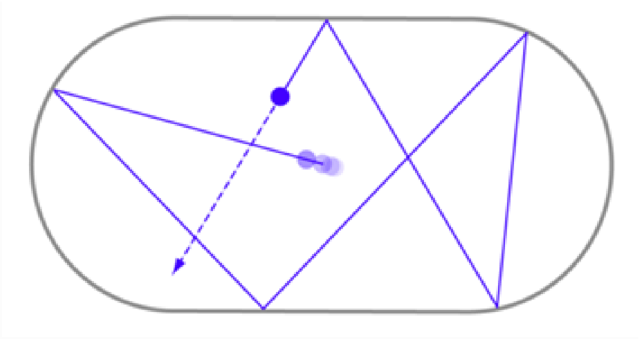
Billiards and Caustics



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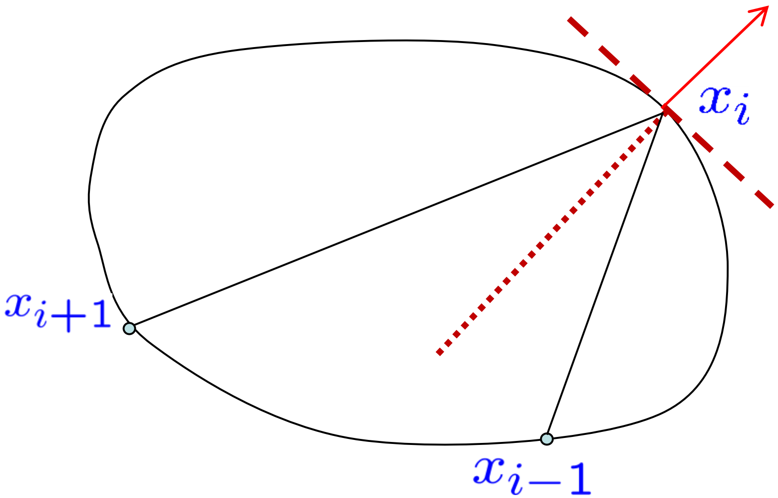
Joint w: Karasev and Ostrover
Yoav Nir
Florentin, Ostrover and Rosen
Pazit Chayim

Euclidean and Minkowski billiards

Move in straight lines + reflection law



1913, Birkhoff: under convexity and smoothness: many periodic trajectories

If not smooth: should explain what to do when we hit a "vertex"

The reflection law corresponds to critical points of the length functional

$$|x_{i+1} - x_i| + |x_i - x_{i-1}|$$

under $x_i \in \partial K$

[Gutkin and Tabachnikov] (K, T)-billiard trajectories correspond to critical points of

$$\sum h_T(x_{i+1} - x_i) \ , \ x_i \in \partial K$$

$$h_T(u) = \sup\{\langle x, u \rangle : x \in T\}$$

Points of view on billiards

Billiards as discrete dynamical systems, on $\partial K \times [0, \pi]$

Billiards as Hamiltonian systems with respect to $H(p,q) = \frac{1}{2}|p|^2 + V(q)$

where $V(q) = 0, \infty$ on K and its complement respectively.

Use approximations.

$$\dot{q} = -\frac{\partial H}{\partial p}$$

$$\dot{p} = \frac{\partial H}{\partial q}$$

The symplectic approach $K \times B_2^n$

$$\partial(K \times B_2^n) = K \times \partial B_2^n \cup \partial K \times B_2^n$$

Travel along the boundary of this 2n-dimensional convex body.

The Hamiltonian is $H(p,q) = \max\{\|q\|_K, \|p\|_{B_2^n}\} = \|(q,p)\|_{K \times B_2^n}$

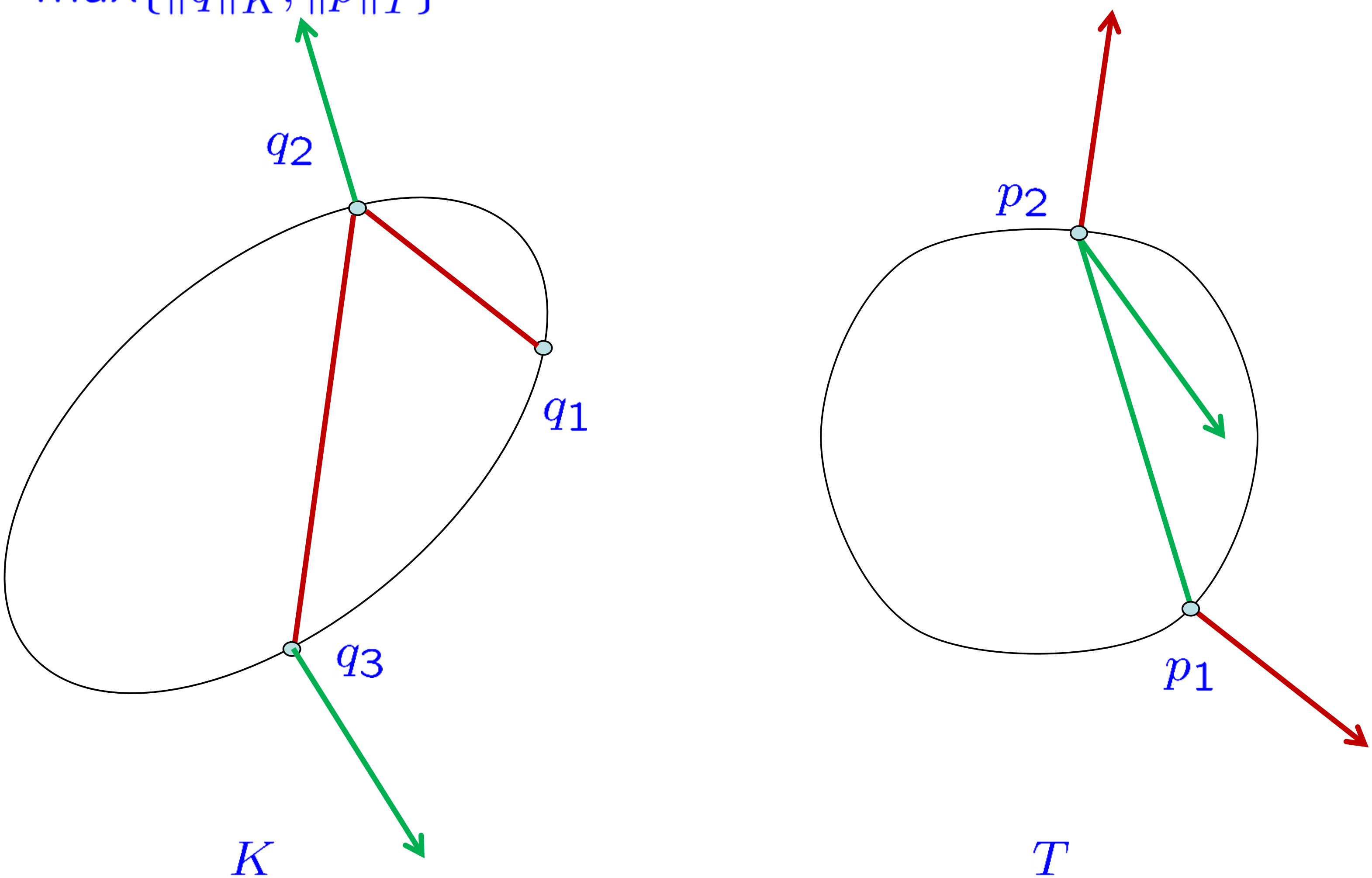
The reflection law for Minkowski billiards

