The fifth moment of modular L-functions

Matthew P. Young
Texas A&M University
Joint work with Eren Mehmet Kiral

May 2017

Statement of result

- Let q be prime, and $\kappa \in \{4, 6, 8, 10, 14\}$.
- Let $H_\kappa(q)$ be the set of weight κ Hecke eigenforms on $\Gamma_0(q)$ with trivial nebentypus.

Statement of result

- Let q be prime, and $\kappa \in \{4, 6, 8, 10, 14\}$.
- Let $H_\kappa(q)$ be the set of weight κ Hecke eigenforms on $\Gamma_0(q)$ with trivial nebentypus.

Theorem (Kiral, Y.)

We have

$$\sum_{f \in H_\kappa(q)} L(1/2, f)^5 \ll q^{1+\theta+\varepsilon}$$

where $\theta = \frac{7}{64}$.

Remarks

- The conditions on κ and q ensure that every element of $H_\kappa(q)$ is a newform.

Remarks

- The conditions on κ and q ensure that every element of $H_\kappa(q)$ is a newform.
- The central values are nonnegative, by Waldspurger/Kohnen-Zagier/Kohnen.

Remarks

- The conditions on κ and q ensure that every element of $H_\kappa(q)$ is a newform.
- The central values are nonnegative, by Waldspurger/Kohnen-Zagier/Kohnen.
- Subconvexity:

$$L(1/2, f) \ll q^{\frac{1+\theta}{5}+\varepsilon}.$$

Previous results

- Duke, Friedlander, Iwaniec (1992) bounded the amplified fourth moment, obtaining

$$L(1/2, f) \ll q^{\frac{47}{192}+\varepsilon}.$$

- Kowalski, Michel, VanderKam (2000) obtained an asymptotic formula with a power-saving error term for the twisted fourth moment.
- Balkanova, Frolenkov (2016) improved the error term in the twisted fourth moment, obtaining

$$L(1/2, f) \ll q^{\frac{27-30\theta}{112-128\theta}+\varepsilon}$$

- Many other results in different aspects varying other parameters or generalizing to other settings.

Higher moments

- The fourth moment is the "barrier" to subconvexity. It is easy to bound, and leads only to convexity. Amplifying gives a way to break past this barrier.

Higher moments

- The fourth moment is the "barrier" to subconvexity. It is easy to bound, and leads only to convexity. Amplifying gives a way to break past this barrier.
- Analogous problem: Conrey and Iwaniec (2000) showed

$$\sum_{f \in H_\kappa(q)} L(1/2, f \otimes \chi_q)^3 \ll q^{1+\varepsilon},$$

where χ_q is the quadratic character.

Higher moments

- The fourth moment is the “barrier” to subconvexity. It is easy to bound, and leads only to convexity. Amplifying gives a way to break past this barrier.
- Analogous problem: Conrey and Iwaniec (2000) showed

$$\sum_{f \in H_\kappa(q)} L(1/2, f \otimes \chi_q)^3 \ll q^{1+\varepsilon},$$

  where χ_q is the quadratic character.
- Here the second moment is the barrier to subconvexity.

Higher moments

- The cubic moment also has generalizations varying other aspects (Ivic, Z. Peng, Petrow, Y., Petrow-Y.)

Higher moments

- The cubic moment also has generalizations varying other aspects (Ivic, Z. Peng, Petrow, Y., Petrow-Y.)
- Other than these examples, and now the fifth moment, there are no known sharp bounds on a moment that is one larger than the barrier to subconvexity.

Zeta

- We know
$$\int_0^T |\zeta(1/2+it)|^4 dt \ll T^{1+\varepsilon}.$$
- We don't know
$$\int_0^T |\zeta(1/2+it)|^6 dt \ll T^{1+\varepsilon}.$$

Moment identities

We have a rough formula of the shape

$$\sum_{f \in H_\kappa(q)} L(1/2, f)^5 \leftrightarrow \sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^{\varepsilon}}} \lambda_j(q) L(1/2, u_j)^4 + \dots,$$

and the RHS can be bounded using $|\lambda_j(q)| \leq 2q^\theta$, and the spectral large sieve.

