Orbit method and analysis of automorphic forms

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May 4, 2017

- This is an informal account of work (still being written) with Paul Nelson.

- Every L-function of degree n has an attached modular form on GL_n, e.g. $\zeta(s)^2$ corresponds to

$$f(x+iy) = \sum_{n\neq 0} \sigma(n)\cos(2\pi nx)\sqrt{y}K_0(2\pi ny) + (\text{ term for } n=0),$$

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- Extract the L-function $L = \sum \frac{a_n}{n^s}$ from f by integration:

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The analogs of this identity in higher rank are sometimes not true; when true they involve products of many Γ functions and are only rarely proven. (For the purpose of proving the analytic properties and functional equation, one can avoid this.)

Here is a remarkable result of Staude; the previous result is the case $n = 2$, $\mathbf{a} = 0$ of this:

$$\int \underbrace{W_{n-1,\mathbf{b}}(y_1, \ldots, y_{n-2})}_{\text{Whit. fn on } \mathrm{GL}_{n-1}} \underbrace{W_{n,\mathbf{a}}(y_1, \ldots, y_{n-1})}_{\text{Whit. fn on } \mathrm{GL}_n} \prod (\pi y_j)^{2js} (2y_j^{-j(n-j-1/2})\frac{dy_j}{y_j}$$

$$= \prod_{j=1}^{n-1} \prod_{k=1}^{n} \Gamma(s + b_j + a_k,)$$

This is a collapsed form of an integral over GL_{n-1}. The GL_{n-1}-translates of W_{n-1} span an irreducible representation indexed by $\mathbf{b} = (b_1, \ldots, b_{n-1})$, and similarly for W_n.

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$$\text{integral of automorphic form(s)}$$
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(a) To exactly evaluate L one needs to evaluate $\mathcal{I}$

(b) To study L analytically one needs to compute $\mathcal{I}$ asymptotically (the focus of our talk).

- I am hopeful that some of the algebraic structures used for (b) may also be relevant for (a).

Spherical harmonics $Y_{\ell,m}$: an orthonormal basis for $L^2(S^2)$

- We have $L^2(S^2) = \bigoplus_{\ell \in \mathbb{N}} V_\ell$, where

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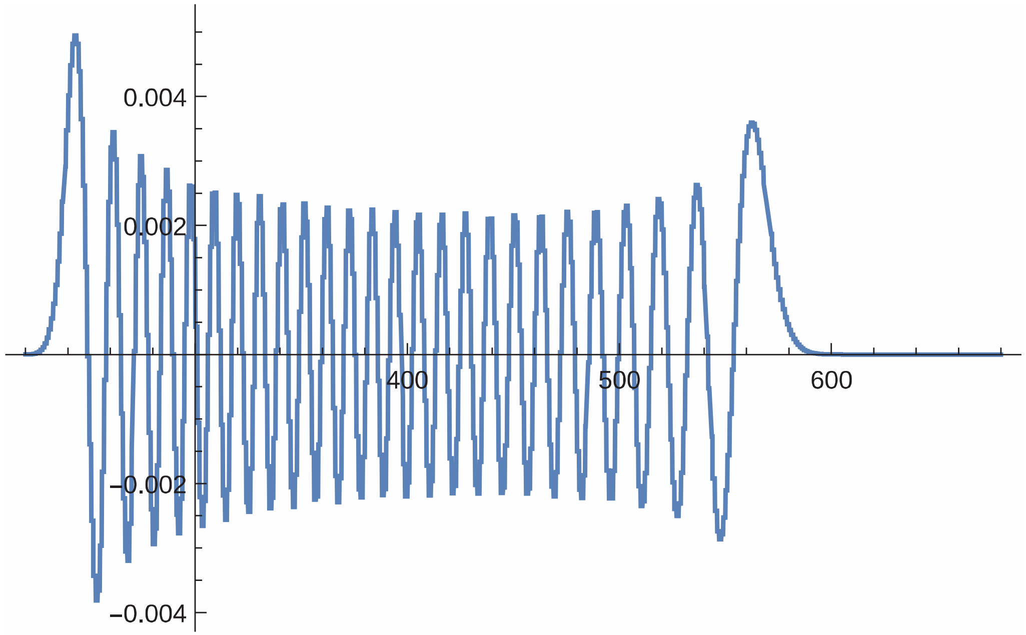
- Further break up V_ℓ into eigenfunctions for z-axis rotations, giving $Y_{\ell,m}$. e.g. $Y_{\ell,\ell} = (x + iy)^\ell$.
- Thus, ℓ indexes an irreducible representation of SO_3, and m tells you where in that representation.