$$\partial(K \times T) = K \times \partial T \cup \partial K \times T$$

$$H(p,q) = \max\{\|q\|_K, \|p\|_T\}$$

$$\dot{q} = -\frac{\partial H}{\partial p}$$

$$\dot{p} = \frac{\partial H}{\partial q}$$

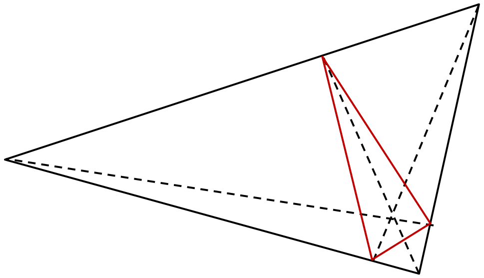


Natural questions

The existence of (classical) periodic orbits (with any number of reflection points, with prescribed rotation number if in the plane).

Does every triangle in $\mathbb{R}^2$ have a classical periodic billiard trajectory?

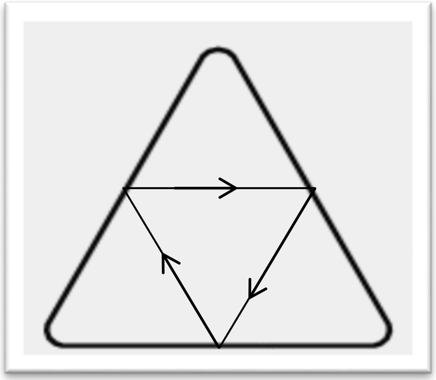
If all angles are acute – Yes and it is simply described [Fagnano]



Same true for "acute" bodies in $\mathbb{R}^n$ [Akopyan and Balitskiy]

What is the length of the shortest periodic billiard trajectory?

It is not necessarily the width:



[A-A, Ostrover]: an interpretation as a well known capacity, we showed

$$\xi_B(K) = c_{EH}(K \times B)$$

$$\xi_T(K) = c_{EH}(K \times T) = \xi_K(T)$$

[A-A,Ostrover and Karasev] For centrally symmetric convex bodies

$$\xi_K(T) = \xi_T(K) = c_{EH}(K \times T) = 4 inrad_{T^\circ}(K)$$

In particular $\xi_K(K^\circ) = 4$ which gives a tight connection with Mahler's conjecture in the symmetric case, which can be restated as

$$c_{EH}(K \times T) \leq (n!\text{Vol}(K \times T))^{1/n}$$

A more elementary approach [Bezdek and Bezdek] motivated [Y. Nir (M.Sc. Thesis)] and [Akpoyan, Balitskiy, Karasev, and Sharipova]

The latter deduce from this method that for non-symmetric (sharp):

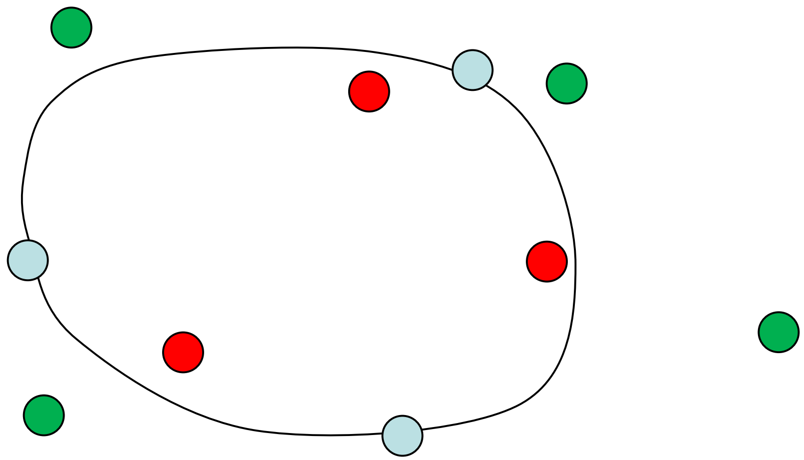
$$\xi_K(K^\circ) \geq 2 + 2/n$$

In a recent paper [Balitskiy] shows that for the simplex and permutahedron

$$\xi_{\triangle_n^\circ}(P_n) = (n+1)^2 = (n!\text{Vol}(\triangle_n^\circ)\text{Vol}(P_n))^{1/n}$$

Shortest length: geometric approach

Consider $P_m(K) = \{(q_1, \ldots, q_m) : \{q_1, \ldots, q_m\}$ does not fit in $\alpha K + x$ with $\alpha \in (0,1)\}$



Theorem: [Bezdek and Bezdek] The length of the shortest closed billiard trajectory in a convex body equals

$$\xi_B(K) = \min_{m \geq 2} \min_{P \in P_m(K)} \sum_{i=1}^{m} |q_{i+1} - q_i|$$

Moreover, the minimum is attained for some $m \leq n+1$.

