Local to global principles in integral circle packings

(joint work with K. Stange and X. Zhang)

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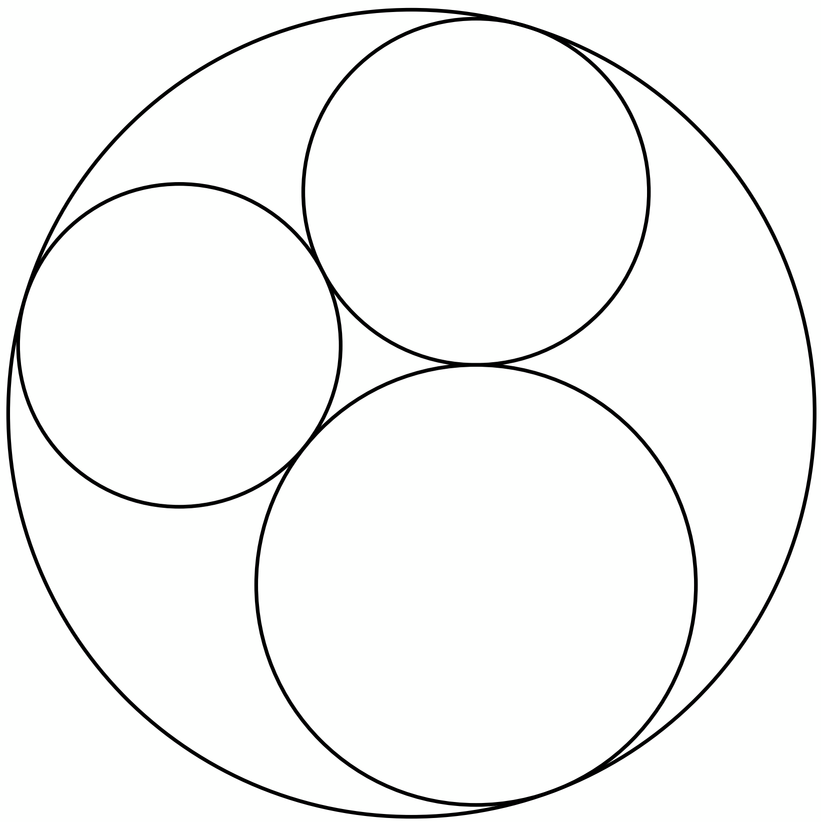
Motivating problem: Apollonian Circle Packings (ACP's)



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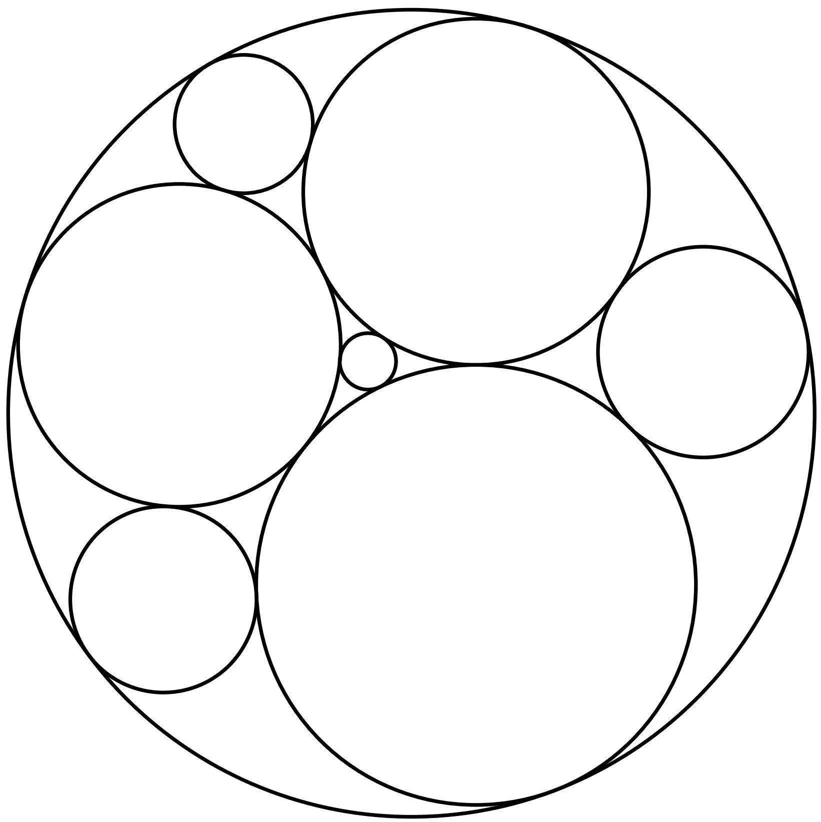

