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Bombieri-Vinogradov for general multiplicative functions

w/ A. Granville, S. Drappeau

I. Bombieri-Vinagrodov Background

f arith. fn., $\mathbb{N} \to \mathbb{C}$

For $a \bmod q$, $(a,q)=1$, define

~~$\Delta(f, x; q, a)$~~

$$\Delta(f,x;q,a) := \sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ q)}} f(n) - \frac{1}{\phi(q)} \sum_{\substack{n \le x \\ (a,q)=1}} f(n)$$

Concentration mod a vs. Average

Trivial bound: $|\Delta(f,x;q,a)| \ll \frac{x}{\phi(q)}$ if $\|f\|_\infty \le 1$

- When $f = \Lambda$, von Mangoldt:

Siegel-Walfisz: For $(a,q)=1$, $A > 0$

$$|\Delta(\Lambda, x; q, a)| \ll_A \frac{x}{(\log x)^A}$$

"Primes equidistributed mod q for $q \le (\log x)^A$"

Bombieri-Vinogradov (B-V) For $A>0$, $\exists$ $B=B(A)$ s.t.

$$\sum_{q \le X^{1/2}/(\log X)^B} \max_{(a,q)=1} |\Delta(\Lambda, x; q, a)| \ll \frac{X}{(\log X)^A}$$

"Primes equidistributed mod q for almost all $q \le X^{1/2-\delta}$"

- Under GRH: Siegel-Walfisz bound true for all $q \le X^{1/2-\delta}$

"Bombieri-Vinogradov is GRH on average"

Similar results for $f = \mu, \lambda$, smooth #'s

(Fouvry-Tenenbaum, Harper)

Goal: Prove B-V for "general" multiplicative f.

Subsequently ~~all~~ f is a mult. fn. w/ $\|f\|_\infty \le 1$.

II. Formulating the results

Let $\mathcal{C}$ be mult. fns, w/ $|\Lambda_f(n)| \le \Lambda(n)$, where

$\Lambda_f(n)$ is defined by $-\frac{F'(s)}{F(s)} = \sum \frac{\Lambda_f(n)}{n^s}$,

and $F(s) := \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1} = \sum \frac{f(n)}{n^s}$

Defn: f satisfies A-Siegel-Walfisz criterion (A-S.W.) if

$$|\Delta(f, x; q, a)| \ll \frac{X}{(\log X)^A} \quad \forall \quad (a,q)=1$$

Cor (G.-S.) $f \in \mathcal{C}$. Fix $\delta, \varepsilon > 0$. If f satisfies 1-S.W. criterion, then for $Q \le x^{1/2-\delta}$

$$\sum_{q \sim Q} \max_{(a,q)=1} |\Delta(f, x; q, a)| \ll \frac{x}{(\log x)^{1-\varepsilon}}$$

Remarks 1) If $f = \chi \bmod r$, $r > 1$, then

$$|\Delta(f, x; q, a)| \gg \frac{x}{\phi(q)} \quad \text{when } r \mid q.$$

So $$\sum_{\substack{q \sim Q \\ \uparrow \\ \boxed{Q \le q \le 2Q}}} \max |\Delta(f, x; q, a)| \gg x/r$$

2) Cannot save more than $\log x$, even if A-S.W. holds. (due to large, "independent" primes $x/2 \le p \le x$)

In particular, $\mathcal{P} = \{ p \in [x/2, x] \mid p-1 \equiv 0 \ (q) \text{ for some } q \sim Q \}$

[Know that $|\mathcal{P}| = o\left(\frac{x}{\log x}\right)$ for ~~fixed~~ large q (Ford)]

Take $f = 1_{\mathcal{P}}$. Then

$$\Delta(f, x; q, 1) = \underbrace{\sum_{\substack{p \in \mathcal{P} \\ p \equiv 1 \ (\mathrm{mod}\ q)}} 1} - \frac{1}{\phi(q)} \sum_{p \in \mathcal{P}} 1$$

$$= \sum_{p \equiv 1 \ (\mathrm{mod}\ q)} 1 - \frac{1}{\phi(q)} \sum_{p \in \mathcal{P}} 1$$

by defn. of $\mathcal{P}$

$$\gg \frac{x}{\phi(q) \log x} \quad \text{(Dirichlet)}$$

Taking into account exceptional chars.

For ψ mod r, define $S_f(x,\psi) := \sum_{n \le x} f(n)\overline{\psi(n)}$,

$$\sigma_f(x,\psi) := \sup_{x^{1/2} \le X \le x} \frac{|S_f(X,\psi)|}{X}$$

Order primitive ψ mod r by σ (with $r \le \log x$)

$$\sigma_f(x,\psi_1) \ge \sigma_f(x,\psi_2) \ge \cdots$$

Recall $\Delta(f,x;q,a) = \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \ne \chi_0}} \chi(a) S_f(x,\chi)$

Define for $k \ge 1$

$$\Delta_k(f,x;q,a) := \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ not induced by } \psi_1,\dots,\psi_k}} \chi(a) S_f(x,\chi)$$

- Note when $\psi_1 =$ trivial, $\Delta_1 = \Delta$

Thm: (G-S) $f \in \mathcal{C}$, $\delta > 0$, $k \ge 1$. For $Q \le x^{1/2-\delta}$,

$$\sum_{q \sim Q} \max_{(a,q)=1} |\Delta_k(f,x;q,a)| \ll \frac{x}{(\log x)^{1-1/\sqrt{k+1}+o(1)}}$$

