Refinement of Gaussian concentration of norms - Tuesday 14$^{\text{th}}$ November, 2017

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Remark: These are supplementary to the lecturers notes, which diverge from the actual lectures in some points. The references can be found in the Arxive papers cited below. We refer to the arxive papers instead of the ultimately published ones.

Main references: Arxiv paper by Paouris and Valettas: "Dichotomies, structure, and concentration results". `https://arxiv.org/abs/1708.05149`
Arxiv paper by Paouris and Valettas: "On Dvoretzkys theorem for subspaces of L_p". `https://arxiv.org/pdf/1510.07289.pdf`

Arxiv paper by Paouris and Valettas: "Variance Estimates and Almost Euclidean Structure". `https://arxiv.org/pdf/1703.10244.pdf`

Arxiv paper by Paouris, Valettas and Zinn: "Random version of Dvoretzkys theorem in ℓ_p^n". `https://arxiv.org/pdf/1510.07284.pdf`

For the more detailed references below, consult the above papers, especially the first and second ones.

1 Introductory

Throughout this note $K \subset \mathbb{R}^n$ will be a bounded (compact) convex set. By B_p^n we denote the unit ball with respect to the ℓ_p^n-metric.

Definition: We say that K is in John's position if the Euclidean unit ball $B_2^n \subset K$ and for any ellipsoid $\mathcal{E} \subset K$ we have $|\mathcal{E}| \leq |B_2^n|$, i.e. the unit ball is maximal in volume among the inscribed ellipsoids.
Some historical results. (For a more detailed discussion see the first paper in the main references). The idea of all of these is that one can find large dimensional subsets where the restricted convex slice can be efficiently sandwiched between two known shapes.

1. Dvoretzky-Rogers 1950: $\exists F$ with $\dim(F) = \sqrt{m}$ and $B_2^F \subset K \cap F \subset \sqrt{3}B_\infty^F$, where B_p^F is unit ball in the ℓ_n^p metric with respect to some basis.

2. Grothendick asked in 1953, if for every $\epsilon > 0$ we can find a large dimensional F with $(1-\epsilon)B_2^F \subset K \cap F \subset (1+\epsilon)B_2^F$

3. Shown to hold in 1961 by Dvoretzky, where $k = \dim(F) \sim \frac{\epsilon\sqrt{\log(n)}}{\log\log(n)}$. Sharp?

4. '71 Milman: $k \sim \frac{\epsilon^2 \log(n)}{\log(\epsilon)}$. This is sharp in terms of the dependence in n, for fixed ϵ, as seen by $K = B_\infty^n$. However, not sharp with respect to ϵ.

5. '85 Gordon $k \sim \epsilon^2 \log(n)$

6. '87 Schechtman $k \sim \epsilon^2 \log(n)$

7. Both Gordon and Schechtman show that a randomized subspace will do.

8. 2009 Schechtman $k \sim \frac{\epsilon \log(n)}{\log(1/\epsilon)^2}$. Existential, not random. In fact, these are different questions. Here focus on random.

At the core, many of these results rely on Concentration of measure.

Theorem 1 (Concentration of Measure, Milman)**.** *Let $(\mathbb{R}^n, ||\cdot||_2, \gamma_n)$ be Gauss space and f an L-Lipschitz function on $\mathbb{R}^n$, then for all $t > 0$*

$$\P(|f(Z) - \mathbb{E}(f(Z))| > t) \leq 2e^{-t^2/(2L^2)}.$$

Sharp for linear functions, but far from sharp for others. Here interest is in the observation that for some norms the inequality is not sharp and they are so called super concentrated.

Consider now for K the Minkowski functional $||\cdot||_K$. Let $X = (\mathbb{R}^n, ||\cdot||_K)$, and $b(X) = \text{Lip}_{||\cdot||_2}(||\cdot||_K) = \max_{\theta\in S^{n-1}} ||\theta||_K$. Then the concentration of measure gives

$$\P(||Z||_K - \mathbb{E}(||Z||_K)| > t\mathbb{E}(||Z||_K)))) \leq 2e^{-ct^2 k(X)},$$

where $k(X) \sim \frac{\mathbb{E}(||Z||_K)^2}{b(X)^2}$. This is however not sharp.