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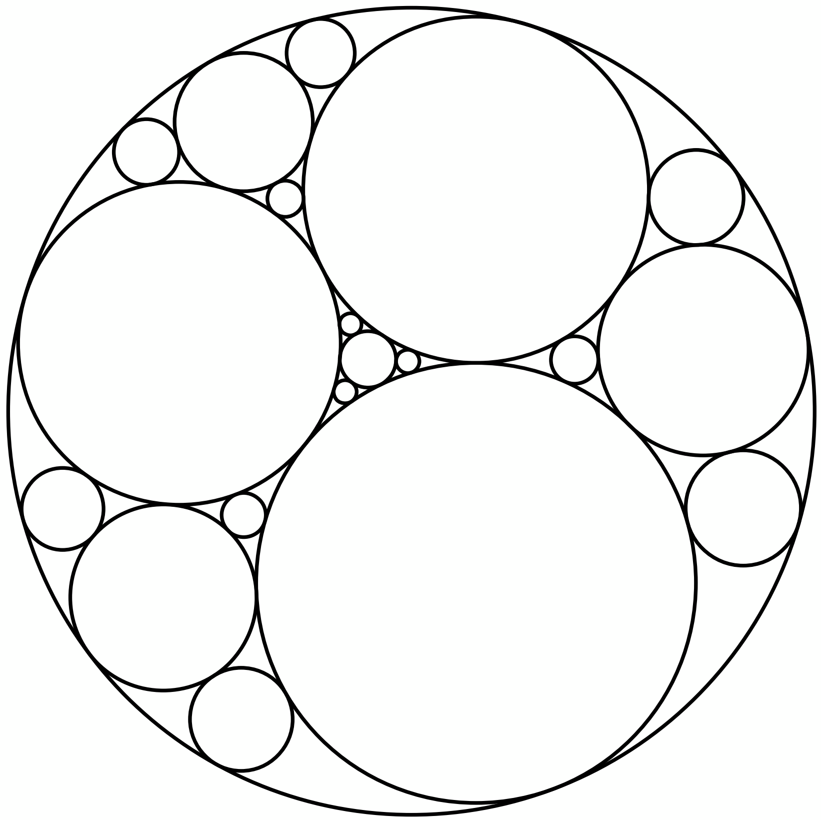
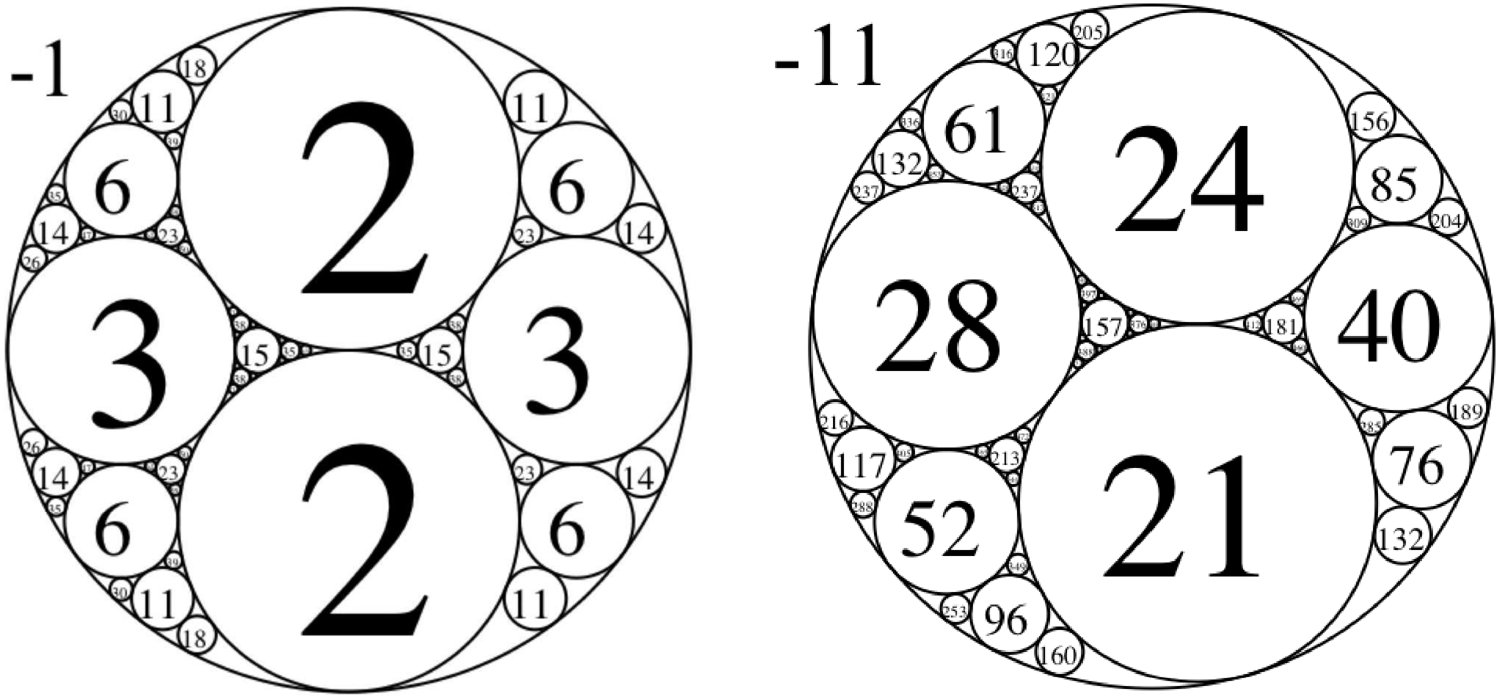


- Start with four pairwise tangent circles, one on the outside
- Into each interstice inscribe a new circle
- This process continues indefinitely, resulting in a packing of infinitely many circles

- If any quadruple of pairwise tangent circles in an ACP have integer curvatures, all circles in the packing have integer curvature $(1/\mathrm{radius})$.



