Rigidity of the 3D hierarchical Coulomb gas

Sourav Chatterjee

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- Called hyperuniformity in the physics literature.

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- Numerous contributors:
  - **Eigenvalues and determinantal processes:** Borodin, Bourgade, Deift, Diaconis, Erdős, Evans, Forrester, Guionnet, Johansson, Pastur, Rider, Shcherbina, Soshnikov, Tao, Virág, Vu, Yau, ...
  - **Coulomb gas and other interacting gases:** Bauerschmidt, Ben Arous, Bourgade, Chafaï, Leblé, Majumdar, Radin, Serfaty, Yau, Zeitouni, ...
  - **Zeros of random analytic functions:** Ghosh, Lebowitz, Nazarov, Peres, Sodin, Volberg, ...
  - **Orthogonal polynomial ensembles:** Bardenet, Berman, Breuer, Duits, Hardy, Johansson, Lambert, ...
  - + many others (see my preprint on arXiv for a survey).

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$$\frac{1}{Z}\exp\biggl(-\beta \sum_{1\le i<j\le n} w(x_i, x_j) - \beta n \sum_{i=1}^n V(x_i)\biggr),$$

where w is a symmetric function, V is any function with sufficient growth at infinity, β is the "inverse temperature" parameter, and Z is the normalizing constant.

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- Coulomb gas: V is arbitrary (usually $V(x) = |x|^2$) and

$$w(x,y) = \begin{cases} |x-y| & \text{if } d = 1,\\ -\log|x-y| & \text{if } d = 2,\\ |x-y|^{2-d} & \text{if } d \ge 3.\end{cases}$$

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- One key motivation for studying interacting gases comes from random matrix theory.
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- Eigenvalues of Ginibre random matrices are 2D Coulomb gases.
- The 1D log gas and the 1D and 2D Coulomb gases are known to be rigid (more later).
- However, the most physically relevant interacting gas is the 3D Coulomb gas. No connection with random matrices or determinantal point processes. Very few rigorous mathematical results are known about it. In particular, it is believed to be rigid but there is no rigorous proof.

Rigidity of interacting gases

- Rigidity of the 1D log gas follows from the works of many authors, e.g. Costin & Lebowitz (1995), Diaconis & Evans (2001), Wieand (2002), Pastur (2006), Bourgade, Erdős & Yau (2012) and Tao & Vu (2013).

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- For the 2D Coulomb gas with general V, rigidity was recently established through contributions from Sandier & Serfaty (2015), Rougerie & Serfaty (2015), Bauerschmidt, Bourgade, Nikula & Yau (2016) and Leblé & Serfaty (2016).

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- The most promising results available at this time are due to Serfaty and collaborators, who have obtained very precise informations about normalizing constants and large deviations for Coulomb gases in general dimensions.

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- For the 3D Coulomb gas, however, this conjecture is still open.
- In a recent preprint, I proved this conjecture (up to logarithmic factors) for a closely related model, known as the 3D hierarchical Coulomb gas. This is the subject of this talk.

The 3D hierarchical Coulomb gas

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- We will take $d = 3$, and

$$V(x) = \begin{cases} 0 & \text{if } x \in [0,1]^3, \\ \infty & \text{if } x \notin [0,1]^3, \end{cases}$$

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- Then $w(x,y)$ "behaves like" the 3D Coulomb potential $|x-y|^{-1}$ when x is close to y, up to constant factors.

Dyson's hierarchical scheme

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- The hierarchical version of the Coulomb gas was introduced in the physics literature by Benfatto, Gallavotti & Nicolò (1986) and subsequently studied by many authors.

Main result: Upper bound

Theorem (C., 2017)

Consider the n-particle 3D hierarchical Coulomb gas in the unit cube. Take any $A \subseteq [0,1]^3$ *with a two-dimensional boundary (in the Minkowski sense) and let* $N(A)$ *be the number of particles falling in A. Then* $\mathbb{E}(N(A)) = \mathrm{vol}(A)n$ *and*

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(There is another similar result in the preprint for the case where A is shrinking with n. Rigidity is proved at all scales.)