Shortest length: geometric approach

This easily generalizes to Minkowski billiards, taking another norm with respect to which the length is minimized.

Consider $P_m(K) = \{(q_1, \dots, q_m) : \{q_1, \dots, q_m\} \text{ does not fit in } \alpha K + x \text{ with } \alpha \in (0,1)\}$

Then $$\xi_T(K) = \min_{m \geq 2} \min_{P \in P_m(K)} \sum_{i=1}^{m} h_T(q_{i+1} - q_i)$$

Moreover, the minimum is attained for some $m \leq n+1$.

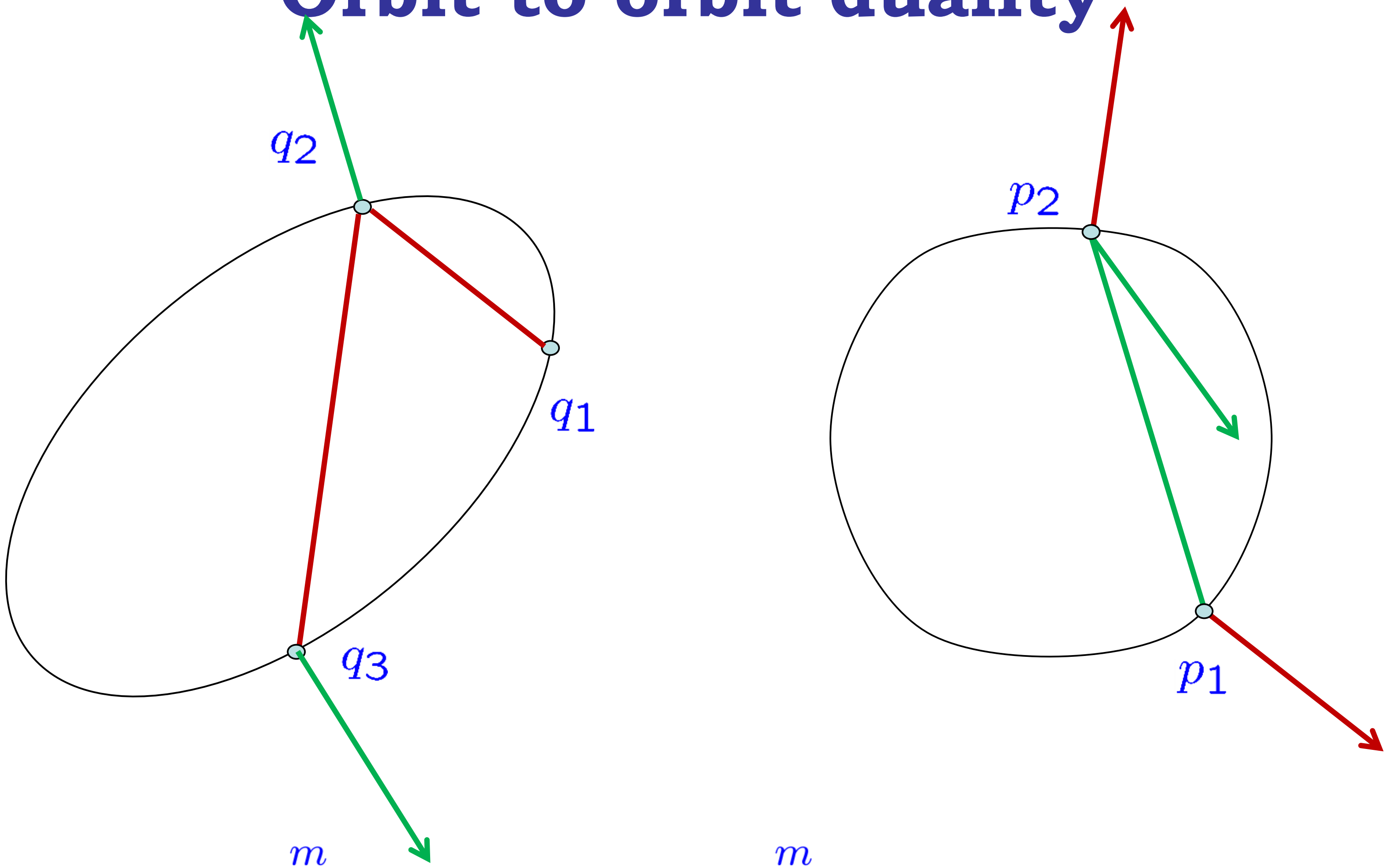
From this one get immediately

Monotonicity: $K_1 \subset K_2 \quad \text{implies} \quad \xi_T(K_1) \leq \xi_T(K_2)$