- In the packings above, curvatures allowed to be $2, 3, 6, 11, 14, 15, 18, 23$ and $0, 4, 12, 13, 16, 21 \pmod{24}$, respectively.
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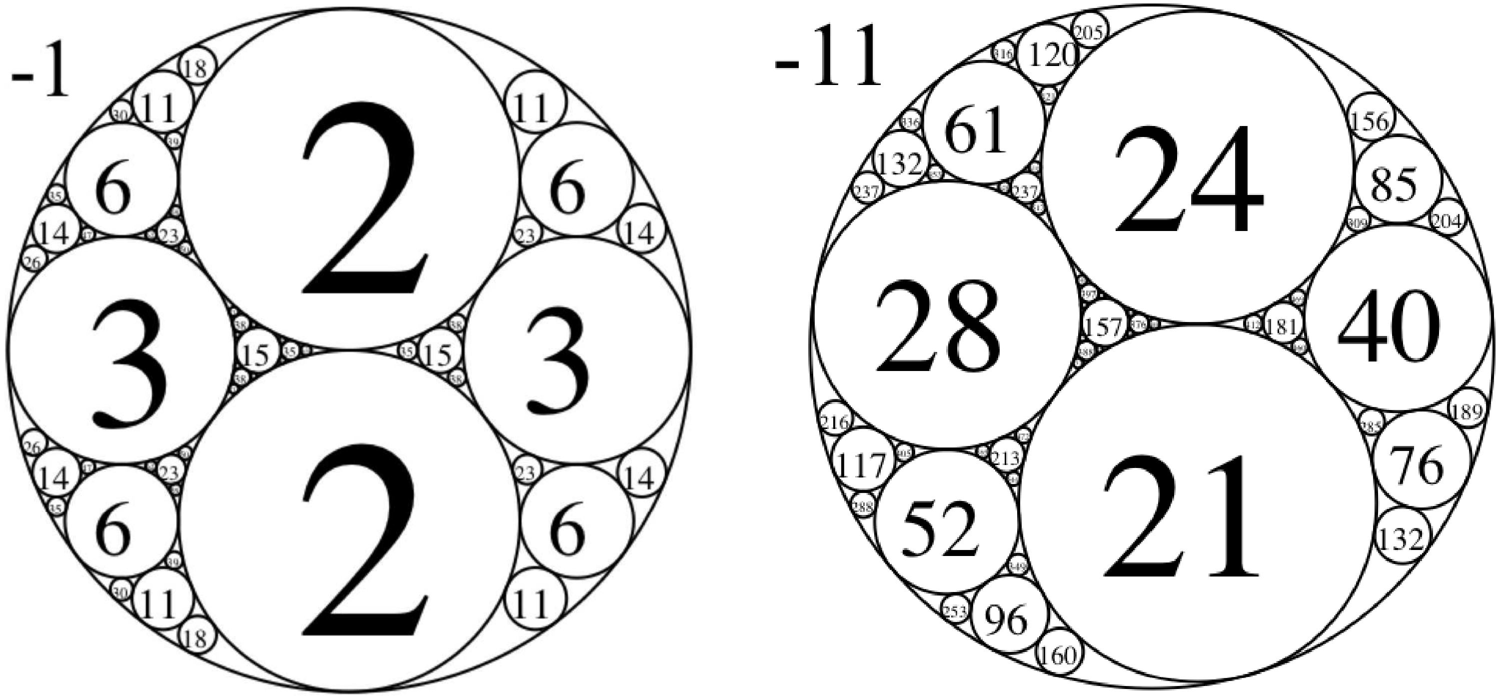

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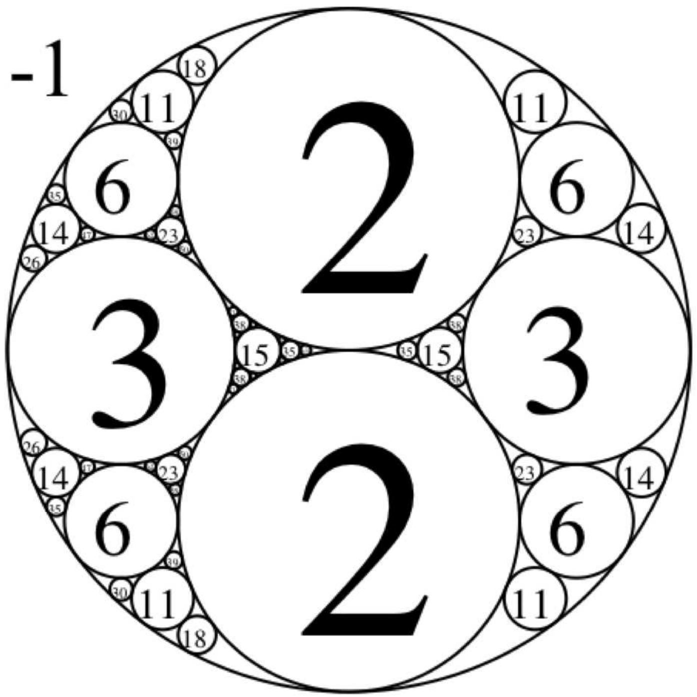