Example: Let $K = B_\infty^n$, then

$$A(K,\epsilon) = \P(||Z||_K - \mathbb{E}(||Z||_K)| > \epsilon\mathbb{E}(||Z||_K)))),$$

and we have by Schechtman 2006 that

$$C' e^{-c'\epsilon \log(n)} \leq A(K,\epsilon) \leq Ce^{-c\epsilon \log(n)},$$

that is we only have an ϵ, and not ϵ^2.

For Dvoretzky's theorem we can take $k \sim \epsilon \log(n)/\log(1/\epsilon)$ (Schechtman 2007, Tikhomirov 2014) (with a net argument).

2 Main Question

Is it true that if $K \subset \mathbb{R}^n$ there always exists $T \in GL_n$ such that

$$A(TK,\epsilon) \leq A(B_\infty^n, \epsilon)?$$

Or that

$$\frac{\text{Var}(||TZ||_K)}{(\mathbb{E}(||Z||_K))^2} \leq \frac{\text{Var}(||Z||_\infty)}{(\mathbb{E}(||Z||_\infty))^2},$$

where Z is a standard Gaussian in $\mathbb{R}^n$.

Bad news:

1. T'17: No to main question when K in John's position. $\exists K$ such that $A(K,\epsilon) \sim e^{-\epsilon^2 \log(n)}$. In fact, can take K = ball times a cube.

2. V'17: $\forall K \exists K_1$ such that
$$||\cdot||_{K_1} \leq ||\cdot||_K \leq 4||\cdot||_{K_1}$$
and $A(K_1, \epsilon) \sim e^{-\epsilon^2 k(X)}$ and $\mathrm{Var}(||Z||_K) \sim Lip(||\cdot||_K)^2$. In other words, can always perturb to fail.

3 Main results and computation

Computational result For norms of ℓ_p^n, corresponding to $K = B_p^n$, we have
$$\mathrm{Var}(||Z||_p) \sim \frac{2^p}{pn^{1-2/p}}$$
for $1 \leq p \leq c_0 \log(n)$,
and
$$\mathrm{Var}(||Z||_p) \sim \frac{1}{\log(n)}$$
for $p \geq c_1 \log(n)$. In between a complicated regime. This is discussed in detail in the main references, where the result of the computation is written somewhat differently.

This estimate was found using functional inequalities and the so called $L_1 - L_2$-Talagrand inequalities.

Remark: The asymptotics are very sensitive, by varying the norm a little the result can change a lot.

Good news:

1. P,V '15: If $X \subset L_p$ is a finite dimensional subspace an if K is in Lewis position in X then $A(K, \epsilon) \leq A(l_p^n, \epsilon)$ for all $p \leq c_0 \log(n)$
2. T'17. K is 1-unconditional if K is in l-position, then $A(K, \epsilon) \leq A(B_\infty^n, \epsilon)$.

Intuition, despite the bad news several positive results hold.

4 Some Proof Ideas

Why is concentration of measure not enough. It gives for an L-Lipschitz f that

$$\mathrm{Var} f(Z) \leq \mathbb{E}(||\nabla f||_2)^2 \leq L^2.$$

For small p, the second inequality is not sharp, and for large p the first one. For the large p we say that $f(Z)$ is super concentrated. This indicates that standard concentration of measure is not enough and other techniques are needed.
Another main result

Theorem 2 ($L_1 - L_2 - Talagrand$)**.** *With respect to Gaussian measure γ_n on $\mathbb{R}^n$,*

$$\mathrm{Var}(f) \leq C \sum_{i=1}^{n} \frac{||\partial_i f||_{L_2(\gamma_n)}}{1 + \log(\frac{||\partial_i f||_{L_2(\gamma_n)}}{||\partial_i f||_{L_1(\gamma_n)}})}$$

Remark: Left hand rotationally invariant, but right not, and this can be used to our advantage.
In applying Talagrand, we want each term to contribute roughly the same. This need not hold, but can be attained by a linear transformation.

Lemma 1. *Assume f smooth. Then there exists $\lambda_1, \ldots, \lambda_n > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, with property*

$$||\partial_i (f \circ \Lambda)|| \leq \frac{a_\Lambda}{n},$$

where we denote $a_\Lambda = Lip_{||\cdot||_\infty}(f \circ \Lambda)$ and $b_\Lambda = Lip_{||\cdot||_2}(f \circ \Lambda)$.

The proof is based on Borsuk-Ulam/a topological argument. No classical position seems to work. Due to the proof technique very little or no direct control is given for λ_i. This is relevant for other parts of the proof.