Moment identities

We have a rough formula of the shape

$$\sum_{f \in H_\kappa(q)} L(1/2, f)^5 \leftrightarrow \sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^\varepsilon}} \lambda_j(q) L(1/2, u_j)^4 + \dots,$$

and the RHS can be bounded using $|\lambda_j(q)| \leq 2q^\theta$, and the spectral large sieve.

Reminiscent of other moment identities of Kuznetsov/Motohashi, but in an approximate form.

Initial steps

Using approximate functional equations, write

$$L(1/2, f)^2 \approx \sum_{m \leq q} \frac{\lambda_f(m)\tau(m)}{\sqrt{m}},$$

and

$$L(1/2, f)^3 \approx \sum_{n \leq q^{3/2}} \frac{\lambda_f(n)\tau_3(n)}{\sqrt{n}},$$

where $\lambda_f(n)$ is the nth Hecke eigenvalue of f.

Petersson

The Petersson trace formula says

$$\sum_{f\in H_\kappa(q)} \omega_f \lambda_f(m)\lambda_f(n)$$
$$= \delta_{m=n} + 2\pi i^{-\kappa} \sum_{c\equiv 0 \pmod q} \frac{S(m,n;c)}{c} J_{\kappa-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

Here ω_f are Petersson weights which satisfy $\omega_f = q^{-1+o(1)}$, and the Kloosterman sum is

$$S(m,n;c) = \sum_{x \pmod c}^{*} e\Big(\frac{xm+\overline{x}n}{c}\Big),$$

where $x\overline{x} \equiv 1 \pmod c$.

Applying Petersson

AFE and Petersson give:

$$\sum_{f\in H_k(q)} \omega_f L(1/2,f)^5 \approx q^{o(1)}$$

$$+\sum_{c\equiv 0 \pmod q}\sum_{m\leq q}\sum_{n\leq q^{3/2}} \frac{\tau(m)\tau_3(n)S(m,n;c)}{c\sqrt{mn}} J_{\kappa-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

Applying Petersson

AFE and Petersson give:

$$\sum_{f \in H_k(q)} \omega_f L(1/2, f)^5 \approx q^{o(1)}$$

$$+ \sum_{c \equiv 0 \pmod q} \sum_{m \le q} \sum_{n \le q^{3/2}} \frac{\tau(m)\tau_3(n) S(m, n; c)}{c\sqrt{mn}} J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

- The Weil bound only gives $\ll q^{7/8+\varepsilon}$, so a lot of cancellation is required.

Applying Petersson

AFE and Petersson give:

$$\sum_{f \in H_k(q)} \omega_f L(1/2, f)^5 \approx q^{o(1)}$$

$$+ \sum_{c \equiv 0 \pmod q} \sum_{m \le q} \sum_{n \le q^{3/2}} \frac{\tau(m)\tau_3(n)S(m,n;c)}{c\sqrt{mn}} J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

- The Weil bound only gives $\ll q^{7/8+\varepsilon}$, so a lot of cancellation is required.
- The Bessel function $J_{\kappa-1}(x)$ is largest in absolute value when $x \asymp 1$, so let's assume

$$m \asymp q, \quad n \asymp q^{3/2}, \quad c \asymp \sqrt{mn} \asymp q^{5/4}.$$

Voronoi summation

The usual technique when studying the (amplified/mollified) fourth moment in this family is to apply Voronoi summation, which (roughly) says

$$\sum_{m \asymp q} \frac{\tau(m)}{\sqrt{m}} e\left(\frac{\overline{x}m}{c}\right) \approx \sum_{m \asymp \frac{c^2}{q}} \frac{\tau(m)}{\sqrt{m}} e\left(\frac{-xm}{c}\right).$$

Voronoi summation

The usual technique when studying the (amplified/mollified) fourth moment in this family is to apply Voronoi summation, which (roughly) says

$$\sum_{m \asymp q} \frac{\tau(m)}{\sqrt{m}} e\left(\frac{\overline{x}m}{c}\right) \approx \sum_{m \asymp \frac{c^2}{q}} \frac{\tau(m)}{\sqrt{m}} e\left(\frac{-xm}{c}\right).$$

- Since $\frac{c^2}{q} \asymp q^{3/2}$, the dual sum is longer than the original one, and so even more cancellation is required after applying this summation formula.