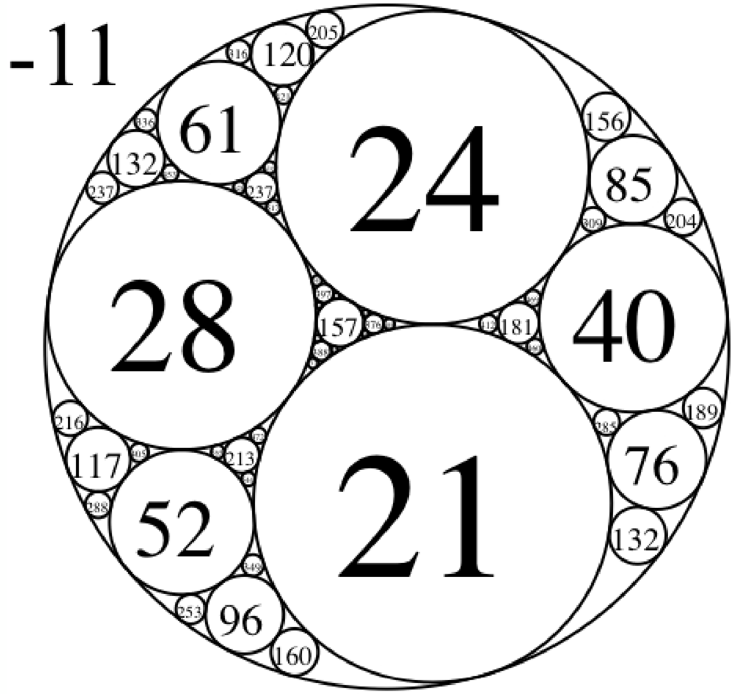

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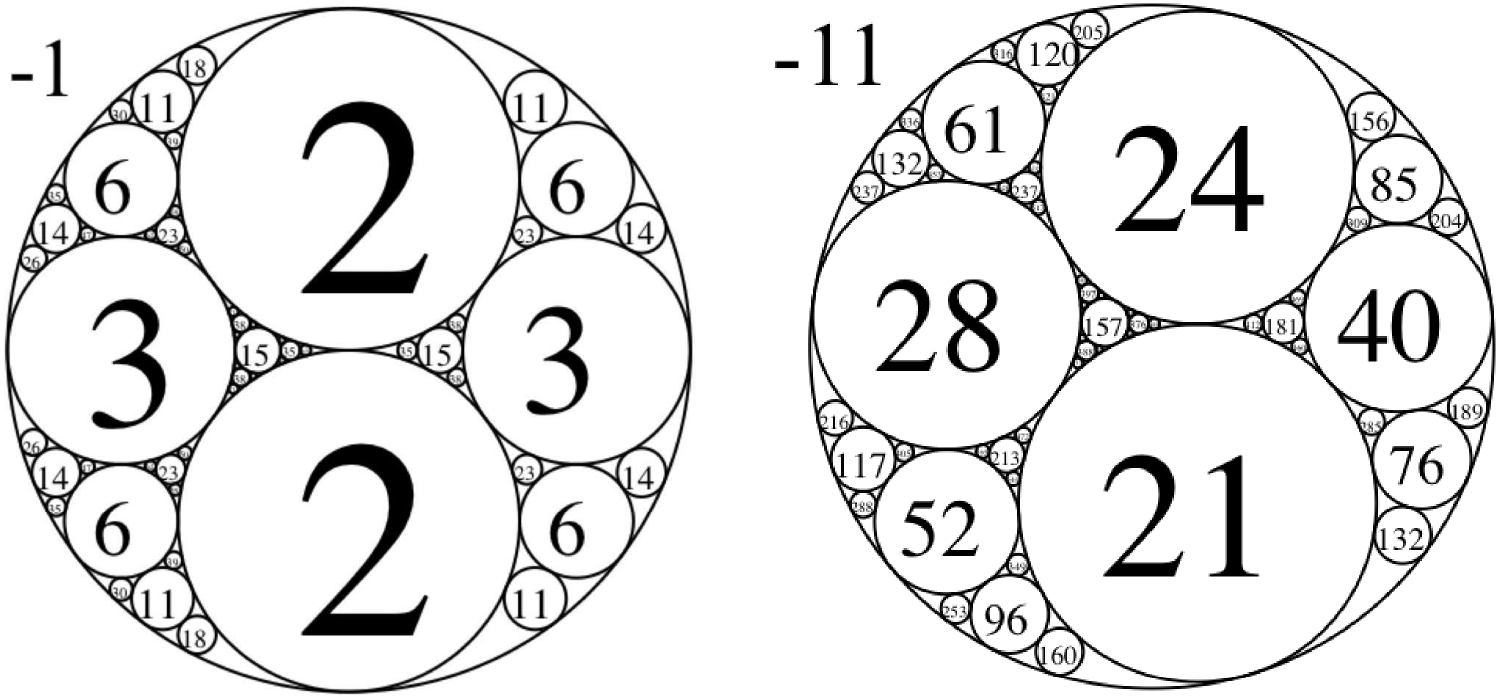

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Theorem 1 (Descartes, 1653)

If $a, b, c,$ and d are curvatures of four pairwise tangent circles, we have

$$Q(a,b,c,d) = 2(a^2+b^2+c^2+d^2) - (a+b+c+d)^2 = 0$$

where a circle which is internally tangent to the others is taken to have "negative" curvature.

- If $\mathbf{v} = (a,b,c,d)$ is a vector of curvatures of four pairwise tangent circles in an ACP, the set of all quadruples of pairwise tangent circles in the packing is precisely the orbit $A\mathbf{v}^T$ of a subgroup $A \subset \mathrm{O}_Q(\mathbb{Z})$ acting on $\mathbf{v}^T$.

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The group A (called the *Apollonian group*) has the following properties:

- It is of infinite index in $O_Q(\mathbb{Z})$.
- A is Zariski dense in $O_Q(\mathbb{C})$.
- Its preimage under the spin homomorphism $\rho : SL_2(\mathbb{C}) \to SO_{\mathbb{R}}(3,1)$ is a subgroup of $SL_2(\mathbb{Z}[i])$ and contains $\Gamma(2)$.
- This preimage reduced modulo p is onto $SL_2(\mathbb{F}_p \times \mathbb{F}_p)$ (or $SL_2(\mathbb{F}_p) \times SL_2(\mathbb{F}_p)$) for all primes $p \neq 2, 3$.

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$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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Conjecture 1 (Local to Global for ACPs, Graham-Lagarias-Mallows-Wilks-Yan 2005, F-Sanden 2011)

Let $\mathcal{O} = A\mathbf{v}$ be an orbit of A corresponding to an integer ACP. Let $\mathcal{O}_{24}$ be the set of residue classes mod 24 of coordinates of points in $\mathcal{O}$. There exists $X_{\mathcal{O}} \in \mathbb{Z}$ such that any $x > X_{\mathcal{O}}$ whose residue mod 24 lies in $\mathcal{O}_{24}$ is in fact a coordinate of some point in $\mathcal{O}$.

First step towards this:

Theorem 1 (Bourgain-F 2011)

For $\mathcal{O}$ as above, let $\kappa(\mathcal{O}, X)$ denote the number of distinct integers up to X occurring as coordinates of points in $\mathcal{O}$. Then we have

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Theorem 2 (Bourgain-Kontorovich, 2014)

A positive integer is said to be *admissible* in an Apollonian orbit $\mathcal{O}$ if it is in $\mathcal{O}_{24}$ mod 24. Let E_N be the set of admissible integers less than N which are not coordinates of some point in $\mathcal{O}$. Then $|E_N| \ll N^{1-\epsilon}$ where $\epsilon > 0$.

Crucial to both of these theorems is an observation of Sarnak's:

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\begin{array}{ccc}
A & \xhookrightarrow{\text{thin}} & O_{\mathbb{R}}(3,1) \\
\uparrow & & \\
A_1 & & (Q(x_1,x_2,x_3,x_4)\ \text{Descartes}) \\
\text{change} \downarrow \text{variables} & & \\
A_1' & \xhookrightarrow{\text{arithmetic}} & O_{\mathbb{R}}(2,1)\ (Q'(A,B,C) = B^2 - AC) \\
\rho \uparrow & & \uparrow \rho \\
\Gamma(2) & \xhookrightarrow{\text{arithmetic}} & SL_2(\mathbb{R})
\end{array}
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Given $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(2)$, one has

$$\rho(g) = \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix}$$

which, after the change of variables indicated on previous slide, gives that the integers appearing in the orbit $A_1(a_0, b, c, d)^T$ are of the form

$$f_{a_0}(\zeta, \nu) - a_0 = A_0\zeta^2 + 2B_0\zeta\nu + C_0\nu^2 - a_0,$$

where A_0, B_0, C_0 can be written in terms of a_0, b, c, d.
Possible question: to what other thin subgroups of $O_{\mathbb{R}}(3,1)$ do these arguments apply?