- Cor follows by $k \to \infty$
- Granville-Harper-Soundararajan: True for individual q w/ $\log q = (\log x)^{o(1)}$

III. Key step in proof: "large moduli"

<u>Notation</u> Let $w \geq 2$. For $q \geq 1$, $\exists!$ $q = q_r \cdot q_s$

"rough" part — q_r; w-smooth part, only $p \leq w$ — q_s

<u>Prop</u>: Let $f \in \mathcal{C}$. Fix $\delta > 0$, $Q \leq x^{1/2 - \delta}$

$$\sum_{q \sim Q} \max_{(a,q)=1} \left| \sum_{\substack{n \leq x \\ n \equiv a \ (\mathrm{mod}\ q)}} f(n) - \frac{1}{q_r} \sum_{\substack{n \leq x \\ n \equiv a \ (\mathrm{mod}\ q_s)}} f(n) \right|$$

$$\ll \frac{x}{w^{1/2}} + \frac{x \log\log x}{\log x}$$

True for all $w \geq 2$, best choice $w \sim (\log x)^2$

<u>Ex</u>: q prime, $q_s = 1$, reduces to original defn.

- Proven by Green, uses ideas of Katai, Bourgain-Sarnak-Ziegler, Montgomery-Vaughan, Ramaré, Matomäki-Radziwiłł-Tao

<u>For general proof</u>: Choose ξ_q, $a_q \bmod q$ s.t. max is achieved, phase ξ_q:

$$\max_{(a,q)=1} | \cdot | = \xi_q \cdot \Bigg(\sum_{\substack{n \leq x \\ n \equiv a_q (q)}} f(n) - \frac{1}{q_r} \sum_{\substack{n \leq x \\ n \equiv a_q (q_s)}} f(n) \Bigg)$$

Need to show $\displaystyle\sum_{n \leq x} f(n)\, g(n) \ll \frac{x}{w^{1/2}} + \frac{x \log\log x}{\log x}$,

where $\displaystyle g(n) = \sum_{q \sim Q} \xi_q \left(\mathbb{1}_{n \equiv a_q (q)} - \mathbb{1}_{n \equiv a_q (q_s)} \cdot \frac{1}{q_r} \right)$

Let $Y < Z$. Consider Ramaré's weight fn:

$$w(n) := \left(\# \{ Y \le p \le Z \mid p \mid n \} + 1 \right)^{-1}$$

Use identity $\displaystyle\sum_{\substack{Y \le p \le Z \\ p \mid n}} w\left(\tfrac{n}{p}\right) = \begin{cases} 1 & p \mid n \text{ for some } Y \le p \le Z \\ 0 & \text{o.w.} \end{cases}$

So $$\sum_{n \le x} f(n) g(n) = \underbrace{\sum_{\substack{n \le x \\ \text{No } p \mid n,\ Y \le p \le Z}} | f(n) g(n) |}_{\text{Upper-bound sieve}} + \sum_{n \le x} f(n) g(n) \sum_{\substack{Y \le p \le Z \\ p \mid n}} w\left(\tfrac{n}{p}\right)$$

$\ll x \dfrac{\log Y}{\log Z}$, Choose $Y = (\log x)^4$, $Z = x^{\delta/2}$

Second term is (setting $m = n/p$)

$$\sum_{p, m} f(pm)\, g(pm)\, w(m)$$

$$\le \sum_{m} |w(m) f(m)| \cdot \Big| \sum_{p} g(pm) \Big|$$

$$\le \sum_{m} |w(m)| \cdot \Big| \sum_{p} g(pm) \Big|$$

Cauchy-Schwarz: Bound $\displaystyle\sum_{m} |w(m)|^2$ and $\displaystyle\sum_{m} \Big| \sum_{p} g(pm) \Big|^2$

Now need to bound $\displaystyle\sum_{p, p'} g(pm)\, g(p'm)$

IV. Further results

a) Beyond $x^{1/2}$-barrier

Thm (Drappeau-Granville-S.) $f \in \mathcal{C}$, Fix $\varepsilon > 0$, $A \geq 1$.
Suppose f supp'd. on y-smooth numbers, where
$\exp((\log x)^{1/2+\varepsilon}) \leq y \leq x^{\delta}$ (Some small $\delta > 0$).
If satisfies 1 or A-S.W., then

$$\sum_{q \leq x^{3/5-\varepsilon}} |\Delta(f, x; q, 1)| \ll \frac{\Psi(x,y)}{(\log x)^A}$$

or

$$\sum_{\substack{q \leq Q \\ (a,q)=1}} |\Delta(f, x; q, a)| \ll \frac{\Psi(x,y)}{(\log X)^A}$$

$1 \leq |a| \leq Q$, $Q \leq x^{3/5-\varepsilon}$

- with $f = 1$, Fouvry-Tenenbaum
  Drappeau $(\log x)^c \leq y \leq x^{\delta}$

b) Thm Let $F * G = 1$. The following are equiv:

1) f, g satisfies A-S-W, ~~$f * 1_p$, $g * 1_p$~~ $f \cdot 1_p$, $g \cdot 1_p$ satisfy B.-V.

2) f, g satisfies B.-V.