Main result: Lower bound

Theorem (C., 2017)

Suppose that A is nonempty, connected and open, and ∂A is a smooth, closed, orientable surface. Then there exist $n_0 \geq 1$, $c_1 > 0$ and $c_2 < 1$, depending only on A and β, such that for any $n \geq n_0$ and any $-\infty < a \leq b < \infty$ with $b - a \leq c_1 n^{1/3}$,

$$\mathbb{P}(a \leq N(A) \leq b) \leq c_2.$$

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- This can be partly validated for eigenvalues of Hermitian random matrices, where we can talk about the k^{th} largest eigenvalue and its fluctuations, for $k = 1, 2, \ldots, n$.

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- However, it is not clear how to make use of this intuition to construct proofs in higher dimensions.
- I will now give a different intuition, using a toy example involving balls and boxes. The proof for the 3D hierarchical Coulomb gas is a generalization of this toy proof.

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- Thus, if n_1 and n_2 are the numbers of balls falling in boxes 1 and 2, then the total potential energy is

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- A configuration with n_1 balls in box 1 and n_2 balls in box 2 is assigned a probability proportional to $e^{-\beta H(n_1,n_2)}$, where β is the inverse temperature parameter, as usual.

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- Answer: If $a > b$, then N_1 has fluctuations of order 1 as $n \to \infty$.
- Let me now explain how to see this.

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- Moreover, for any n_1, n_2 such that $n_1 + n_2 = 2n$,
$$\mathbb{P}(N_1 = n_1,\ N_2 = n_2) = \frac{Z(n_1, n_2)}{Z}.$$

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- Thus, if $k \ll \sqrt{n}$, then

$$\frac{Z(n+k, n-k)}{Z(n,n)} \sim e^{-\beta k^2(a-b)+o(1)}.$$

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- This gives, for $k \ll \sqrt{n}$,

$$\begin{aligned}
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- The above argument generalizes to any finite number of boxes, as long as the matrix of potentials is strictly positive definite.

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- Here, we inspect the *changes in energy and entropy due to small changes in the ground state*. The calculations become more delicate.
- In some sense, this is a combination of energy-entropy competition and the cavity method.

Back to the 3D hierarchical Coulomb gas

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$$\frac{Z(n+1)}{Z(n)} = \mathbb{E}\exp\left(-\beta\sum_{i=1}^{n} w(U, X_i)\right),$$

where $(X_1, \ldots, X_n)$ is a realization of the n-particle system, and $U \sim Unif[0,1]^d$.

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- Jensen's inequality gives $Z(n+1)/Z(n) \geq e^{-\beta\alpha n}$, where $\alpha = \iint w(x,y)dxdy$, since each X_i is uniformly distributed.

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$$\frac{Z(n)}{Z(n+1)} \geq \exp\left(\beta\sum_{i=1}^{n} \mathbb{E}(w(X_{n+1}, X_i))\right).$$
- But by the symmetry between $X_1, \ldots, X_{n+1}$, this equals
$$\exp\left(\frac{\beta n}{\binom{n+1}{2}} \sum_{1\leq i<j\leq n+1} \mathbb{E}(w(X_i, X_j))\right)$$
$$= \exp\left(\frac{\beta n}{\binom{n+1}{2}}\mathbb{E}(H_{n+1}(X_1, \ldots, X_{n+1}))\right).$$

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- Thus,

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- Combined with the lower bound on $Z(n+1)/Z(n)$, this shows that
$$e^{-\beta\alpha n} \le \frac{Z(n+1)}{Z(n)} \le e^{-\beta\alpha n + Cn^{1/3}}.$$

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- A multi-scale generalization of this argument leads to the proof.

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- For each dyadic cube D, let n_D be the number of points falling in D.
- The energy of the 3D hierarchical Coulomb gas can be written as a multi-scale χ^2 statistic:

$$H_n(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \sum_{D \in \mathcal{D}_j} 2^j \binom{n_D}{2} + 2\binom{n}{2}.$$

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- Thus, for any k,

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- Thus,

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- This gives

$$H_n(x_1, \ldots, x_n) \geq \frac{7}{3}\binom{n}{2} - \frac{n^2}{6}4^{-k} - 2^k n.$$

- A simple calculation gives $\alpha = 7/3$.
- Choosing k such that $n^{1/3} \leq 2^k \leq 2n^{1/3}$ gives the desired lower bound.

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- Proof technique is based on a general approach that combines energy-entropy competition and the cavity method.
- The corresponding result for the 3D Coulomb gas, predicted by Jancovici, Lebowitz and Manificat in 1993, remains an open problem.