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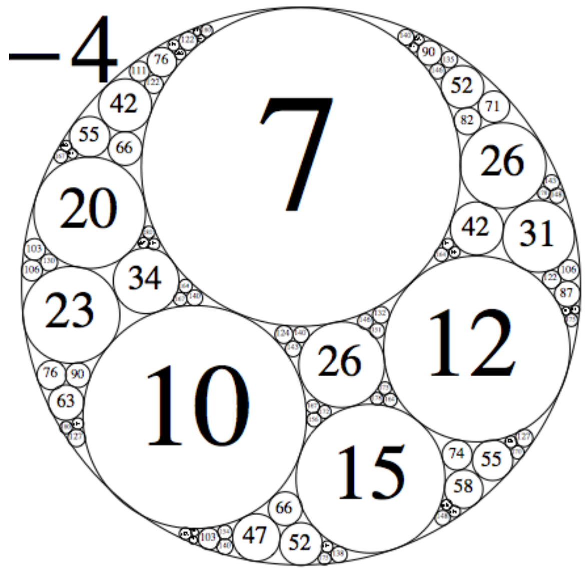

Zhang proved a local to global theorem for such packings.

The relevant group sits in $O_{\tilde{Q}}(\mathbb{Z})$ where

$\tilde{Q}(x_1, x_2, x_3, w) = w^2 - 2w(x_1 + x_2 + x_3) + x_1^2 + x_2^2 + x_3^2$.

If a, b, c, d, e, f are curvatures of a sextuple of touching circles,

$a + d = b + e = c + f = 2g$ and $\tilde{Q}(a, b, c, g) = 0$.

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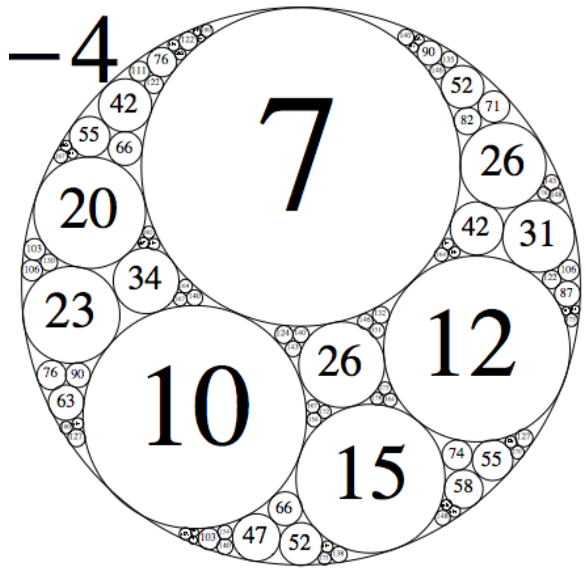

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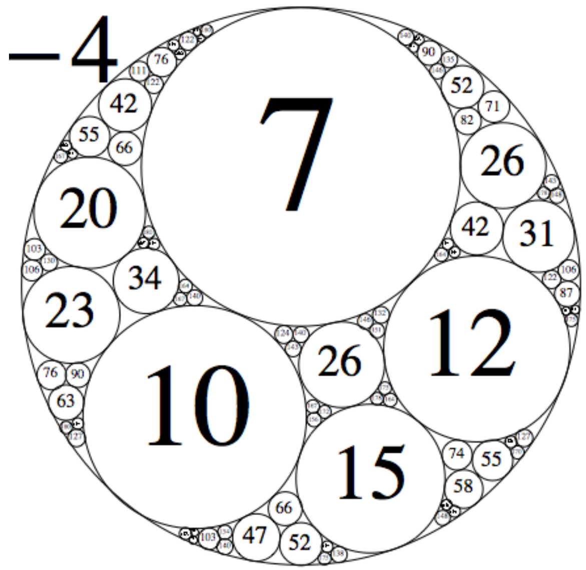

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Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and let $\mathcal{O}_K$ be its ring of integers.

Let $\mathcal{A} \leq \mathrm{SL}_2(\mathcal{O}_K)$ be a thin group. We assume

- $\mathcal{A} \not\subset \mathrm{SL}_2(\mathbb{R})$.
- $\mathcal{A}$ is Zariski dense in $\mathrm{SL}_2(\mathbb{C})$.
- $\mathcal{A}$ is geometrically finite.
- $\mathrm{SL}_2(\mathbb{Z}) \cap \mathcal{A}$ contains a principal congruence subgroup of level

$\mathrm{SL}_2(\mathcal{O}_K)$ acts on $\mathbb{C} \cup \infty$ via Möbius transformations.

$\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$ sends $\mathbb{R} + \frac{\sqrt{\Delta}}{2}$ to a circle or line in the complex plane.

Here Δ is the discriminant of K.

By a circle packing associated to $\mathcal{A}$, we mean circles that are the images of $M\mathcal{A}$ acting on $\mathbb{R} + \frac{\sqrt{\Delta}}{2}$, where $M \in \mathrm{SL}_2(\mathcal{O}_K)$ is fixed.