Brunn-Minkowski: $\xi_T(K_1 + K_2) \geq \xi_T(K_1) + \xi_T(K_2)$

$$\xi_{K_1+K_2}(T) \geq \xi_{K_1}(T) + \xi_{K_2}(T)$$

Orbit to orbit duality

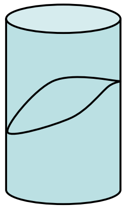


$$\sum_{i=1}^{m} h_T(q_{i+1} - q_i) = \sum_{i=1}^{m} h_K(p_{i+1} - p_i)$$

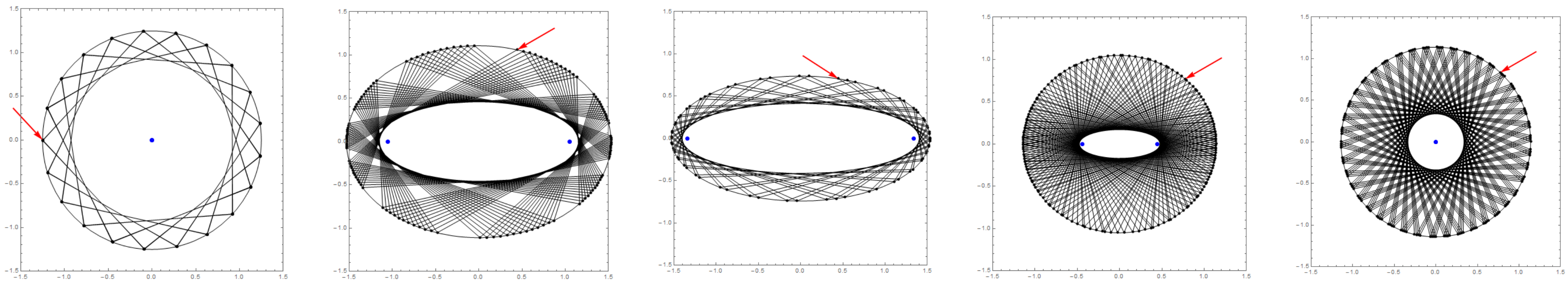
$$\xi_T(K) = \xi_K(T) = c_{EH}(K \times T)$$

Invariant circles and caustics

A convex caustic in a billiard table $K \subset \mathbb{R}^n$ is a convex set $C \subset K$ such that if a billiard trajectory is tangent to it once, it is tangent to it after each reflection.



Euclidean case



Dan S. Reznik "Dynamic Billiards in Ellipse"
http://demonstrations.wolfram.com/DynamicBilliardsInEllipse/

[Lazutkin] Near the boundary of a sufficiently (553, improved to 6 by Douady) smooth convex body in the plane, with positive curvature, one may find caustics.

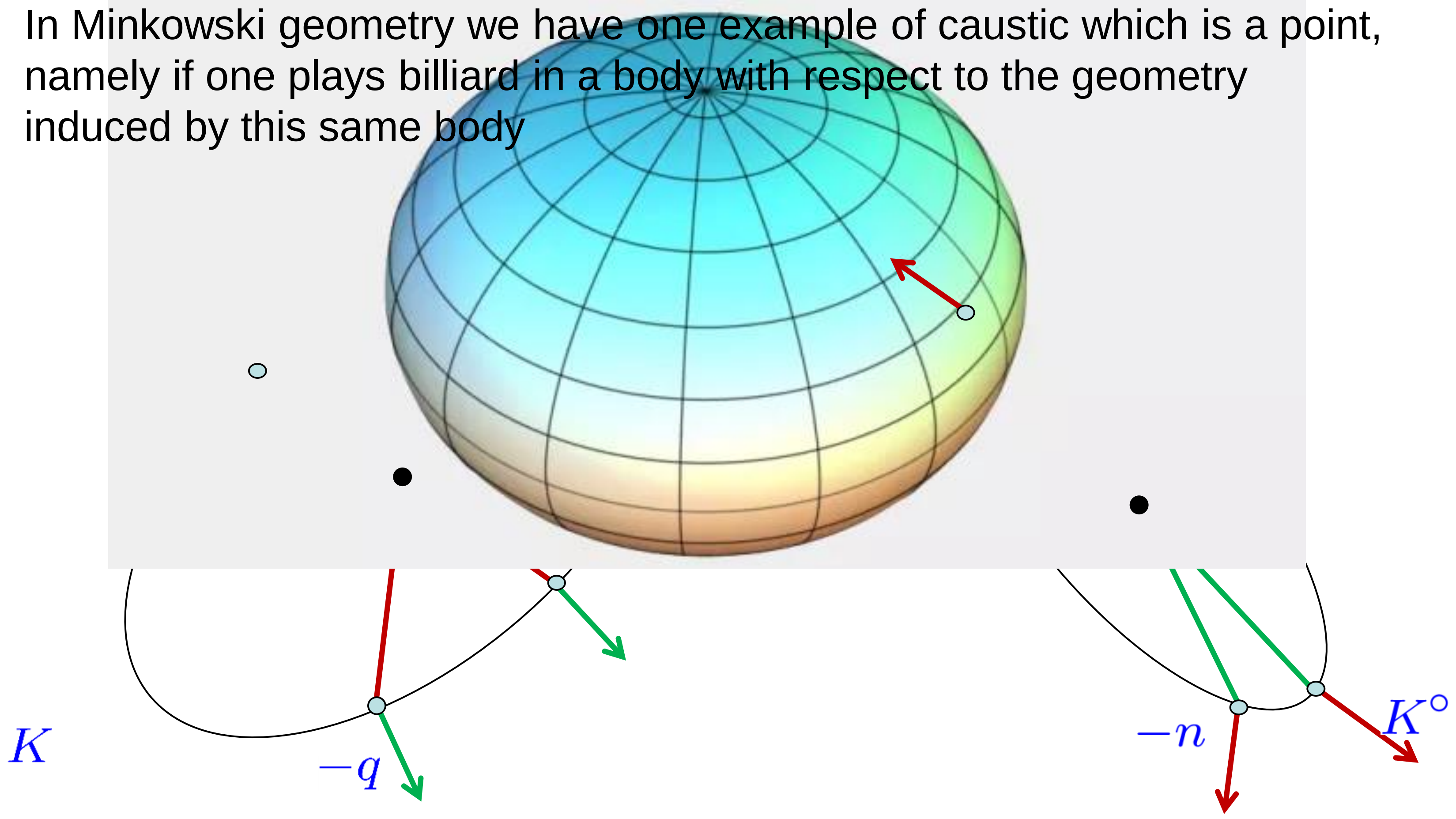
[Mather] For a caustic to exist, the table must be strictly convex,
[Gutkin and Katok] quantitative.