Using $L_1 - L_2$-Talagrand and the previous Lemma, we get

$$\mathrm{Var}(f \circ \Lambda) \leq C \sum_{i=1}^{n} \frac{||\partial_i f||_{L_2(\gamma_n)}}{1 + \log(\frac{||\partial_i f||_{L_2(\gamma_n)}}{||\partial_i f||_{L_1(\gamma_n)}})} \leq C \frac{b_\Lambda^2}{1 + \log(n b_\Lambda / a_\Lambda)}.$$

The ratio in the logarithm can be bounded by

$$\begin{aligned}
\frac{a_\Lambda}{b_\Lambda} &= \max_{\epsilon_i=\pm1} || \sum_{i=1}^{n} \epsilon_i \lambda_i e_i || && (1)\\
&= \mathbb{E}_\epsilon || \sum \epsilon_i \lambda_i e_i || && (2)\\
&\le LC\mathbb{E} || \sum g_i \lambda_i e_i || = C\sqrt{K(X)}L, && (3)
\end{aligned}$$

where $L = d(K, \mathrm{Unc})$ (see paper for proper definition and calculation). This gives the desired variance estimate, and the final proposition.

Theorem 3. *For any convex K there exists Λ a linear map such that*

$$\frac{\mathrm{Var}(||\Lambda Z||_K)}{\mathbb{E}(||\Lambda Z||_K)^2} \le \frac{c}{(\log(n/L^2))^2}.$$

and

$$A(\Lambda K, \epsilon) \le Ce^{-c\epsilon \log(n/L^2)}.$$

Useful if L can be controlled. In the worst case though $L \le \sqrt{n}$. But by Figiel-Johnson, there exists a subspace F of $\dim(F) = n/2$ with better bounds for $d(K \cap F, Unc)$.
Using the following theorem by Alon-Milman one can further get a slight improvement for the Dvoretzkys theorem.

Theorem 4. *K in John position, $\exists e_i$ orthonormal basis with $1/4 \le ||e_i||_K \le 1$ and $k(X) = \mathbb{E}||\sum g_i e_i||$. Either this is big, or if it small, then there exists a subspace F of dimension similar to $\sqrt{n}$ where $d(K \cap F, B_\infty^n)$ is small (in a controlled way).*

Gives slight improvement (due to P.,V.) for the existential version of Dvoretzkys theorem of the dimension of the almost-Euclidean subspace of the form $k \sim \epsilon \frac{\epsilon \log(n)}{\log(1/\epsilon)}$.

- K is in John's position if $B_2^n \subseteq K$.
  and $\mathcal{E} \subseteq K$ then $|\mathcal{E}| < |B_2^n|$

- D-R ('50) $\exists$ F $\dim F \simeq \sqrt{n}$: $B_2^F \subseteq B_X \cap F \subseteq \sqrt{3}\, B_\infty^F$

- Grothendieck.

①

Dvoretzky Theorem

1. Introduction

Birth point: Dvoretzky - Rogers Lemma (1950)
If B_2^n is the ellipsoid of max. vol in B_X then there exists $x_1, \dots, x_m$, $m \simeq \sqrt{n}$ s.t.

$$\frac{1}{\sqrt{3}} \max_{i \le m} |\alpha_i| \le \Big\| \sum_{i=1}^{m} \alpha_i x_i \Big\| \le \Big(\sum_{i=1}^{m} |\alpha_i|^2 \Big)^{1/2}$$

Geometrically, $\exists$ F with $\dim F = m \simeq \sqrt{n}$ s.t.

$$B_F \subseteq B_X \cap F \subseteq \sqrt{3}\, Q_F$$

Grothendieck: Can we replace the cube by the ball and still the $\dim F \to \infty$ with n?

? $N(k,\varepsilon) \sim$ SB.

Dvoretzky '61: $\forall\, \varepsilon \in (0,1)$, $\forall\, k \ge 1$ $\exists\, N(k,\varepsilon)$ s.t. (→ ∞ as $k \to \infty$)
for any normed space X with $\dim X = n \ge N$
there exists subspace $F \le X$ with $\dim F = k$
and $d(F, \ell_2^k) < 1+\varepsilon$
or $\exists\, T : \ell_2^k \to X$ s.t.

$$\|x\|_2 \le \|Tx\|_X \le (1+\varepsilon)\|x\|_2$$

$\forall x \in \ell_2^k$.