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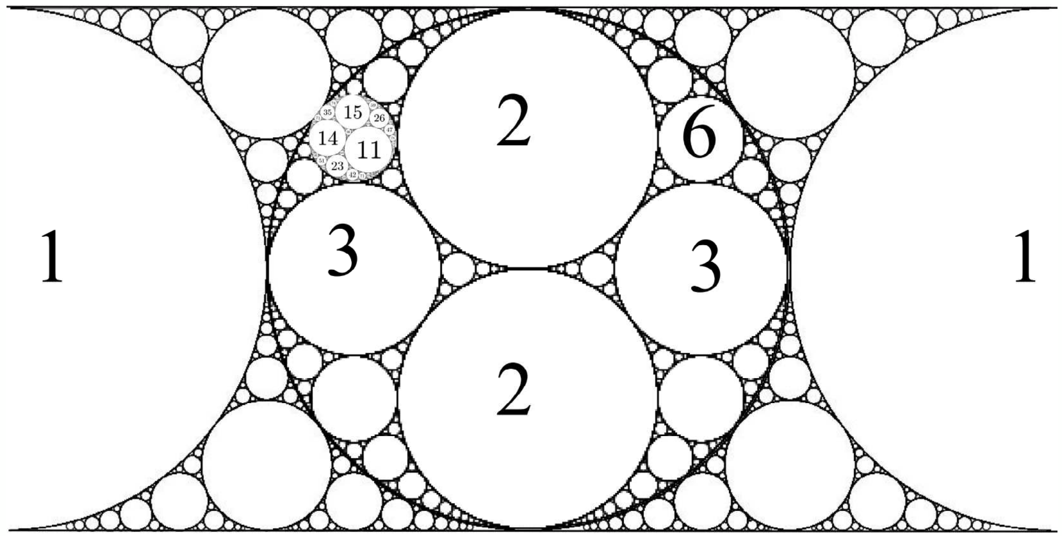

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$$\{k \mid k\sqrt{-\Delta} \text{ is the curvature of } M\gamma(\mathbb{R} + \tfrac{\sqrt{\Delta}}{2}) \text{ for some } \gamma \in \mathcal{A}\}$$

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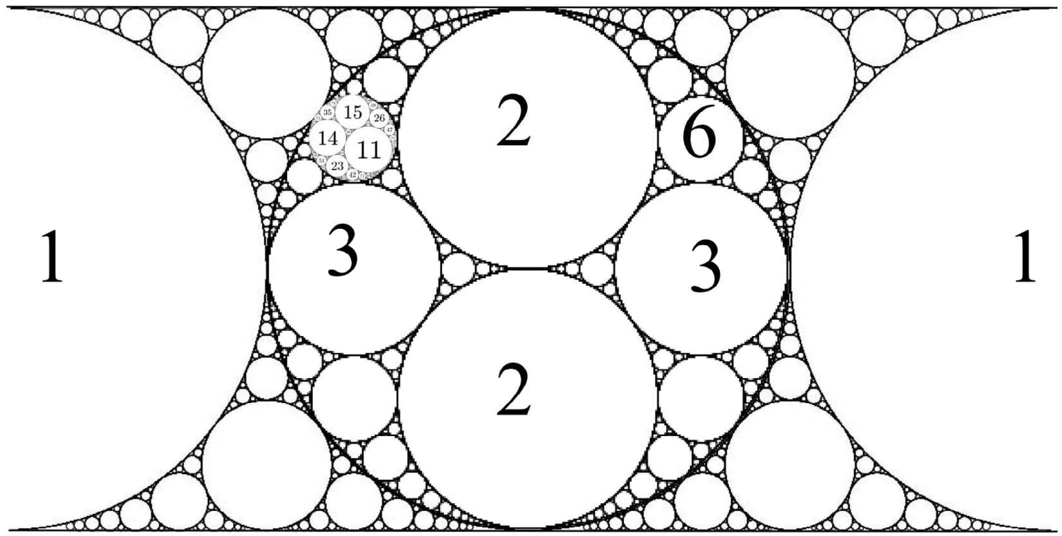

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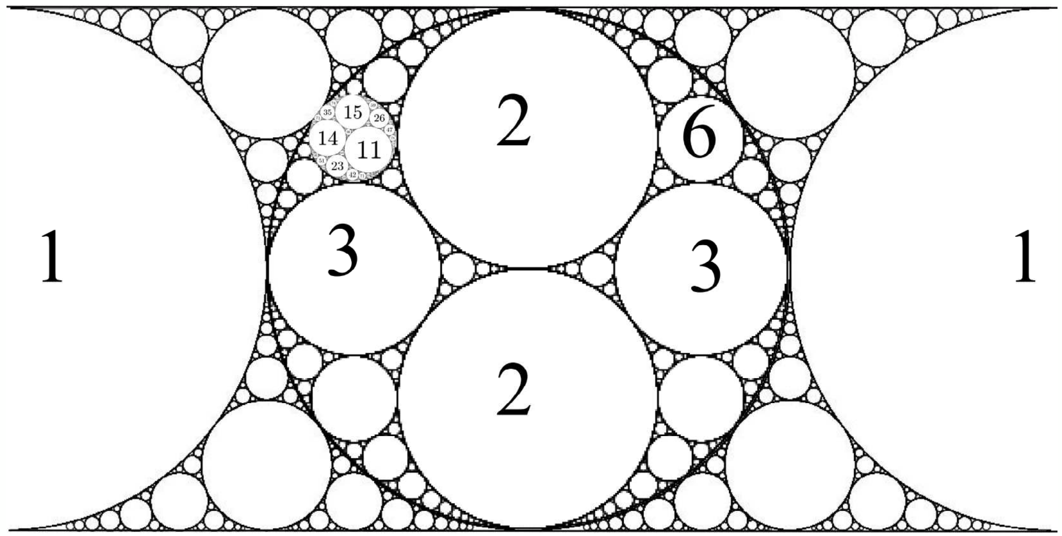

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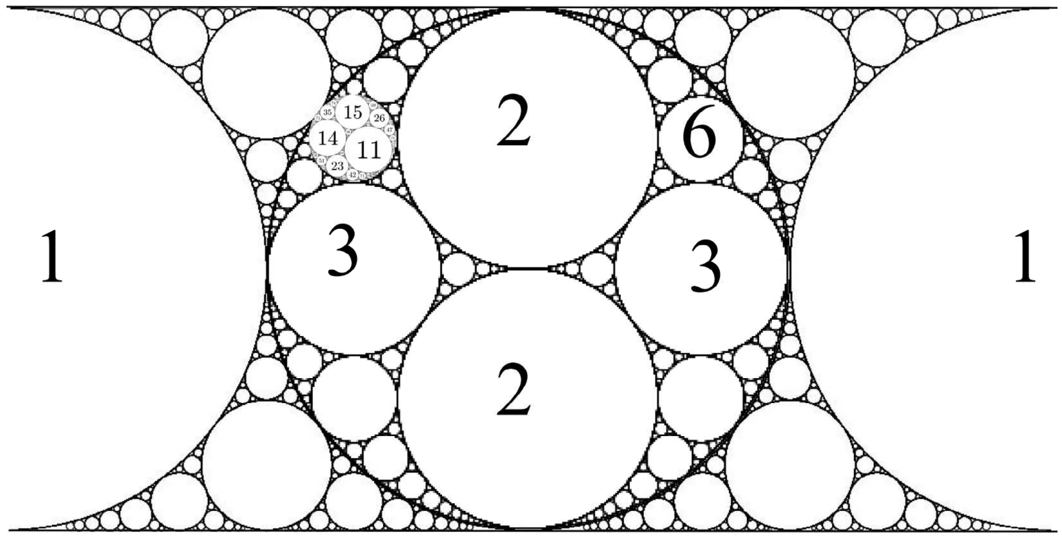

