Fluctuations of random tilings and discrete Beta-ensembles

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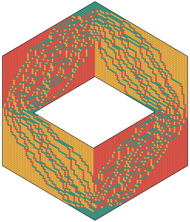
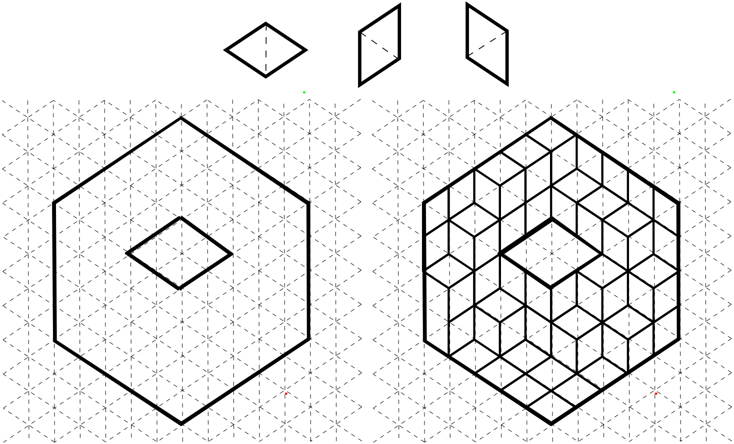
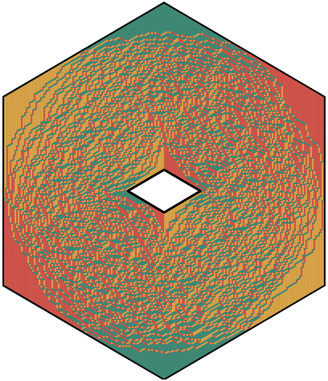
Joint work with A. Borodin, G. Borot, V. Gorin, J.Huang



Consider an hexagon with a hole and take a tiling at random. How does it look ?

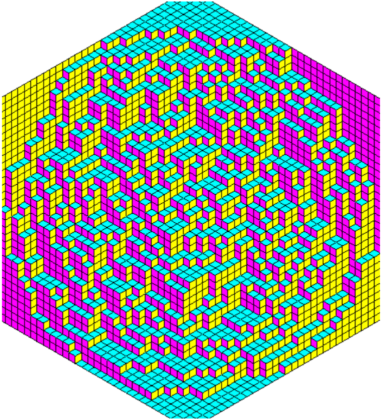




Petrov's picture.

When the mesh of the tiling goes to zero, one can see a "frozen" region and a "liquid" region. Limits, fluctuations ?

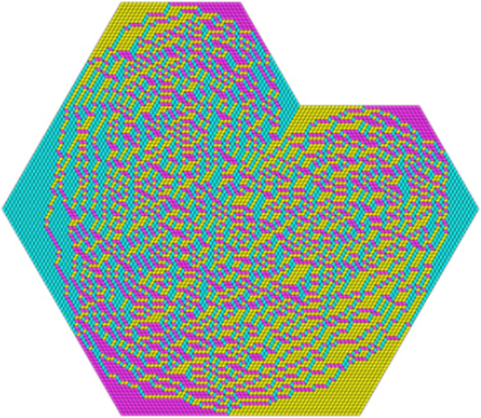
Tiling of the hexagon



Kenyon's picture.

Cohn, Larsen,Propp 98': When tiling an hexagon, the shape of the tiling converges almost surely as the mesh goes to zero.

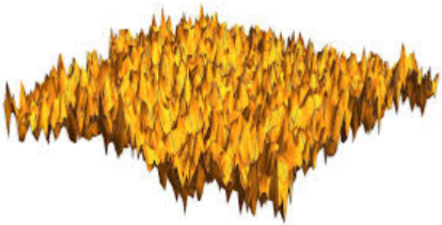
General domains



Kenyon-Okounkov's picture.

Cohn-Kenyon-Propp 00' and Kenyon- Okounkov 07': The shape of the tiling (e.g the height function) converges almost surely for a large class of domains.

