Damaris Schindler: (Utrecht)

On integral points on degree four del Pezzo surfaces

Thm (Hasse-Minkowski) Let $Q(x_1,\dots,x_n) \in \mathbb{Z}[x_1,\dots,x_n]$ be a homog. quad. form. Then Q represents 0 (nontrivially in $\mathbb{Z}$) iff Q represents 0 "locally":

Q represents 0 over $\mathbb{R}$, $\mathbb{Z}/p\mathbb{Z}$ $\forall$ prime p.

Qu: what about non-homog. forms?

Ex: (Schulze-Pillot, Xu) $n,m \in \mathbb{N}$, $(n,m)=1$, $k \geq 1$,

$n \equiv 5 \pmod 8$, $2 \,\|\, m$.

$C_{n,m} : m^2x^2 + n^{2k}y^2 - nz^2 = 1$

Then $C_{n,m}(\mathbb{R}) \neq \emptyset$ and $C_{n,m}(\mathbb{Z}/p\mathbb{Z}) \neq \emptyset$ $\forall p$,

but $C_{n,m}(\mathbb{Z}) = \emptyset$.

• Local points: $(\frac{1}{m}, 0, 0), (0, \frac{1}{n^k}, 0)$

At least 1 exists mod p.

• No global points: Consider Azumoya algebra gen'd by $(1+n^k y, n)$ (in fn. field)

If $(x,y,z) \in C_{n,m}(\mathbb{Z})$,

$$\prod_{v \in \Omega_{\mathbb{Q}}} (1+n^k y, n)_v = 1$$

Places in $\mathbb{Q}$ — Hilbert symbol

But w/ an integer soln., Hilbert symbol is $\begin{cases} 1 & v \neq 2 \\ -1 & v = 2 \end{cases}$

so no local pts. satisfy global relation.

Defn: (Integral Hasse principle)

Let $f_1, \dots, f_R \in \mathbb{Z}[x_1, \dots, x_n]$, $\underline{X} \subseteq \mathbb{A}^n_{\mathbb{Z}}$ given by (or more properly, a scheme $\underline{X}_{\mathbb{Z}}$)

$$f_i(x_1, \dots, x_n) = 0 \quad \forall \quad 1 \leq i \leq R$$

The integral Hasse principle (HP) holds for X

~~if $X(\mathbb{Z}) \neq \emptyset$ as soon~~

if $\underline{X}(\mathbb{R}), \underline{X}(\mathbb{Z}/p) \neq \emptyset \implies \underline{X}(\mathbb{Z}) \neq \emptyset$.

Qn: When does HP hold?

When not, what are the obstructions?

Colliot-Thélène, Xu (2009): Examples above explained by "integral Brauer-Manin obstruction":

$X = \underline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ (i.e. take variety $/\mathbb{Q}$)

Brauer group $Br(X) = H^2_{et}(X, \mathbb{G}_m)$

There is a pairing

$$Br(X) \times X(\mathbb{A}_{\mathbb{Q}}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$(x_v)_v$, $x_v \in X(\mathbb{Q}_v)$ $\forall v$, and $x_v \in \underline{X}(\mathbb{Z})$ for almost all v.

$$(\alpha, (x_v)_{v \in \Omega_{\mathbb{Q}}}) \longmapsto \sum_{v \in \Omega_{\mathbb{Q}}} ev_{\alpha, v}(x_v)$$

evaluation map

Let $X(\mathbb{A}_\mathbb{Q})^{Br} := \{ (x_v) \in X(\mathbb{A}_\mathbb{Q}) : \sum_{v \in \Omega} ev_{\alpha,v}(x_v) = 0 \ \forall \alpha \in Br(X) \}$

Then. $X(\mathbb{Q}) \subseteq X(\mathbb{A}_\mathbb{Q})^{Br} \subseteq X(\mathbb{A}_\mathbb{Q})$

Rem: If X is projective, $X(\mathbb{A}_\mathbb{Q}) = \prod_v X(\mathbb{Q}_v)$,

so local obstructions are not an issue.

Obstructions at ∞

Ex $2x^2 + 3y^2 + 4z^2 = 1$ violates HP, but not explained by Brauer-Manin obstrucns. (Colliot-Thélène)

Note $X(\mathbb{R})$ is compact.

- To have HP, should have all real components unbounded.

Ex: $x^2 - y^2 = 3$. Real points satisfy ~~$|x+y| \le \sqrt{3}$~~ $|x-y|$ or $|x+\sqrt{y}| \le \sqrt{3}$

$$C \in \mathbb{Z} \setminus \{0\}, \quad F, G \in \mathbb{Z}[x_1, \dots, x_n]$$

$$F(x_1, \dots, x_n) \cdot G(x_1, \dots, x_n) = C$$

integral points restricted to proper subvarieties

Ex: (Harpaz 2015)

$U: ((11x+5)y + 3)z = 3x + 1$ violates HP, not via Brauer-Manin

If $(x,y,z) \in U(\mathbb{Z})$, then $z \neq 0$, and

$$|11x+5| > |3x+1| + 3 > |(11+5)y + 3| + 3$$
$$> |11x+5| \cdot |y|$$

$\Rightarrow y = 0$, but no such solns!