Moreover, $N(k,\varepsilon) \ge \exp(c\varepsilon^{-2} k^2 \log^2 k)$ or equiv.
$\forall n$ $\exists$ $k \le c\varepsilon \sqrt{\log n} / \log\log n$ s.t. $\ell_2^k \underset{1+\varepsilon}{\hookrightarrow} X$.

V. Milman '71: $k_n \le c\varepsilon^2 \log n / \log\frac{1}{\varepsilon}$

Gordon '85 (Minmax), Sch '87 (Chaining): $k_n \le c\varepsilon^2 \log n$

Random version due V. Milman

Random approach: Concentration of Measure + ...

Sch '06: $k \le c\varepsilon \log n / (\log\frac{1}{\varepsilon})^2$ ⟶

• var

[~~$p=1$, $p=\infty$ γνωστά αλλά πώς συμπερι~~]

var. of a norm.

$$\mathrm{var}(\|\cdot\|) \leq \mathbb{E}\|\nabla \|\cdot\|\|_2^2 \leq L_\cdot^2.$$

Bad news ① Tychomirov '17 Av. John

$\simeq e^{-\varepsilon^2 \log n}$.

② Valettas '17 Av $\|\cdot\|_K$ $\exists$ $|||\cdot|||_K$ 2-isom

: $A(K,\varepsilon) \geq e^{-k_* \varepsilon^2}$.

Examples 1) ℓ_p^n, $1 \leq p \leq \infty$

$$\mathrm{Var}(\|Z\|_p) \simeq \begin{cases} \frac{2^p}{p n^{1-\frac{2}{p}}}, & 1 \leq p \leq c_0 \log n \\ \frac{1}{\log n}, & C_0 \log n \leq p \leq \infty \end{cases}$$

Talagrand's L_1-L_2 estimate

αλλαγή Var αν προσθέσουμε $|x_1|$

LT '17 $c_0 \log n \leq p \leq C_0 \log n$ $\mathrm{Var}(\|\cdot\|)$

2) n-dim. subspaces of L_p στη θέση Lewis

conc τόσο κακό όσο του ℓ_p^n. (q,q)-Poincaré ineq

Δ Positive results (στην Εφαρμογή)

1). Για $\forall n$ $\forall p \leq c_0 \log n$ $\forall$ n-διάστατο $X \leq L_p$ στη Lewis

$$A(X,\varepsilon) \leq C e^{-c \varepsilon \log n}, \quad \varepsilon > 0.$$

2) (Tik '17) $\forall$ X 1-uncond. στην ℓ-θέση το ίδιο

2. Concentration of measure due V. Milman

Gauss' space: $(\mathbb{R}^n, \|\cdot\|_2, \gamma_n)$ f L-Lipschitz on $\mathbb{R}^n$

$$(*) \quad \mathbb{P}\left(|f(Z) - \mathbb{E}f(Z)| > t\right) \le 2e^{-t^2/2L^2}, \quad t>0.$$

Applying (*) for $f = \|\cdot\|$ and $t = \varepsilon\,\mathbb{E}f(Z)$ we get:

$$(2) \quad \mathbb{P}\Big(\underbrace{\big|\|Z\| - \mathbb{E}\|Z\|\big| > \varepsilon\mathbb{E}\|Z\|}_{A(K,\varepsilon).}\Big) \le 2\exp\left(-\tfrac{1}{2}\varepsilon^2 k(X)\right)$$

where $k(X) := \left(\dfrac{\mathbb{E}\|Z\|}{\mathrm{Lip}(\|\cdot\|)}\right)^2$, $\quad \mathrm{Lip}(\|\cdot\|) = \max\limits_{\|\theta\|_2=1} \|\theta\|$

critical dimension

Note - In the large deviation regime, i.e $\varepsilon > 1$. (2) is tight.
- Dvoretzky for $\varepsilon = \frac{1}{2}$ is sharp the rdm formulation is sharp up to $k(X)$; that is ....
- Choose the linear structure so as to make $k(X)$ large
  In John's position $k(X) \ge c\log n$
  → This sharp for the cube

However, for the cube the concentration in the small deviation regime, i.e. $0<\varepsilon<1$ the tail estimate is sub-exponential

$$ce^{-C\varepsilon\log n} \le \mathbb{P}\Big(\cancel{A(K,\varepsilon)}\Big) \le Ce^{-c\varepsilon\log n}, \quad 0<\varepsilon<1$$

↓ Tal '94 + Sch '04

Net argument → $k \ge c\varepsilon\log n / \log\frac{1}{\varepsilon}$

Tik '13 $\quad k \le C\varepsilon\log n / \log\frac{1}{\varepsilon}$.