Many open questions about caustics (Birkhoff's conjecture), connected with integrability and ergodicity.

Gruber/Berger theorem

Among all convex billiard tables in $K \subset \mathbb{R}^n$ for $n \geq 3$ only ellipsoids have convex caustics, and these are the confocal ellipsoids contained in them (+ the intersection of all these confocal ellipsoids).

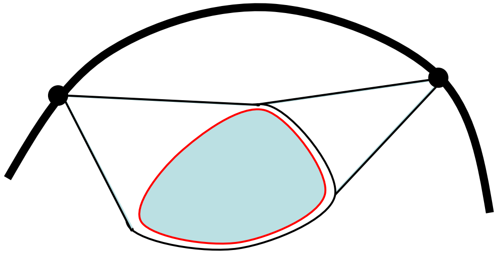
In Minkowski geometry we have one example of caustic which is a point, namely if one plays billiard in a body with respect to the geometry induced by this same body



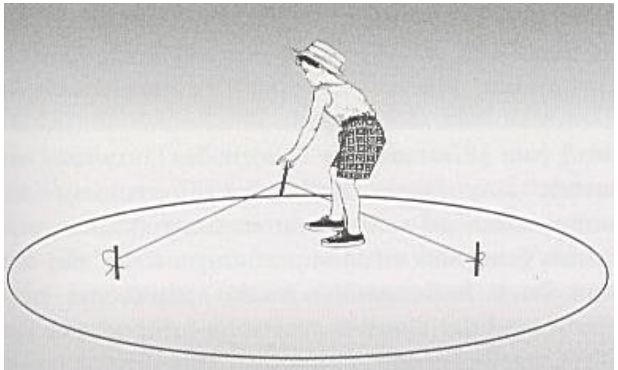
The gardener's construction

Given a convex planar set $C \subset \mathbb{R}^2$ one can take a string of length $l \geq |\partial C|$ and build the body of all points

$$\{x \in \mathbb{R}^2 : |\partial(\mathrm{conv}(x, C))| \leq l\}$$



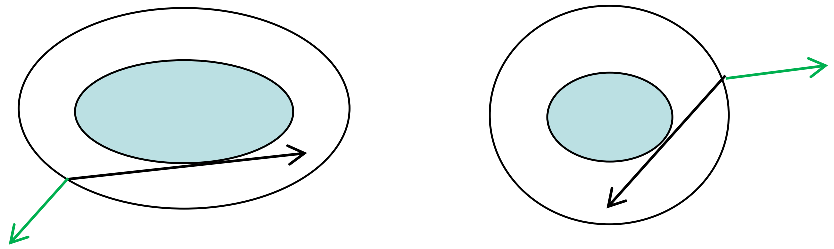
Classical parameters of a caustic: string length, rotation number, perimeter.



It is not difficult to show that (also when length is measured with respect to some non-euclidean norm) one always gets a convex set, and the original set is a caustic of the new one [Gutkin and Tabachnikov].

On caustic duality

A T-caustic in K and a K-caustic in T will be called **dual** if trajectories remain tangent to both.



Theorem [A-A, Florentin, Ostrover, and Rosen]

Assume the body K is centrally symmetric strictly convex and C^1 smooth.

Then every convex B-caustic in K has a dual K-caustic in B.

Moreover – they have the same perimeter, string length, rotation number.

Theorem [A-A, Florentin, Ostrover, and Rosen]

There exist two centrally symmetric infinitely smooth convex sets K and T such that K has a convex T-caustic with no dual convex caustic.