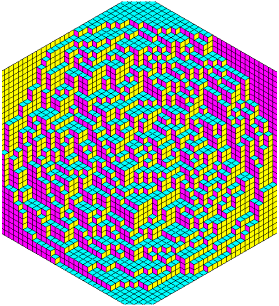
Fluctuations of the surface



Conjecture.(Kenyon-Okounkov) The recentered height function converges to the **Gaussian Free Field** in the liquid region in general domains.

- (Kenyon-06') A class of domains with no frozen regions
- (Borodin–Ferrari-08') Some *infinite* domains with frozen regions
- (Boutillier-de Tilière-09', Dubedat-11') On the torus
- (Petrov-12', Bufetov-G.-16') A class of simply-connected polygons
- (Berestycki-Laslier-Ray-16+) *Flat* domains, some manifolds
- Borodin-Gorin-G.- 16', and Bufetov-Gorin.-17' Polygons with holes — *trapezoid gluings.*

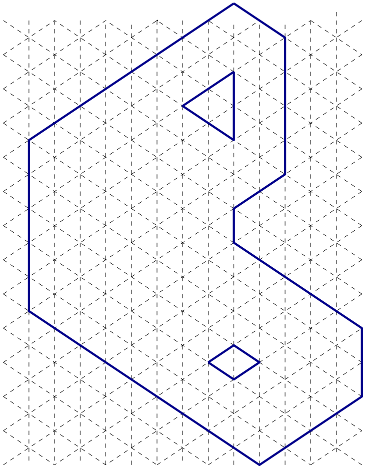
Local fluctuations of the boundary of the liquid region



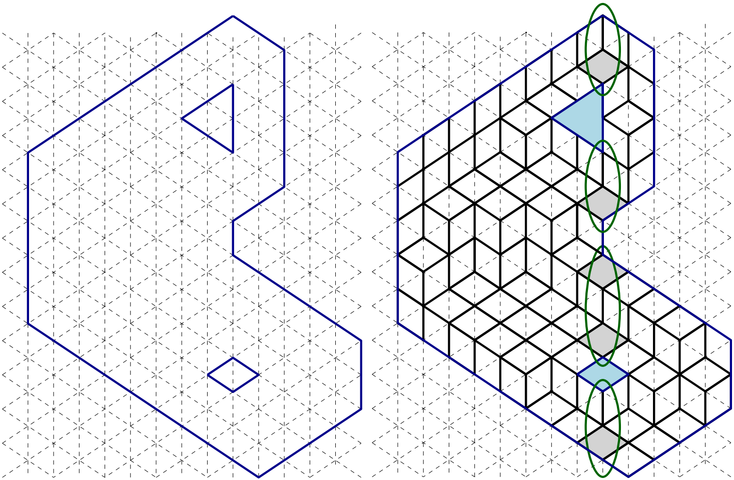
Ferrari-Spohn 02', Baik-Kriecherbauer-McLaughlin-Miller 03': appropriately rescaled, a generic point in the boundary of the liquid region converges to the Tracy-Widom distribution in the random tiling of the hexagon, the distribution of the fluctuations of the largest eigenvalue of the GUE.

G-Huang. -17': This extends to polygonal domains obtained by trapezoid gluings (on the gluing axis).

Trapezoids gluings

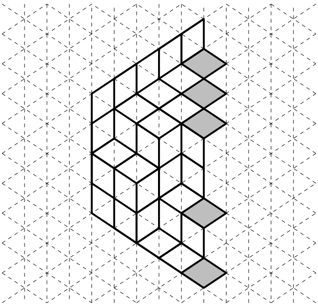


Trapezoids gluings



- We can glue arbitrary many trapezoids, where we may cut triangles or lines,
- Always along a **single** vertical axis

What is good about trapezoids?