Question Is it true that if $\|\cdot\|$. $\quad \varepsilon \in (0,1)$

$\exists\, T \in GL_n$: $A(TK, \varepsilon) \le e^{-\varepsilon\log n}\left(A(\cdot, B_\infty^n, \varepsilon)\right)$. $\quad \dfrac{\mathrm{var}(\|TX\|)}{\varepsilon\,\|TX\|^q} \le (\ )$

$$\frac{C_1}{n^q \varepsilon^2} < e^{-\varepsilon \log n}$$

3) $\exists\, T \in GL(n)$ st.

$$P(- -] = A(TK, \varepsilon) \leq C e^{-c\varepsilon^2 \log(n/L^2)}$$

$0 < \varepsilon < 1$

What is L? $L \leq d(X, \ell_\infty^n)$.

Tools. (a) Talagrand's $L_1 - L_2$
(b) Borsuk-Ulam Thm. — Dion.

(E) Existential form ~~of~~ Dvoretzky's.

~~#~~ ℓ_∞-structure ~~builds in the approach arises naturally~~ builds in the approach in a natural way due to the use of Talagrand's L_1-L_2 ~~bounds~~ bounds on. More precisely, ~~the~~ the L_1 norms of ~~the~~ $\partial_i f$ can be equiv. interpreted as. Lipschitz condition wrt ℓ_∞-metric via duality. ~~Which~~ This in turn yields. a deviation estimates. which takes into account ~~both ℓ_2 and ℓ_∞ Lipschitz and which leads to the above concentration~~. the Lipschitz condition in both ℓ_2 and ℓ_∞ sense and which leads to the above concentratio

Talagrand's $L_1 - L_2$ estimate

$$\mathrm{Var}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_{L_2}^2}{1+\log\left(\frac{\|\partial_i f\|_{L_2}}{\|\partial_i f\|_{L_1}}\right)}$$

Recall that $|f(x) - f(y)| \leq M \|x-y\|$

equiv.

$\|\nabla f(x)\|_* \leq M. \longleftrightarrow$ f is M Lipschitz wrt $\| \ \|$.

Hence, bounds on $\|\partial_i f\|_{L_2}$, $\|\partial_i f\|_{L_1}$ "indicate" Lip. conditions wrt. ℓ_2 and ℓ_∞ metric respecenely

e.g. Say that we want to achive the right fluctuations

Using Borsuk we "balance" the ~~...~~ $\|\partial_i f\|_{L_1}$. Then,

$$\mathrm{Var}(f\circ\Lambda) \leq C \sum_{i=1}^{n} \frac{\|\partial_i (f\circ\Lambda)\|_{L_2}^2}{1+\log\left(\frac{n\|\cdot\|_{L_2}}{a_\Lambda}\right)}$$

$$t \mapsto \frac{t^2}{1+\log\left(\frac{t}{u}\right)} \qquad \mathrm{Var}(f\circ\Lambda) \leq \frac{b_\Lambda^2}{1+\log\left(\frac{n b_\Lambda^2}{a_\Lambda^2}\right)}$$

Prop. $\exists\ \Lambda$:

$$\frac{\mathrm{Var}(f\circ\Lambda)}{\mathbb{E}(f\circ\Lambda)} \leq \frac{C}{\log(n/L^2)}$$

$L = d(H_K, \mathcal{U}_K)$.

Figiel - Lindenstrauss - Pic

Figiel - Johnson.

$\exists T$

Theorem $M(K,\varepsilon) \leq \exp\left(-\varepsilon \log(n/L^2)\right)$.

Gowers-Milman.

Alon-Milman (Talagrand).

Given $x_1, \dots, x_n$ in $X = (\mathbb{R}^n, \|\cdot\|)$ we can extract $\sigma \subseteq [n]$.
$\|x_i\| \geq 1$

such $(x_i)_{i\in\sigma}$ $8M_n$-equivalent to natural basis of ℓ_∞^σ.