Voronoi, continued

- However, the exponential sum would convert to a Ramanujan sum eventually leading to the study of a shifted divisor sum

$$\sum_n \tau_3(n)\tau(n+h).$$

- Work of Hooley (1957), Deshouillers (1981), Pitt (1995), Munshi (2013), Topacogullari (2016), ... potentially allow one to attack the problem with this type of approach.

Poisson summation

Open up $\tau(m) = \sum_{m_1 m_2 = m} 1$, and suppose $m_1 \leq m_2$. Then apply Poisson in m_2 only:

$$\sum_{m_2 \asymp M_2} \frac{1}{\sqrt{m_2}} S(m_1 m_2, n; c) \approx \sqrt{c} \sum_{k \asymp \frac{c}{M_2}} \frac{1}{\sqrt{k}} e\Big(\frac{m_1 n \overline{k}}{c}\Big).$$

Poisson summation

Open up $\tau(m) = \sum_{m_1 m_2 = m} 1$, and suppose $m_1 \leq m_2$. Then apply Poisson in m_2 only:

$$\sum_{m_2 \asymp M_2} \frac{1}{\sqrt{m_2}} S(m_1 m_2, n; c) \approx \sqrt{c} \sum_{k \asymp \frac{c}{M_2}} \frac{1}{\sqrt{k}} e\left(\frac{m_1 n \overline{k}}{c}\right).$$

- Now $\frac{c}{M_2} \approx \frac{q^{5/4}}{q^{1/2}} = q^{3/4}$.

Reciprocity

- We can use the reciprocity formula:

$$e\left(\frac{m_1 n\overline{k}}{c}\right) = e\left(\frac{-m_1 n\overline{c}}{k}\right) e\left(\frac{m_1 n}{ck}\right)$$

.

- We have $\frac{m_1 n}{ck} \approx \frac{q^{1/2} q^{3/2}}{q^{5/4} q^{3/4}} = 1$ so this part is not very oscillatory.
- The modulus is greatly reduced, from $c \approx q^{5/4}$ to $k \approx q^{3/4}$.

Voronoi/Poisson

Now that the modulus is reduced, we may apply Voronoi summation to $\sum_n \tau_3(n) e(-\frac{m_1 n\bar{c}}{k})$. This is equivalent to opening $\tau_3(n) = \sum_{n_1 n_2 n_3 = n} 1$ and applying Poisson in each n_i.

Voronoi/Poisson

Now that the modulus is reduced, we may apply Voronoi summation to $\sum_n \tau_3(n)e(-\frac{m_1 n\bar{c}}{k})$. This is equivalent to opening $\tau_3(n) = \sum_{n_1 n_2 n_3 = n} 1$ and applying Poisson in each n_i.

The moment gets converted to

$$\sum_{m_1 \approx q^{1/2}} \sum_{\substack{c \equiv 0 \pmod q \\ c \approx q^{5/4}}} \sum_{k \approx q^{3/4}} \sum_{p \approx q^{3/4}} \frac{\tau_3(p)}{k\sqrt{m_1 p c}} S(p, c\overline{m_1}; k).$$

Voronoi/Poisson

Now that the modulus is reduced, we may apply Voronoi summation to $\sum_n \tau_3(n) e(-\frac{m_1 n \bar{c}}{k})$. This is equivalent to opening $\tau_3(n) = \sum_{n_1 n_2 n_3 = n} 1$ and applying Poisson in each n_i.

The moment gets converted to

$$\sum_{m_1 \approx q^{1/2}} \sum_{\substack{c \equiv 0 \pmod q \\ c \approx q^{5/4}}} \sum_{k \approx q^{3/4}} \sum_{p \approx q^{3/4}} \frac{\tau_3(p)}{k\sqrt{m_1 p c}} S(p, c\overline{m_1}; k).$$

- Now the Weil bound gives $O(q^{5/8})$.
- Poisson in m_1 would give some more savings, but the Kloosterman sum would become hyper.

Kuznetsov

Deshouillers-Iwaniec (1982), building on Bruggemann (1978) and Kuznetsov (1980), showed

$$\sum_c \frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;c)}{c} f\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_j h_f(t_j)\overline{\nu}_{\mathfrak{a},j}(m)\nu_{\mathfrak{b},j}(n) + \dots.$$

Here $\mathfrak{a}, \mathfrak{b}$ are cusps for Γ, and $\nu_{\mathfrak{a},j}(m)$ is the m-th Fourier coefficient of a Maass form u_j, expanded around the cusp $\mathfrak{a}$.