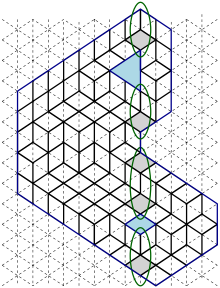


Fact: The total number of tilings of trapezoid with **fixed** along the border horizontal lozenges $\ell_N > \cdots > \ell_1$ is proportional to

$$\prod_{i<j} \frac{\ell_j - \ell_i}{j - i}$$

Indeed Tilings = Gelfand–Tsetlin patterns, enumerated through combinatorics of Schur polynomials or characters of unitary groups

Distribution of horizontal tiles



$H = 4$ cuts

The distribution of horizontal lozenges $\{\ell_i^h\}$ along the axis of gluing has the form: $\ell_{i+1}^h \geq \ell_i + 1$

$$P_N^{\Theta,w}(\ell) = \frac{1}{Z_N^{\Theta,w}} \prod_{i<j} (\ell_i - \ell_j)^{2\Theta[h(i),h(j)]} \prod_{i=1}^{N} w(\ell_i)$$

$h(i)$ — number of the **cut**. Θ — symmetric $H \times H$ matrix of 1's, 1/2's, and 0's with 1's on the diagonal.

Discrete β-ensembles ($\beta = 2\theta$)

For configurations ℓ such that $\ell^h_{i+1} - \ell^h_i - \theta_{h,h} \in \mathbb{N}$, $\ell^h_i \in [a_h N, b_h N]$, $b_{h-1} < a_h < b_h < a_{h+1}$, it is given by:

$$P^{\theta,w}_N(\ell) = \frac{1}{Z^{\theta,w}_N} \prod_{1\le h\le h'\le H} \prod_{\substack{1\le i\le N_h \\ 1\le j\le N'_h, i<j}} I_{\theta_{h,h'}}(\ell^{h'}_j, \ell^h_i) \prod w_h(\ell^h_i),$$

$$\text{where } I_\theta(\ell', \ell) = \frac{\Gamma(\ell' - \ell + 1)\Gamma(\ell' - \ell + \theta)}{\Gamma(\ell' - \ell)\Gamma(\ell' - \ell + 1 - \theta)}$$

Note that $I_\theta(\ell', \ell) \simeq |\ell' - \ell|^{2\theta}$ with $=$ if $\theta = 1, 1/2$.

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- We can study the convergence, global fluctuations of the empirical measures

$$\hat{\mu}_N^h = \frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N}, 1 \le h \le H$$

and fluctuations of the extreme particles of the liquid region.

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- Bufetov-Gorin 17': the fluctuations of the surface of the whole tiling follows from the fluctuations on the gluing axis.

Discrete β-ensembles ($\beta = 2\theta$): law of large numbers

Assume $w_h(x) \simeq e^{-NV_h(x/N)}$, V_h C^0, $(\theta_{h,h'})_{h,h'} \geq 0$, $\theta_{h,h} > 0$.

- Fixed heigths :$N_h/N \mapsto \varepsilon_h$, Then

$$\lim_{N\to\infty} \frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N} \to \mu_\varepsilon^h \quad a.s\,,$$

- Random heigths : $\theta = \Lambda\Lambda^T$ with ΛN fixed. Then $N_h/N \to \varepsilon_h^*$ and

$$\lim_{N\to\infty} \frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N} \to \mu_{\varepsilon^*}^h \quad a.s\,,$$

Indeed

$$P_N^{\theta,w}(\ell) \simeq e^{-N^2\mathcal{E}(\frac{1}{N_h}\sum_{i=1}^{N_h}\delta_{\ell_i^h/N}, N_h/N, 1\leq h\leq H)}$$

where $\mathcal{E}$ has a unique minimizer.