A good toy model

- $$\int_{S^2} Y_{\ell_1,m_1} Y_{\ell_2,m_2} Y_{\ell_3,m_3}$$



- The asymptotic has the form

$$\frac{1}{2\pi Q}\cos^2(S),$$

$$S = \sum(\ell_i + 1/2)\Lambda_i + M + \pi(\sum \ell_i) - 3\pi/4$$

$$Q = \frac{\sqrt{A-2B}}{2}, \quad A = \sum_{i\neq j}(\ell_i+1/2)^2(\ell_j+1/2)^2 - \sum(\ell_i+1/2)^4$$

$$B = -\sum_{\{i,j,k\}=\{1,2,3\}}(\ell_i+1/2)^2 m_j^2 m_k^2$$

$$M = m_1(\pi - \cos^{-1}\left(\frac{(\ell_1+1/2)^2 - m_1^2 + (\ell_2+1/2)^2 - m_2^2 - (\ell_3+1/2)^2 + m_3^2}{2\sqrt{((\ell_1+1/2)^2 - m_1^2)((\ell_2+1/2)^2 - m_2^2)}}\right)) + \text{ similar.}$$

$$\Lambda_3 = \tan^{-1}\left(m_1\frac{(\ell_2+1/2)^2 + (\ell_3+1/2)^2 - (\ell_1+1/2)^2}{2Q(\ell_3+1/2)} - m_2\frac{(\ell_1+1/2)^2 + (\ell_3+1/2)^2 - (\ell_2+1/2)^2}{2Q(\ell_3+1/2)}\right.$$

$$\Lambda_1, \Lambda_2 = \text{ similar, by cyclic permutation.}$$

History of the $3j$ asymptotic

- This asymptotic is due to Ponzano and Regge (1968); the amplitude was already observed by Wigner. Rigorous derivations were given Borodin, Kroshilin, Tolmachev as well as Schulten and Gordon. Most direct method: write it out as an oscillating integral and try to identify critical points, etc. This becomes extremely unwieldy in higher dimension.

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- A good discussion of the surrounding mathematics, and rigorous proof, is *Classical* $6j$ *symbols and the tetrahedron,* by Roberts.
- We will sketch how Kirillov's orbit method helps with this type of problem.

- Microlocal analysis on $\mathbf{R}^n$ amounts to taking a function f and cutting it into "pieces" $f = \sum f_\alpha$, indexed by $\alpha \in \mathbf{R}^n \oplus \mathbf{R}^n$ that are localized in space and frequency simultaneously.

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- Microlocal analysis on a unitary representation V of a Lie G amounts to taking a vector $v \in V$ and cutting it into pieces $v = \sum v_\alpha$ that are localized in the following sense: they are approximate eigenvectors for elements of G near 1.

Example of a localized vector

Consider $v = (x + iy)^\ell$, i.e. the spherical harmonic $Y_{\ell,\ell}$.

- v is an approximate eigenvector: If $g \in G$ is close to e, say $\mathrm{dist}(g, e) < 100/\ell$, we have

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- In fact, the eigenvalue λ_g for $g = \exp(\frac{1}{\ell}\begin{pmatrix} 0 & A & C \\ -A & 0 & B \\ -C & -B & 0 \end{pmatrix})$ is e^{iA}.

One possibility for a formal definition

Say that a sequence of vectors $x_\ell \in V_\ell$ is localized at $(a, b, c) \in \mathbb{R}^3$ if and only if

$$\frac{\|e^{\ell^{-1}X} x_\ell - e^{i(aA+bB+cC)} x_\ell\|}{x_\ell} \to 0.$$

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It turns out

$$a^2 + b^2 + c^2 = 1.$$

There are many different sequences of vectors localized at λ.

How could this help with $\int_{S^2} Y_{\ell_1,m_1} Y_{\ell_2,m_2} Y_{\ell_3,m_3}$?