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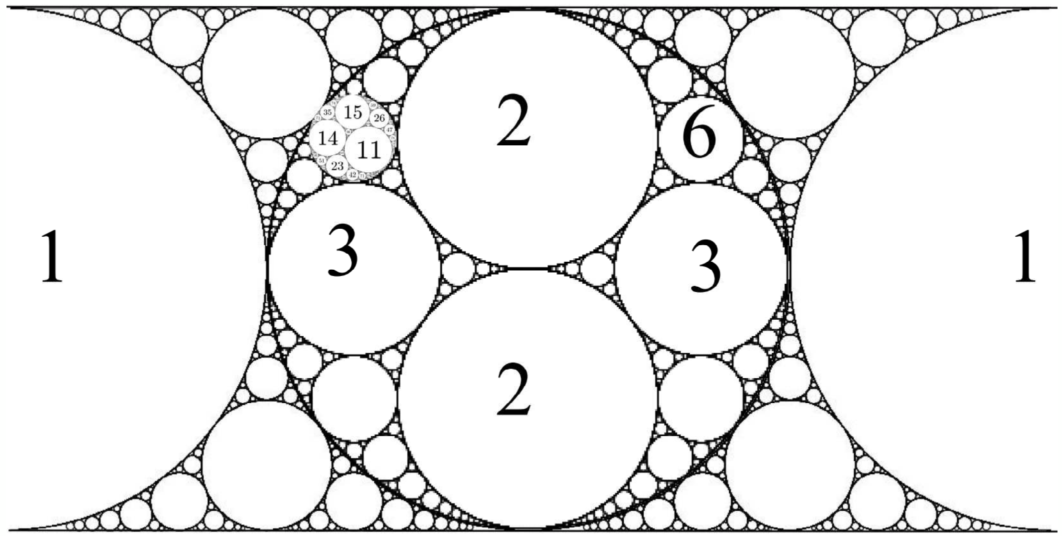

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Let $\mathcal{A} \leq \mathrm{SL}_2(\mathcal{O}_K)$ be as above. Let P be a circle packing associated to $\mathcal{A}$, and let S be the set of integers appearing as curvatures in P. Let A_N be the set of $0 < n < N$ with $n \in S \pmod{m}$ for all $m \geq 1$. Let E_N be the subset of A_N not in S. There exists an $\epsilon > 0$ such that

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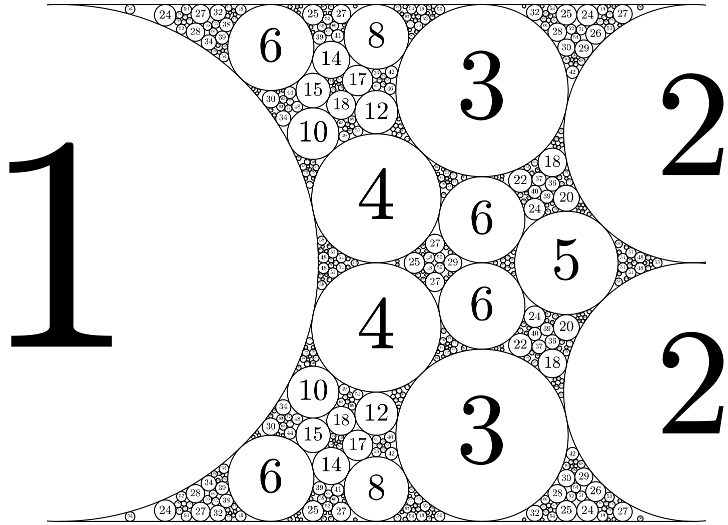

This packing also appears in an upcoming paper of Kontorovich-Nakamura. The corresponding $\mathcal{A}$ is the intersection of $\mathrm{SL}_2(\mathbb{Q}(\sqrt{-6}))$ with the group generated by

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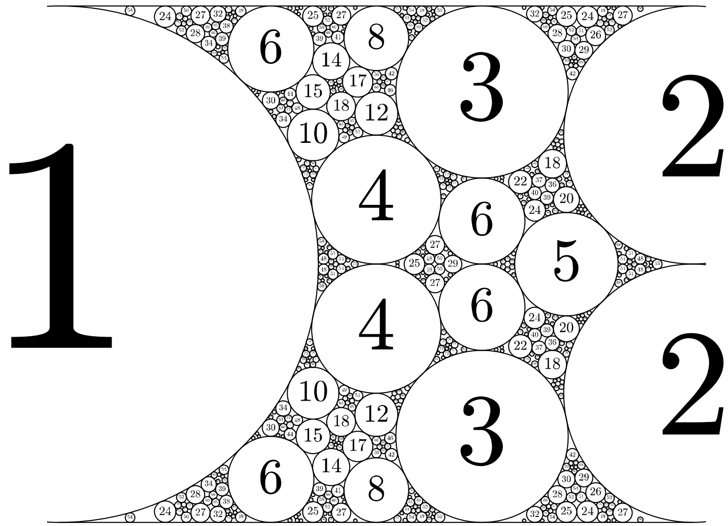

