 & 2 & & -1 \end{pmatrix} \cdots \begin{pmatrix} -1 & -\frac{2}{3} & \frac{2}{3} & & \frac{2}{3} \\ & 1 & & & \\ & & 1 & & \\ & -2 & 2 & & 1 \\ & -3 & 2 & & 2 \\ & -2 & 1 & & 2 \\ & & & & 1 \end{pmatrix}$$

For $S \in \widehat{\Pi}$, $A_S \in \mathbb{Z}$

$$\widetilde{G} = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \\ 1 & -1 & 1 & 5 & 7 & 5 & 1 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 \\ 1 & 1 & -1 & 1 & 5 & 7 & 5 & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 \\ 1 & 5 & 1 & -1 & 1 & 5 & 7 & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 0 \\ 1 & 7 & 5 & 1 & -1 & 1 & 5 & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 0 \\ 1 & 5 & 7 & 5 & 1 & -1 & 1 & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 0 \\ 1 & 1 & 5 & 7 & 5 & 1 & -1 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & -1 & 1 & 5 & 7 & 5 & 1 & 1 \\ 0 & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 1 & -1 & 1 & 5 & 7 & 5 & 1 \\ 0 & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 5 & 1 & -1 & 1 & 5 & 7 & 1 \\ 0 & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 7 & 5 & 1 & -1 & 1 & 5 & 1 \\ 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 5 & 7 & 5 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 1 & 5 & 7 & 5 & 1 & -1 & 1 \\ \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

Non-arithmetic! (à la Deligne-Mostow)