Alternatively, consider $\mathcal{U}(\mathbb{R})$. Any (x,y,z) satisfies $|z|\le 1$ or $|y|\le 1$ or $|x|\le \frac{9}{8}$

<u>Defn</u>: Let $X \subseteq \mathbb{P}^n_{\mathbb{Q}}$ be an irred. Zariski-closed subset, $\ell \in \Gamma(\mathbb{P}^n_{\mathbb{Q}}, \mathcal{O}(1))^{\mathbb{Q}}$ a linear form, $H = V(\ell) \subseteq \mathbb{P}^n_{\mathbb{Q}}$

Set $\mathcal{U} := X \setminus H$. $\mathcal{U}$ is <u>strongly unobstructed at ∞</u> if every component $\mathcal{U}' \subseteq \mathcal{U}(\mathbb{R})$ satisfies:

Given $d>0$, finitely many forms $s_1, \dots, s_r \in \Gamma(X, \mathcal{O}(d))$ that are not scalar mults. of ℓ^d and $c \in \mathbb{R}$,

$\exists\ x \in \mathcal{U}'(\mathbb{R})$ s.t. $\left| s_i(x)/\ell^d(x) \right| \ge c \quad \forall\ 1 \le i \le r$

<u>Remark</u>: This explains all examples above.

- If this condition is violated and $\mathcal{U}$ has only one real component, then integral pts. cannot be Zariski-dense in $\mathcal{U}$

<u>Systems of two quadratic eqns.</u> (w/ J. Jahnel)

Let $Q_1, Q_2 \in \mathbb{Z}[x_0, \dots, x_4]$ be quad. forms

Consider $\mathfrak{U} \subseteq \mathbb{P}^4_{\mathbb{Z}}$ given by $\{Q_1 = Q_2 = 0,\ x_4 \ne 0\}$

- Integral pts. on $\mathfrak{U}$ correspond to solns. $(x_0, \dots, x_4) \in \mathbb{Z}^5$ with $x_4 = \pm 1$

Again, properly a scheme $\mathfrak{U}_{\mathbb{Z}}$

Otherwise, reduce mod p and get $x_4 \equiv 0$.

Let $\mathcal{U} := \mathcal{U} \otimes_{\mathbb{Z}} \mathbb{Q}$

Filtration of Brauer group $Br(\mathbb{Q}) \subseteq Br_1(\mathcal{U}) \subseteq Br(\mathcal{U})$

"algebraic part of Brauer" $\operatorname{Ker}(Br(\mathcal{U}) \to Br(\overline{\mathcal{U}}))$

Thm (Jahnel–S. 2016) $X \subseteq \mathbb{P}^4_{\mathbb{Q}}$ degree four del Pezzo surface, $H \subseteq \mathbb{P}^4_{\mathbb{Q}}$ $\mathbb{Q}$-rational hyperplane s.t. $H \cap X$ is geometrically irred., $\mathcal{U} := X \setminus H$.

Then $Br_1(\mathcal{U}) / Br(\mathbb{Q})$ is isom. to one of the following:

$$0, \quad (\mathbb{Z}/2\mathbb{Z})^k \quad 1 \le k \le 4$$

$$\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\ell} \quad 0 \le \ell \le 2$$

Remark: $Br_1(X)$ calculated by Swinnerton-Dyer:

$$Br_1(X)/Br(\mathbb{Q}) \cong 0, \ \mathbb{Z}/2\mathbb{Z}, \text{ or } (\mathbb{Z}/2\mathbb{Z})^2$$

Qn: Transcendental part of Brauer gp?

Prop: $\mathcal{U}, X, H$ as above. $D := X \cap H$ non-singular. (genus 1)

Then $$Br(\mathcal{U})/Br_1(\mathcal{U}) \hookrightarrow \underset{\text{(If } \exists\, \mathbb{Q} \text{ pt., an ell. curve)}}{Jac(D)}(\mathbb{Q})_{tors}$$

- Constructing individual transcendental Brauer class difficult, but can show they exist in families.

Ex: X: $$\begin{cases} x_0(8x_1+3) + x_2^2 - 3x_3 = 2 \\ (8x_3+2)(16x_1+x_2+8x_3+8) + x_4^2 - 8x_1 = 3 \end{cases}$$

Violates HP w/ transcendental Brauer-Manin obstruction

Obstruction of the form $(8x_1+3, 8x_3+2)$

- $\underline{X}$ strongly unobstructed at ∞
  - no known "direct" proof ~~of~~ that $\underline{X}(\mathbb{Z}) = \emptyset$.

<u>Qn</u>: How often do we expect $Br(U)/Br(\mathbb{Q}) \neq 0$?

Assume X given by two quad. forms

(*) $$x^T A x = x^T B x = 0, \quad A, B \in M_{5\times5}(\mathbb{Q})$$

Let $$f(\lambda,\mu) := \det(\lambda A + \mu B)$$

<u>Thm</u>: If $Br_1(U) \neq Br(\mathbb{Q})$, then $f(\lambda,\mu)$ is reducible $/\mathbb{Q}$.

A model family: Fix $A \in M_{5\times5}(\mathbb{Q})$, $\det(A) \neq 0$

$$N_{Br}(P) := \#\{B \in M_{5\times5}(\mathbb{Z}) : B = B^T, |b_{ij}| \leq P, \text{ (*) smooth codim } 2, Br_1(U) \neq Br(\mathbb{Q})\}$$

<u>Thm (J-S)</u> $$N_{Br}(P) \ll P^{14 + 1/5 + \varepsilon}$$

<u>Note</u>: $$|M_{5\times5}(\mathbb{Z})| = O(P^{15})$$

<u>Qn</u>: Are these the <u>only</u> possible obstructions?

"Two quad. forms on $x_0, \dots, x_4$ are K3-surfaces, so perhaps in this setting"