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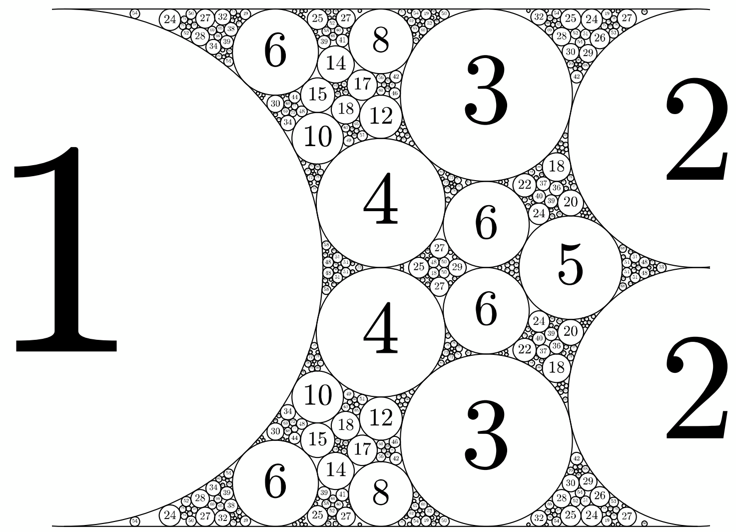

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Here T is a small power of N, and X is $\sqrt{N}/T$.
We want to show that $\mathcal{R}_N(n) > 0$ for almost all n if T is large enough.

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Expander Families of Graphs

It has eigenvalues

$$k = \lambda_0(X_i) > \lambda_1(X_i) \ge \lambda_2(X_i) \ge \cdots \ge \lambda_n(X_i) \ge -k$$

$\{X_i\}_{i \ge 0}$ is an *absolute expander family* if $\exists\, \epsilon > 0$ such that the difference between k and the λ_i closest but not equal to k in absolute value is at least ϵ for all i.

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Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of k-regular, finite, nonempty connected graphs with $|X_i| \to \infty$. Let M_i be the adjacency matrix of X_i. It has eigenvalues

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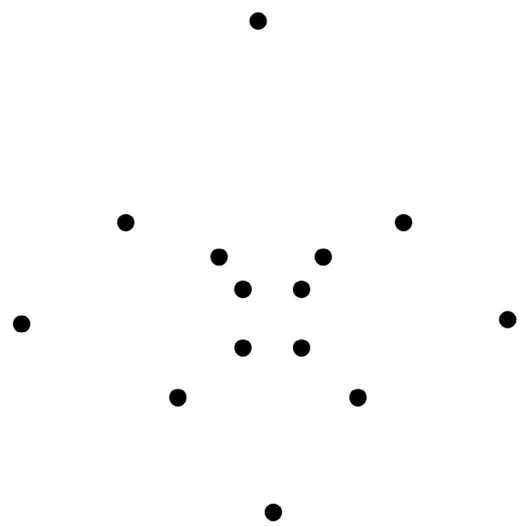
Motivating the definition

Idea: build good models for networks of any size. Want a very connected and cost-efficient network.



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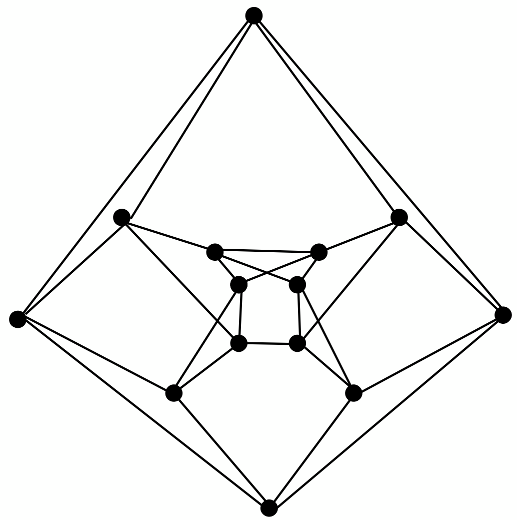

Let X_i be a k-regular finite graph. Fix a vertex v_0 in X_i, take a random walk on X_i starting at v_0. Choose another vertex v in X_i. What is the probability that I am at v after n steps?

One can show: the larger n, the closer the answer will be to $\frac{1}{|X_i|}$.

In a “good” network, this equidistribution should happen quickly. It turns out that

$$\left|P(\text{being at } v \text{ after } n \text{ steps}) - \frac{1}{|X_i|}\right| \le \left(\frac{\lambda_1(M_i)}{k}\right)^n$$

So, the larger the spectral gap, the faster this equidistribution happens for all graphs X_i.

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Theorem 2

There is a finite symmetric collection of generators S and a positive ϵ such that $Cay(\mathcal{A}_q, S_q)$ has a spectral gap of at least ϵ for all positive integers q, where ϵ is independent of q.

This is shown by first proving the statement for the group $\mathcal{A}' := \langle \Gamma, g\Gamma g^{-1} \rangle$, where $g \in \mathcal{A}$ is not in $\mathrm{SL}_2(\mathbb{R})$ or $i\mathrm{SL}_2(\mathbb{R})$, and $\Gamma < \mathcal{A}$ is a Zariski-dense subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Use the fact that the theorem is true (due to Bourgain-Varju) for Γ and $g\Gamma g^{-1}$.

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Lemma 3 (Varju)

Let G be a finite group with a finite symmetric generating set S. Suppose $G_1, \ldots, G_\ell \leq G$, and that for each $g \in G$, there exist $g_i \in G_i$ such that $g = g_1 g_2 \ldots g_\ell$. Then

$$1 - \lambda_1'(G, S) \geq \min_{1 \leq i \leq \ell} \left\{ \frac{|S \cap G_i|}{|S|} \cdot \frac{1 - \lambda_1'(G_i, S \cap G_i)}{2\ell^2} \right\}.$$

With all this, we get a lower bound of

$$\mathcal{M}_N(n) \gg T^{2\delta-2}$$

and

$$\sum_{n\in E_N} |\mathfrak{m}_N(n)| \ll T^{2\delta-2}N^{1-\epsilon}$$

Note that if $n \in E_N$, then $R_N(n) = 0$, and we have

$$|\mathfrak{m}_N(n)| = |R_N(n) - \mathcal{M}_N(n)| = |\mathcal{M}_N(n)| \gg T^{2\delta-2}$$

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