- Dsintegrate $Y_{\ell_1,m_1} = \int c_\alpha v_\alpha$ into localized vectors v_α, localized at $\alpha = (?, ?, \frac{m_1}{\ell_1})$. Then express this integral as a combination of similar integrals

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- There is are two solutions, mirror images of one another; the asymptotic is determined by the geometry of the corresponding triangles.

(To carry this out, one needs to use vectors that are more severely localized than our prior discussion.)

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- It makes sense perfectly well for infinite-dimensional representations. In this case the story is interesting even for a fixed irreducible representation.
- Replace (a, b, c) by an element of the *dual Lie algebra* $\mathfrak{g}^*$, and $a^2 + b^2 + c^2 = 1$ by the *nilpotent cone*.

- Nelson and I develop the basic "calculus" (with estimates) of cutting vectors up into localized pieces, along the same lines as ΨDO: For any function f on $\mathfrak{g}^*$ get an operator

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- This is not surprising. It is just a quantification of the general philosophy of the orbit method, first advanced by Kirillov, which says that the representations are obtained by quantizing conjugacy classes in $\mathfrak{g}^*$.
- However there is an interesting new phenomenon of "microlocal irreducibility."

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G	H	status of formula
GL_n	GL_{n-1}	Jacquet-PS-Shalika
U_n	U_{n-1}	conj. Ichino-Ikeda, partly proved Zhang
SO_n	SO_{n-1}	conj. Ichino-Ikeda and Harris, proved $n \leq 4$

- Nelson and I prove: assuming the validity of the period formula

$$\lim_{T\to\infty} |\mathcal{F}_T|^{-1} \sum_{\varphi\in\mathcal{F}_T} 2^{-\beta} \frac{L(\frac{1}{2}, \Phi \times \varphi)}{L(1, \mathrm{Ad})} = \frac{1}{2},$$

where

- We impose local constraints, forcing in particular the space $[G]$ to be compact – this eliminates GL_n
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  - one expects $\beta = 2$ most of the time.

- This estimate is sufficiently strong that the corresponing estimate for each individual term which is "weakly subconvex." It is "weak" because the application of Ratner's theorem is at present ineffective.

Microlocal calculus

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- The microlocal calculus permits us to analyze satisfactorily $\mathcal{I}$. But, more interestingly, it also gives insight into the right hand side. Namely we can choose Φ to be itself localized at some nilpotent $\lambda \in \mathfrak{g}^*$ [more precisely there are sequences of scales and vectors involved.]

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- We will show $|\Phi|^2 d\mathrm{vol}$ is very close to the uniform measure on $[G]$. One can deduce that $\int_{[H]} |\Phi|^2 \sim \mathrm{vol}([H])$.

Analysis of localized Φ

We claim that for Φ localized at λ, the measure

$$\mu_\Phi = |\Phi|^2 d\mathrm{vol}_X$$

is approximately invariant by the centralizer $Z(\lambda)$.

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- So we want to know that $\mu_\Phi \sim \mu_{\Phi'}$ are approximately the same, whenever Φ', Φ are localized at the same point λ.
- This seems to (surprisingly) be usually true and is a manifestation of the "microlocal irreducibility" mentioned earlier. Although we haven't proven it in general at the moment, we can sidestep the need for it, by averaging over all vectors localized at λ.

Application of symbol calculus

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- Shrinking the support of f to $\{\lambda\}$ (very carefully) one obtains a $Z(\lambda)$-invariant measure on $[G]$.

Endgame

Unlike in the usual QUE problem, $Z(\lambda)$ is a unipotent group.

- Ratner's theorem classifies $Z(\lambda)$-invariant measures; under our local assumptions, this forces

$$\sum_i \mu_{\Phi_i} \to \text{volume measure.}$$

which is enough for the application.

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- In problems where Φ is more degenerate, could get large $Z(\lambda)$, where more effective measure classification results may be available.

Speculations: Microlocal irreducibility

- Why would $\mu_\Phi = \mu_{\Phi'}$ for Φ, Φ' localized at the same λ? (We have checked this for several groups G by hand.)
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- For p-adic groups this can be formalized into a proof, the basic ideas of which are all in Roger Howe's work of the 1970s! We have not proven it for general Lie groups yet, but it seems tractable.