Kloosterman sums with cusps

Question: Given a sum of Kloosterman sums, with various coprimality conditions, congruences, etc., can we recognize it as the sum of $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$?

Kloosterman sums with cusps

Question: Given a sum of Kloosterman sums, with various coprimality conditions, congruences, etc., can we recognize it as the sum of $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$?

- Deshouillers-Iwaniec worked out many cases.
- Blomer-Milicevic (2015) showed how to decompose

$$\sum_{c \equiv a \pmod{q}} S(m, n; c)$$

into $S_{\mathfrak{a}\mathfrak{b}}$'s.

Kloosterman sums with cusps

Goal:

$$\sum_{(k,m_1)=1} \frac{S(p, c\overline{m_1}; k)}{k}.$$

Example.

$$S_{\infty,1/r}(m,n;c) = e\Big(m\frac{\overline{r}}{s}\Big) S(m\overline{s}, n; \ell r),$$

where $N = rs$, $(r,s) = 1$, and ℓ runs over integers coprime to s.

Kloosterman sums with cusps

Goal:

$$\sum_{(k,m_1)=1} \frac{S(p, c\overline{m_1}; k)}{k}.$$

Example.

$$S_{\infty,1/r}(m,n;c) = e\left(m\frac{\overline{r}}{s}\right) S(m\overline{s}, n; \ell r),$$

where $N = rs$, $(r,s) = 1$, and ℓ runs over integers coprime to s.

This is with scaling matrices

$$\sigma_\infty = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \sigma_{1/r} = \begin{pmatrix} \sqrt{s} & \\ r\sqrt{s} & 1/\sqrt{s} \end{pmatrix}.$$

(The Kloosterman sum and the Fourier coefficients both depend on the choice of scaling matrices)

Kloosterman sums with cusps

Observation: Whenever possible, one should choose the scaling matrices to be Atkin-Lehner operators.

Kloosterman sums with cusps

Observation: Whenever possible, one should choose the scaling matrices to be Atkin-Lehner operators.

Since then the Fourier coefficients at an Atkin-Lehner cusp $\mathfrak{a}$ (i.e., a cusp equivalent to ∞ by an Atkin-Lehner involution) will be ± 1 times the Fourier coefficients at ∞, preserving multiplicativity, etc.

Kloosterman sums with cusps

Observation: Whenever possible, one should choose the scaling matrices to be Atkin-Lehner operators.

Since then the Fourier coefficients at an Atkin-Lehner cusp $\mathfrak{a}$ (i.e., a cusp equivalent to ∞ by an Atkin-Lehner involution) will be ± 1 times the Fourier coefficients at ∞, preserving multiplicativity, etc.

"Theorem." Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are Atkin-Lehner cusps. Then we have an explicit formula for $S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \chi)$.

Example. Let $N = rs$ with $(r, s) = 1$, and let $\sigma_{1/r}$ be an appropriate Atkin-Lehner operator. Then

$$S_{\infty,1/r}(m, n; c) = S(\bar{s}m, n; c),$$

and c runs over integers $\equiv 0 \pmod{r}$, coprime to s.

Eisenstein series

We also explicitly calculated the Fourier coefficients of $E_{\mathfrak{a}}(z, s)$, expanded around an arbitrary Atkin-Lehner cusp.

Back to L-functions

After Kuznetsov, we are led to

$$\sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^{\varepsilon}}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \approx q^{5/4}}} \sum_{p_1, p_2, p_3 \ll q^{1/4}} \frac{\nu_j(p_1 c)\nu_j(p_2 p_3)}{\sqrt{p_1 p_2 p_3 c}}.$$

Back to L-functions

After Kuznetsov, we are led to

$$\sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^{\varepsilon}}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \approx q^{5/4}}} \sum_{p_1, p_2, p_3 \ll q^{1/4}} \frac{\nu_j(p_1 c)\nu_j(p_2 p_3)}{\sqrt{p_1 p_2 p_3 c}}.$$

This can be expressed in terms of L-functions as

$$\sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^{\varepsilon}}} |\nu_j(1)|^2 \lambda_j(q) L(1/2, u_j)^4.$$