Assumption on the equilibrium measures towards fluctuations

Note that for all h:

$$0 \leq \frac{d\mu_\varepsilon^h}{dx} \leq \theta_{hh}^{-1}$$

We shall assume

- The liquid regions $\{0 < \frac{d\mu_\varepsilon^h}{dx} < \theta_{hh}^{-1}\}$ are connected,
- The equilibrium measures are off critical: at the boundary of the liquid region they behave like a square root.
-

$$\frac{w_h(x)}{w_h(x-1)} = \frac{\phi_{N,h}^+(x)}{\phi_{N,h}^-(x)}, \quad \phi_{N,h}^\pm \text{ analytic }, \phi_{N,h}^\pm = \phi_h^\pm + \frac{1}{N}\phi_{1,h}^\pm + o(\frac{1}{N})$$

Rmk: Off-criticality should be generically true.

Global fluctuations: fixed heights

Assume $N_h/N \mapsto \varepsilon_h$,

Theorem (Borodin-Gorin-G 15' Borot-Gorin-G 17')

Then for any analytic functions f_h:

$$\left(\sum_{i=1}^{N_h}(f_h(\ell_i^h/N) - \mathbb{E}[f_h(\ell_i^h/N)])\right)_h \Rightarrow N(0, \Sigma(f)) .$$

Global fluctuations: Random Heights

Assume ΛN given. Then [WIP Borot-Gorin-G]

- $$\frac{N_i}{N} \to \varepsilon_i^*$$
- The heights are equivalent to discrete Gaussian ':

$$P_N^{\theta,w}(N_h - \mathbb{E}[N_h] = x) \simeq \frac{1}{Z} e^{-\frac{1}{2\sigma}(x)^2}$$

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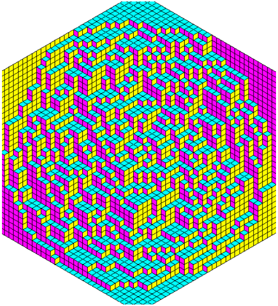
$$P_N^{\theta,w}(N_h - \mathbb{E}[N_h] = x) \simeq \frac{1}{Z} e^{-\frac{1}{2\sigma}(x)^2}$$
- $$\sum_{i,h} (f(\ell_i^h/N) - \mathbb{E}[f(\ell_i^h/N)]) \simeq \sum_k (N_k - \mathbb{E}[N_k]) \partial_{\varepsilon_k} \mu_\varepsilon^h(f)|_{\varepsilon=\varepsilon^*} + G_f$$

  where G_f is a centered Gaussian variable, independent from the filling fractions.

Discrete β-ensembles, edge fluctuations [Huang-G 17']



Under the previous assumptions, the boundary fluctuates like a Tracy-Widom distribution.

If $\frac{1}{N_1}\sum \delta_{\ell_i^1/N}$ converges towards μ^1 with liquid region $[a, b]$, $\mu^1((-\infty, a)) = 0$, then for all t real

$$\lim_{N\to\infty} P_N^{\theta,w}\left(N^{2/3}(\ell_1^1/N - a) \geq t\right) = f_{2\theta_{11}}(t)$$

with $f_{2\theta}$ the 2θ-Tracy-Widom distribution appearing in the **continuous** β-ensembles.

Corollary: Fluctuations of the first rows of Young diagrams under Jack deformation of Plancherel measure.

Continuous β-ensembles

The distribution of continuous β-ensembles is given by

$$dP_N^{\beta,V}(\lambda) = \frac{1_{\lambda_1}}{\tilde{Z}_N^{\beta,V}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta N \sum_{i=1}^N V(\lambda_i)} \prod d\lambda_i \, .$$

- When $V(x) = \frac{1}{4}x^2$, and $\beta = 1$ (resp. $\beta = 2, 4$), $P_N^{\beta,V}$ is the distribution of the eigenvalues of a symmetric (resp. Hermitian, resp. symplectic) $N \times N$ matrix with centered Gaussian entries with covariance $1/N$, the GOE (resp. GUE, resp. GSE) .