Αν δοθεί ορθοκ. βάση $(e_i)_{i=1}^n$ στη θέση John με $1 \geq \|e_i\| \geq \frac{1}{4}$ τότε

είτε $k(X) = \mathbb{E}\|\sum g_i e_i\|$ "μεγάλο" και τελειώσαμε

ή $k(X)$ είναι "μικρό" και το Alon-Milman λέει ότι
$\exists$ $\sigma \subseteq [n]$ "μεγάλο" με $(e_i)_{i\in\sigma}$ $\sqrt{k(X)}$-ισοδύναμη
μιας φυσική βάση του ℓ_∞^σ.

$\Rightarrow$ $L(\mathrm{span}\{e_i : i\in\sigma\}) \ll \sqrt{|\sigma|}$

Talagrand's $L_1 - L_2$

$b = \mathrm{Lip}_{\|\cdot\|_2}(\|\cdot\|)$

$a = \mathrm{Lip}_{\|\cdot\|_\infty}(\|\cdot\|)$.

$$\mathrm{var}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_{L_2}^2}{1+\log\left(e\frac{\|\partial_i f\|_{L_2}}{\|\partial_i f\|_{L_1}}\right)}$$

$$\cdot \left(\leq C_1 \frac{\|\nabla f\|_{L_2}^2}{1+\log\left(n\frac{\|\nabla f\|_2^2}{\|\nabla f\|_\infty^2}\right)} \right) \leq C_1 \frac{b^2}{1+\log\left(\frac{n b^2}{a^2}\right)}$$

<u>Lemma</u> f is a smooth Lipschitz. (not constant on any proper subspace)

then $\exists\ \lambda_i > 0$ $\quad \sum \lambda_i^2 = 1$

and $\|\partial_i (f \circ \Lambda)\|_{L_1} \leq \frac{1}{n} a_\Lambda = \frac{1}{n} \mathrm{Lip}_{\|\cdot\|_\infty}(\|\Lambda \cdot\|)$.

$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

<u>Proof</u> Borsuk-Ulam.

$b_\Lambda = \frac{1}{n} \mathrm{Lip}_{\|\cdot\|_2}(\|\Lambda \cdot\|)$

$$\leq C_1 \sum_{i=1}^{n} \frac{\|\partial_i f\|_{L_2}^2}{1+\log\left(e\, n \frac{\|\partial_i f\|_{L_2}^2}{a_\Lambda}\right)} \leq C_1 \frac{b_\Lambda^2}{1+\log\left(n \frac{b_\Lambda^2}{a_\Lambda^2}\right)}$$

<u>Remark</u> If $\|\cdot\|$ 1-uncond. with respect e_i.

$$a_\Lambda = \max_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i \lambda_i e_i \right\| \leq \mathbb{E}_\varepsilon \left\| \sum \varepsilon_i \lambda_i e_i \right\| \leq c\, \mathbb{E} \|\Lambda G\|$$

$$L \qquad\qquad L$$

If $L = d(K, U_n)$.

$$\sigma((f \circ \Lambda)) \leq C_1 \frac{1}{K} \frac{b_\Lambda^2}{1+\log\left(\frac{n}{L^2}\right)}$$

<u>Proposition</u> If K $\exists\ \Lambda$ such that

$$\frac{\mathrm{var}(\|\Lambda \cdot G\|)}{(\mathbb{E}\|\Lambda G\|)^2} \leq \frac{C}{\left(\log\left(\frac{n}{L^2}\right)\right)^2}$$

Moreover $A(K, \varepsilon) \leq C e \exp\left(-C_1 \varepsilon \log(n/L^2)\right) \quad \varepsilon \in (0,1)$

Alon - Milman (Talagrand) K. $\tilde{a} = \|T : \ell_\infty^n \to X\|$

$M_n = \mathbb{E}_\varepsilon \| \sum \varepsilon_i T e_i \|$

Assume that $\|Te_i\| \geq 1$ $\forall i$ $\exists$ $\sigma \subset [n]$ with.

$\sigma \geq \frac{cn}{a}$.

such that $\frac{1}{2}\max|a_i| \leq \|\sum_{i \in \sigma} a_i T e_i\| \leq 4 M_n \max|a_i|$

Take K in John position.

Fix (e_i) : $1 \geq \|e_i\| \geq 1/4$ (Johnson)

Then either. $k(X) = \mathbb{E}\|\sum g_i e_i\|$ is big $(\cdot n^{1/3})$

" $k(X)$ is "small" and by A-L

$\exists$ $\sigma \subseteq [n]$ big enough $[\sqrt{n}]$

such that distance from B_∞^σ is $\ll \sqrt{|\sigma|}$

$(\sqrt{k(X)}.)$

Then ....