Back to L-functions

After Kuznetsov, we are led to

$$\sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^{\varepsilon}}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \approx q^{5/4}}} \sum_{p_1, p_2, p_3 \ll q^{1/4}} \frac{\nu_j(p_1 c)\nu_j(p_2 p_3)}{\sqrt{p_1 p_2 p_3 c}}.$$

This can be expressed in terms of L-functions as

$$\sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \text{ level } m_1 \\ t_j \ll q^{\varepsilon}}} |\nu_j(1)|^2 \lambda_j(q) L(1/2, u_j)^4.$$

Finally, we use $|\lambda_j(q)| \leq 2q^{\theta}$, and an "easy" bound on the fourth moment in this family, leading to the main theorem.

Oscillatory integrals

Problem 0. How to understand

$$I = I(\phi, w) = \int_{-\infty}^{\infty} e^{i\phi(t)} w(t) dt.$$

E.g. $\phi(t) = A\sqrt{t} - \lambda t$, $w(t) = \omega(t/T)$.

Oscillatory integrals

Problem 0. How to understand

$$I = I(\phi, w) = \int_{-\infty}^{\infty} e^{i\phi(t)} w(t) dt.$$

E.g. $\phi(t) = A\sqrt{t} - \lambda t$, $w(t) = \omega(t/T)$.

Integration by parts: If $|\phi'(t)|$ is not too small inside the support of w, then I is very small by repeated integration by parts.

Oscillatory integrals

Problem 0. How to understand

$$I = I(\phi, w) = \int_{-\infty}^{\infty} e^{i\phi(t)} w(t)dt.$$

E.g. $\phi(t) = A\sqrt{t} - \lambda t$, $w(t) = \omega(t/T)$.

Integration by parts: If $|\phi'(t)|$ is not too small inside the support of w, then I is very small by repeated integration by parts.

In practice this may say that unless A, T, λ are constrained in a certain way (e.g. $A\sqrt{T} \asymp \lambda T$) then I is very small.

Oscillatory integrals

Problem 0. How to understand

$$I = I(\phi, w) = \int_{-\infty}^{\infty} e^{i\phi(t)} w(t) dt.$$

Stationary phase method: If $|\phi''(t)|$ is not too small, and $\phi'(t_0) = 0$ for some t_0, (and additional technical conditions hold), then

$$I \sim c e^{i\phi(t_0)} \frac{w(t_0)}{\sqrt{|\phi''(t_0)|}}.$$

Oscillatory integrals

Problem 0. How to understand

$$I = I(\phi, w) = \int_{-\infty}^{\infty} e^{i\phi(t)} w(t) dt.$$

Stationary phase method: If $|\phi''(t)|$ is not too small, and $\phi'(t_0) = 0$ for some t_0, (and additional technical conditions hold), then

$$I \sim c e^{i\phi(t_0)} \frac{w(t_0)}{\sqrt{|\phi''(t_0)|}}.$$

Problem 1. The error term is not small enough for some applications. How to improve?

Oscillatory integrals

Solution 1. (Blomer, Khan, Y., 2013): There exists a "nice" function F so that

$$I = ce^{i\phi(t_0)} \frac{F(t_0)}{\sqrt{|\phi''(t_0)|}},$$

plus a very small error term.

Oscillatory integrals

Solution 1. (Blomer, Khan, Y., 2013): There exists a "nice" function F so that

$$I = ce^{i\phi(t_0)} \frac{F(t_0)}{\sqrt{|\phi''(t_0)|}},$$

plus a very small error term.

What is F? Complicated: subtract off the quadratic Taylor polynomial expansion to ϕ centered at y, include the rest into w, and call this function G. Then

$$F(y) = \sum_n p_n(y), \qquad p_n(y) = c_n(\phi''(y))^{-n} G^{(2n)}(t)|_{t=y}.$$

Oscillatory integrals

Solution 1. (Blomer, Khan, Y., 2013): There exists a "nice" function F so that

$$I = ce^{i\phi(t_0)} \frac{F(t_0)}{\sqrt{|\phi''(t_0)|}},$$

plus a very small error term.

What is F? Complicated: subtract off the quadratic Taylor polynomial expansion to ϕ centered at y, include the rest into w, and call this function G. Then

$$F(y) = \sum_n p_n(y), \qquad p_n(y) = c_n(\phi''(y))^{-n} G^{(2n)}(t)|_{t=y}.$$

Problem 2. If ϕ and w depend on other variables, how does F depend on these variables?