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- when $w(x) \simeq e^{-\beta N V(x/N)}$, and $H = 1$,

$$Z_N^{\theta,w} P_N^{\theta,w}(\ell) \simeq \tilde{Z}_N^{2\theta,V} \frac{dP_N^{2\theta,V}}{d\lambda}(\ell/N)$$

Some techniques to deal with β-ensembles

- Integrable systems. In some cases, these distributions have particular symmetries allowing for special analysis, e.g when $\beta = 2\theta = 2$ the density is the square of a determinant, allowing for orthogonal polynomials analysis [Mehta, Deift, Baik, Johansson etc]
- General case.
  -*Dyson-Schwinger (or loop) Equations.* Use equations for the correlators, e.g the moments of the empirical measures, and try to solve them asymptotically [Johansson, Shcherbina, Borot-G, etc]
  -*Universality.* Compare your law to one you can analyze [Erdös-Yau et al, Tao-Vu]. An important step is to prove *rigidity* (particles are very close to their deterministic limit), see [Bourgade-Erdös-Yau]

Continuous β-ensembles: Dyson-Schwinger equations(DSE)

Let

$$\hat{\mu}^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} : \quad \hat{\mu}_N(f) = \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)$$

If f is a smooth test function, then integration by parts yields the DSE:

$$N\mathbb{E}\left[\hat{\mu}_N(V'f) - \frac{1}{2}\int\int \frac{f(x)-f(y)}{x-y} d\hat{\mu}_N(x) d\hat{\mu}_N(y)\right]$$
$$= (\frac{1}{\beta} - \frac{1}{2})\mathbb{E}[\hat{\mu}_N(f_0')]$$

Continuous β-ensembles: Dyson-Schwinger equations

Linearizing this equation around its limit, we get

$$N\mathbb{E}[(\hat{\mu}_N - \mu)(\Xi f)] = (\frac{1}{\beta} - \frac{1}{2})\mathbb{E}[\hat{\mu}_N(f')]$$
$$+\frac{1}{2N}\mathbb{E}[\int\int \frac{f(x)-f(y)}{x-y} dN(\hat{\mu}_N - \mu)(x) dN(\hat{\mu}_N - \mu)(y)]$$

where

$$\Xi f(x) = V'(x)f(x) - \int \frac{f(x)-f(y)}{x-y} d\mu(y)\,.$$

If we can show that the last term is negligible and Ξ is invertible (off-criticality), we can solve asymptotically this equation to get

$$N\mathbb{E}[\hat{\mu}_N - \mu](f) \simeq (\frac{1}{\beta} - \frac{1}{2})\mathbb{E}[\hat{\mu}_N((\Xi^{-1} f)')]$$

We can get an infinite system of DSE for all moments of $\hat{\mu}_N$ and get large N expansion up to any order. This gives the CLT.

Concentration inequalities

- *First concentration inequalities* (Maida–Maurel Segala) Let $\bar\lambda_i = \bar\lambda_{i-1} + \max\{\lambda_i - \lambda_{i-1}, N^{-p}\}, \bar\lambda_1 = \lambda_1$ and $\tilde\mu_N = u_N * \frac{1}{N}\sum \delta_{\bar\lambda_i}$, u_N uniform law on $[0, N^{-p-1}]$. Then

$$P_N^{\beta,V}\left(D(\tilde\mu_N, \mu_V) \geq t\right) \leq e^{CpN\ln N - \frac{\beta}{2}t^2N^2}$$

with

$$\begin{aligned} D(\mu,\nu)^2 &= -\int \ln|x-y| d(\mu-\nu)(x)d(\mu-\nu)(y) \\ &= \int_0^\infty \frac{1}{t}\left|\int e^{itx} d(\mu-\nu)(x)\right|^2 dt \end{aligned}$$

Indeed, one can show that

$$\frac{dP_N^{\beta,V}}{d\lambda} \leq e^{C_pN\ln N - \frac{\beta}{2}N^2 D(\tilde\mu_N,\mu_V)^2}$$

- Dyson-Schwinger equations can be used to improve the concentration inequalities to get a concentration of order $1/N$.