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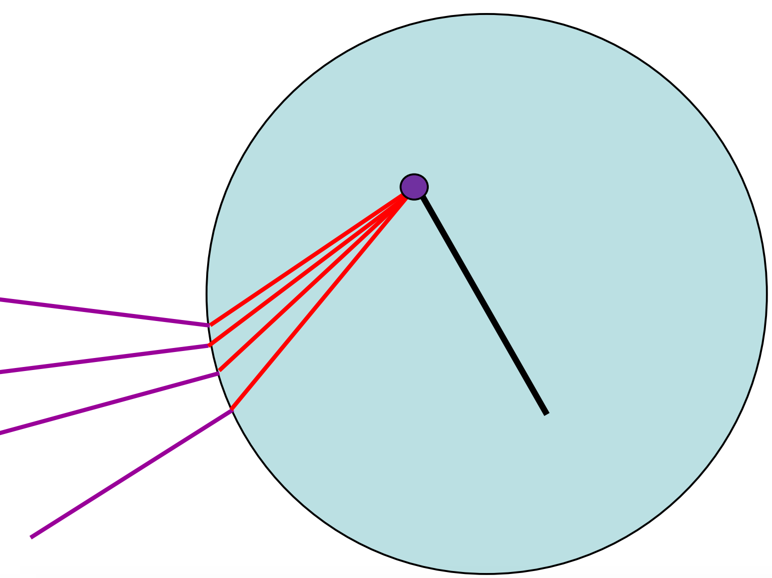
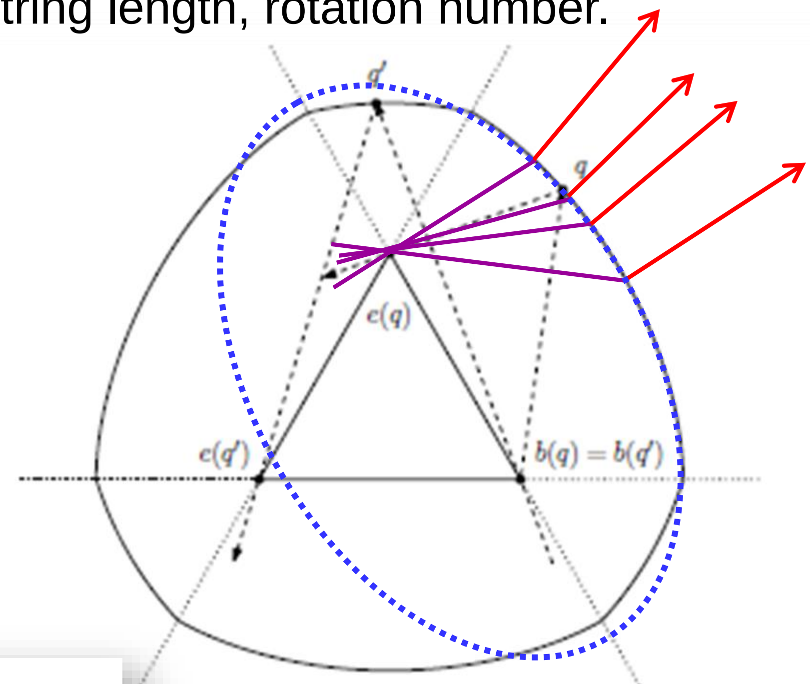
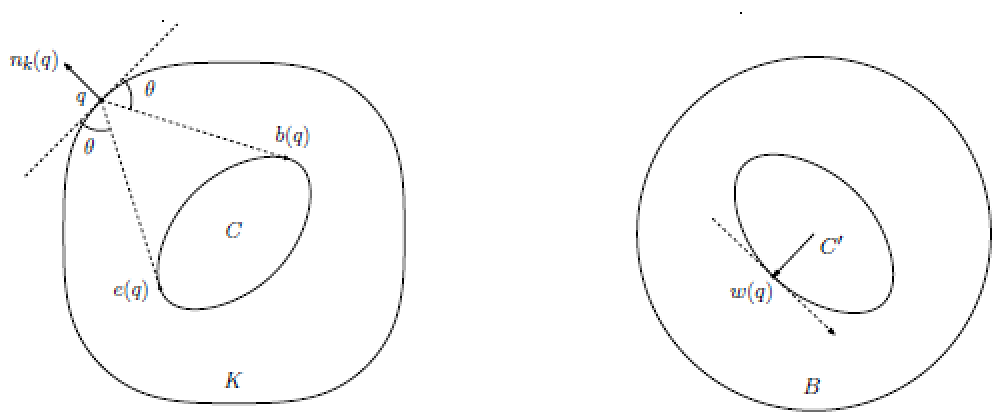