What is a “nice” function?

Suppose that a collection of smooth functions has support on a product of dyadic intervals. Say that the collection is X-inert if for all $j_1, \dots, j_d \geq 0$, there exists a constant $C = C(j_1, \dots, j_d) > 0$ (depending on the family) so that

$$|w^{(j_1,\dots,j_d)}(x_1, \dots, x_d)| \leq \frac{C(j_1, \dots, j_d) X^{j_1+\dots+j_d}}{|x_1|^{j_1} \dots |x_d|^{j_d}}$$

Example: Fix ω supported on $[1,2]^d$, and let $w(x_1, \dots, x_d) = \omega(x_1/X_1, \dots, x_d/X_d)$.

Stationary phase preserves inertness

Solution 2. Y. (2014), Petrow-Y. (2016), Kiral-Y. (2017).

Theorem. Suppose that $w(t_1, \dots, t_d)$ and $\phi(t_1, \dots, t_d)$ are part of an X-inert family of functions, and the conditions for stationary phase hold for t_1. Then $F(t_0) = F(t_0, t_2, \dots, t_d)$ is an X-inert family of function of $t_2, \dots, t_d$.

Main terms

Poisson summation says

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{\nu\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(-\nu x)dx.$$

The zero-frequency $\nu = 0$ is often the source of the main term on the right hand side, amounting to an approximation of the original sum by an integral.

Main terms

Poisson summation says

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{\nu\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(-\nu x)dx.$$

The zero-frequency $\nu = 0$ is often the source of the main term on the right hand side, amounting to an approximation of the original sum by an integral.

In the sketch of the proof so far, all the zero frequencies were ignored.

Fake main terms

In this problem, the zero frequencies (when some or all of p_1, p_2, p_3 are zero), when trivially bounded, give back a bound that is $O(q^{1/4+\varepsilon})$, not $O(q^{\varepsilon})$.

It is a delicate analytic/arithmetic calculation to show that these fake main terms are truly bounded by $O(q^{\varepsilon})$.

Toy example

Suppose that

$$V(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} x^{-s} ds,$$

where G is holomorphic, even, rapid decay in vertical strips, and $G(0) = 1$. (Here V is the type of weight function that arises from an approximate functional equation, and one can choose G subject to these conditions).

Now consider

$$S = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} V\Big(\frac{n}{\sqrt{q}}\Big).$$

Toy example

Suppose that

$$V(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} x^{-s} ds,$$

where G is holomorphic, even, rapid decay in vertical strips, and $G(0) = 1$. (Here V is the type of weight function that arises from an approximate functional equation, and one can choose G subject to these conditions).
Now consider

$$S = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} V\Big(\frac{n}{\sqrt{q}}\Big).$$

Using $V(x) \ll 1$ for $x \ll 1$, and $V(x) \ll x^{-1}$ for $x \gg 1$, shows that $S \ll q^{1/4}$.

Toy example

Applying the definition of V, and reversing the order of summation and integration, then

$$S = \frac{1}{2\pi i} \int_{(1)} q^{s/2} \zeta(1/2 + s) \frac{G(s)}{s} ds.$$

Moving the contour to the left shows

$$S = q^{1/4} \frac{G(1/2)}{1/2} + O(q^{\varepsilon}).$$

Toy example

Applying the definition of V, and reversing the order of summation and integration, then

$$S = \frac{1}{2\pi i} \int_{(1)} q^{s/2} \zeta(1/2 + s) \frac{G(s)}{s} ds.$$

Moving the contour to the left shows

$$S = q^{1/4} \frac{G(1/2)}{1/2} + O(q^{\varepsilon}).$$

So choose G so that $G(1/2) = 0$.

Toy example

Note

$$\frac{G(1/2)}{1/2} = \int_0^\infty \frac{V(x)}{\sqrt{x}} dx.$$

By contrast, one cannot use G to completely cancel the main term from

$$S' := \sum_{n=1}^{\infty} \frac{1}{n} V\left(\frac{n}{\sqrt{q}}\right),$$

since

$$S' = \frac{1}{2\pi i} \int_{(1)} q^{s/2} \zeta(1+s) \frac{G(s)}{s} ds.$$