Continuous β-ensembles: Fluctuations at the edge

Dumitriu-Edelman 02': Take $V(x) = \beta x^2/2$. Then $P_N^{\beta,\beta x^2/2}$ is the law of the eigenvalues of

$$H_N^\beta = \begin{pmatrix} Y_1^\beta & \xi_1 & 0 & \cdots & 0 \\ \xi_1 & Y_2^\beta & \xi_2 & 0 & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_{N-1} & Y_N^\beta \end{pmatrix}$$

where ξ_i are iid $N(0,1)$ and $Y_i^\beta \simeq \chi_{i\beta}$ independent.

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Ramirez-Rider-Viràg 06': The largest eigenvalue fluctuates like Tracy-Widom β distribution.

Bourgade-Erdòs-Yau 11', Shcherbina 13', Bekerman-Figalli-G 13': Universality: This remains true for general potentials provided off-criticality holds.

Discrete β-ensembles: Nekrasov's equation

Recall $\ell_{i+1} - \ell_i - \theta \in \mathbb{N}$ and set

$$P_N^{\theta,w}(\ell) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} \prod w(\ell_i, N)$$

and assume there exists $\phi_N^{\pm}$ analytic so that

$$\frac{w(x,N)}{w(x-1,N)} = \frac{\phi_N^+(x)}{\phi_N^-(x)}\,.$$

Then

$$\phi_N^-(\xi)\mathbb{E}_{P_N^{\theta,w}}\left[\prod_{i=1}^N \left(1 - \frac{\theta}{\xi - \ell_i}\right)\right] + \phi_N^+(\xi)\mathbb{E}_{P_N^{\theta,w}}\left[\prod_{i=1}^N \left(1 + \frac{\theta}{\xi - \ell_i - 1}\right)\right]$$

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is analytic.

$\Rightarrow$ Gives asymptotic equations for $G_N(z) := \frac{1}{N}\sum_{i=1}^N \frac{1}{z - \frac{\ell_i}{N}}$

which can be analyzed as DSE.

Consequences of Nekrasov's equation

- One can estimate all moments of $N(G_N(z) - G(z))$ for $z \in \mathbb{C} \backslash \mathbb{R}$, proving CLT,

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$$P_N^{\theta,w}(\sup_i |\ell_i - N\gamma_i| \geq \frac{N^a}{\min\{i/N, 1 - i/N\}^{1/3}}) \leq e^{-(\log N)^2}$$

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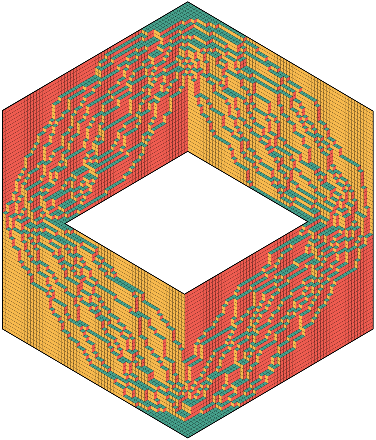
  where $\mu((-\infty, \gamma_i)) = i/N$.
- One can compare the law of the extreme particles, at distance of order $N^{1/3} \gg 1$ (the mesh of the tiling) with the law of the extreme particles for the continuous model and deduce the 2θ-Tracy-Widom fluctuations.

Some open questions

- Critical case. In continuous setting, Bekerman, Leblé, Serfaty 17' derived CLT for functions in $\mathrm{Im}(\Xi)$. What about the discrete case ? Can we get universality of local fluctuations ? How are the fluctuations for functions which are not in $\mathrm{Im}(\Xi)$?
- Universality for CLT. What if we have true Coulomb gas interaction in the discrete case ?
- General domains ?
- Local fluctuations in the bulk. When $\theta = 1$ correlations functions in the bulk converge to discrete Sine process. What about other θ ?



Many thanks to Alexei Borodin, Vadim Gorin and Leonid Petrov for the pictures.

And thank you for your attention!