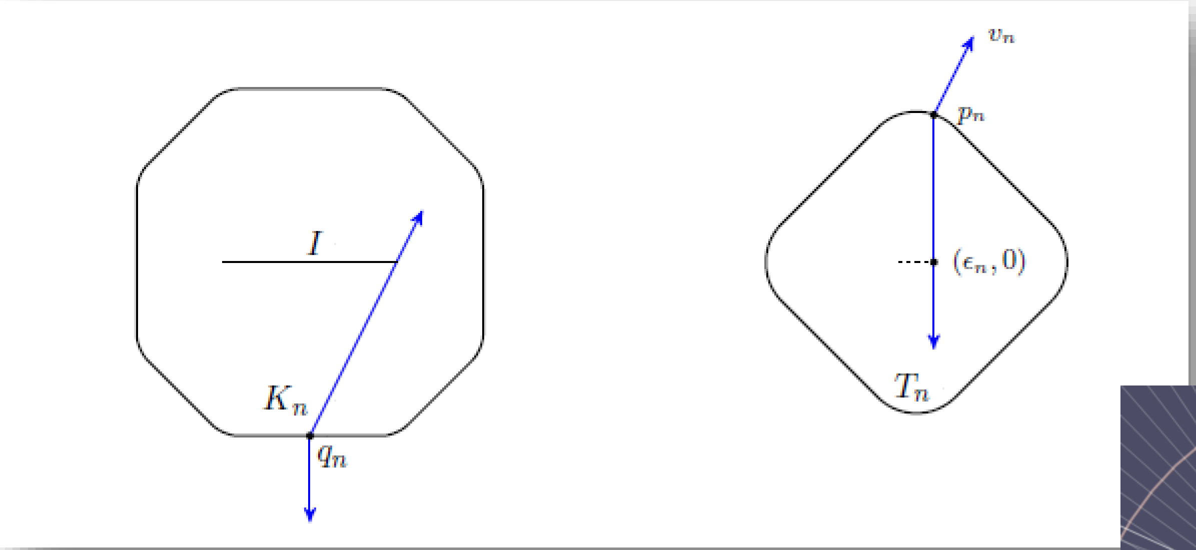


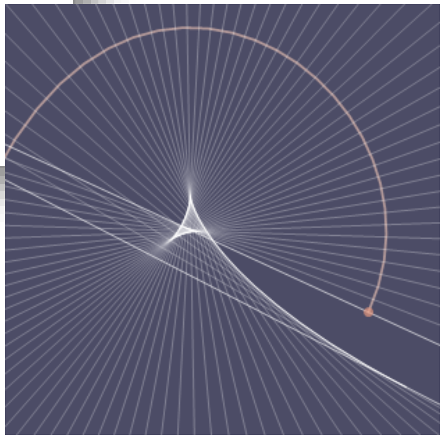
$$w(q) = \frac{e(q) - b(q)}{L(q)}$$

An example with no duality

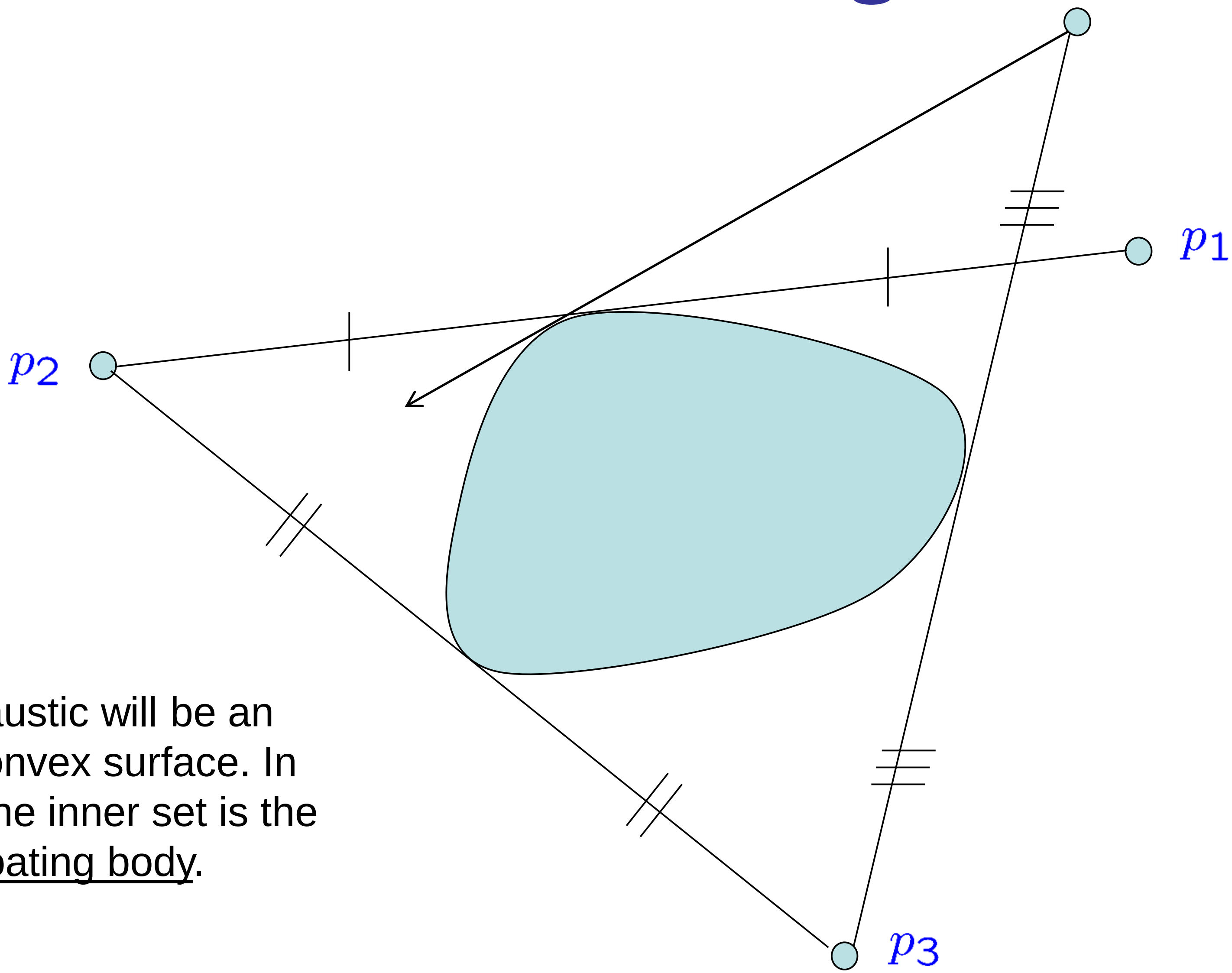
Theorem [A-A, Florentin, Ostrover, and Rosen]

There exist two centrally symmetric infinitely smooth convex sets K and T such that K has a convex T-caustic with no dual convex caustic.

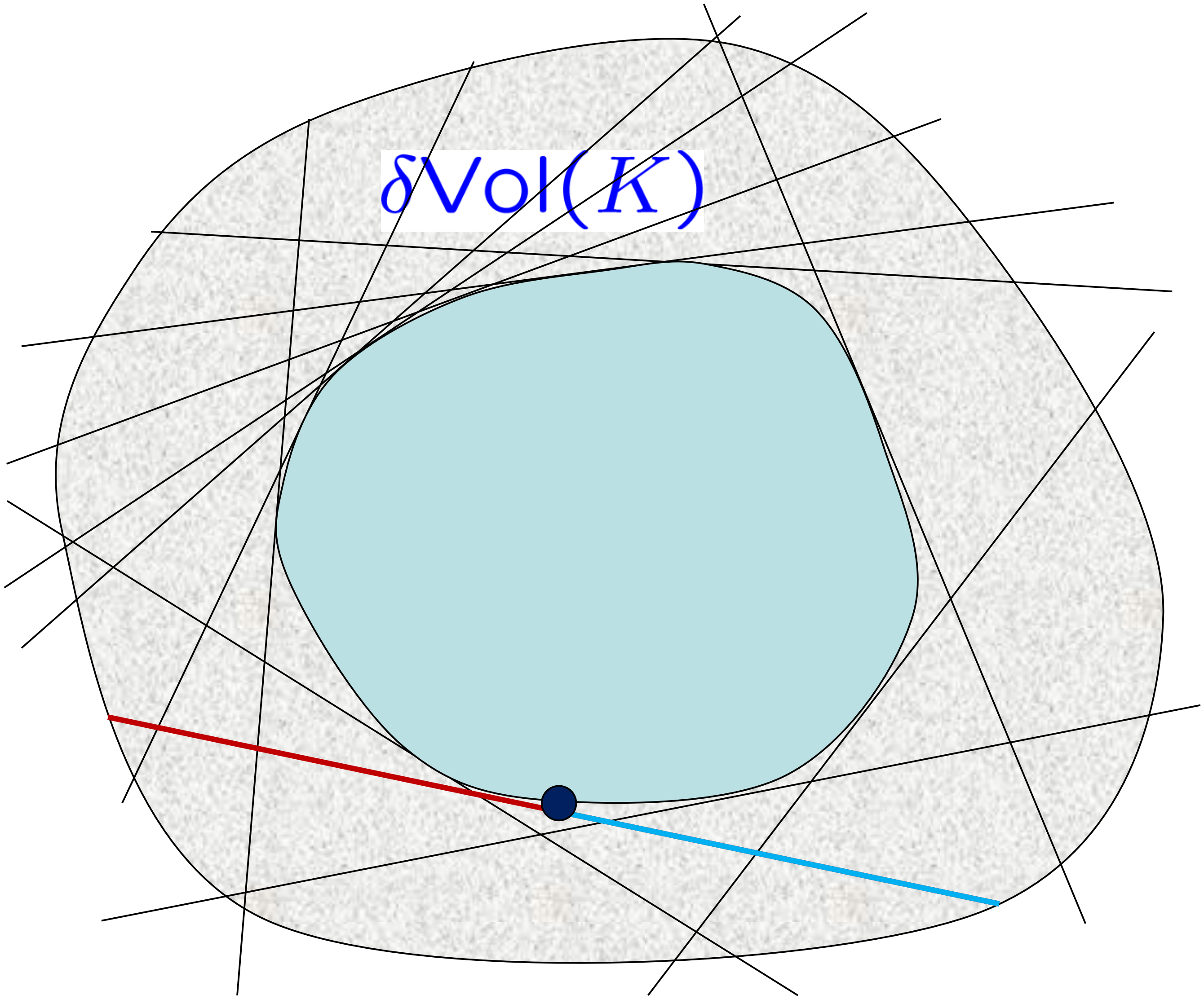




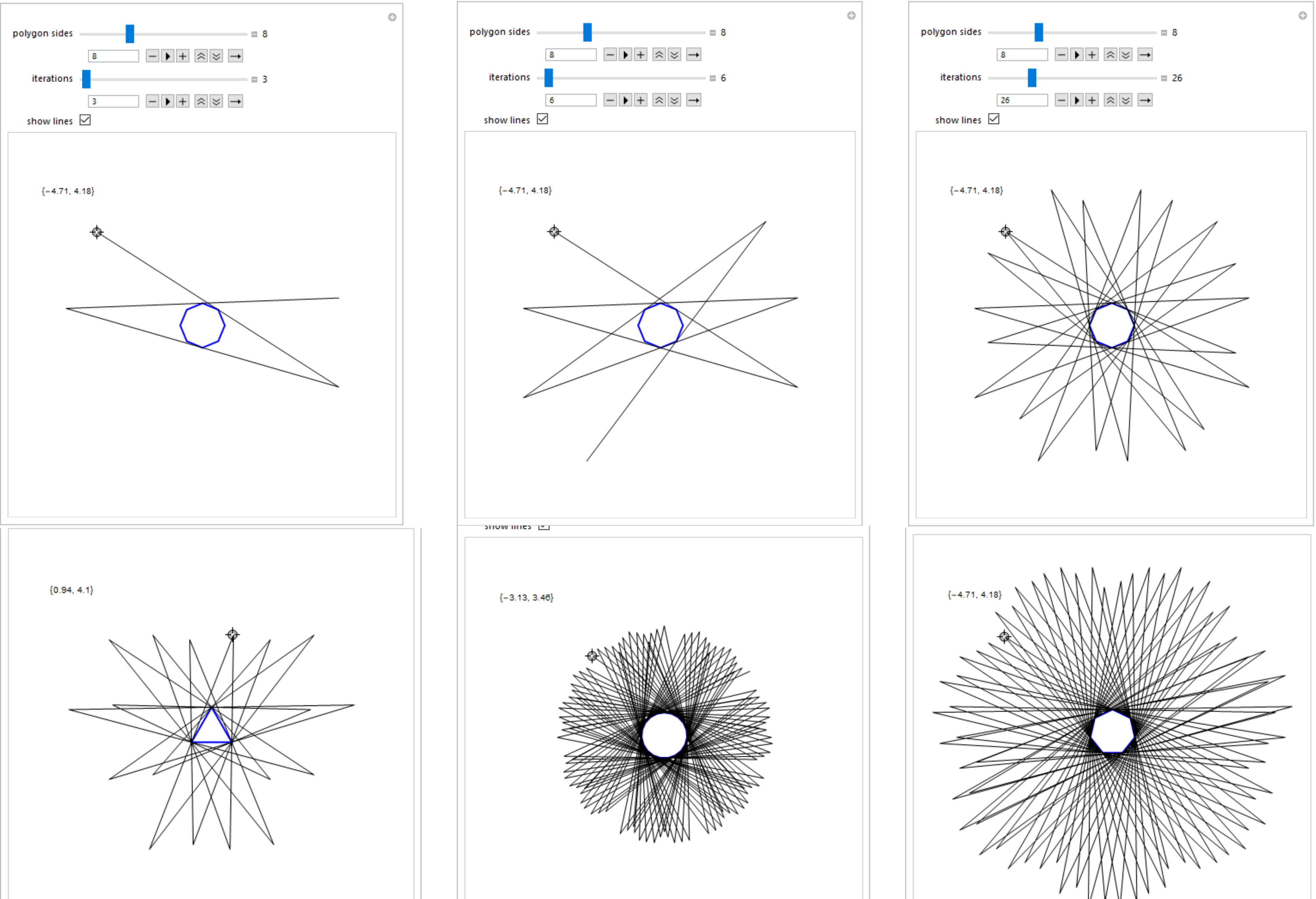
Outer billiards and floating bodies



An outer caustic will be an invariant convex surface. In this case, the inner set is the caustic's floating body.


$\delta\text{Vol}(K)$

Thank you for your attention



[Gordon Hughes](#) "[Outer Billiards on Regular Polygons](#)" [Wolfram Demonstrations Project](#)
http://demonstrations.wolfram.com/OuterBilliardsOnRegularPolygons/

Gruber-Berger theorem:

Among all convex billiard tables in $K \subset \mathbb{R}^n$ for $n \geq 3$ only ellipsoids have convex caustics, and these are the confocal ellipsoids contained in them (+ the intersection of all these confocal ellipsoids).

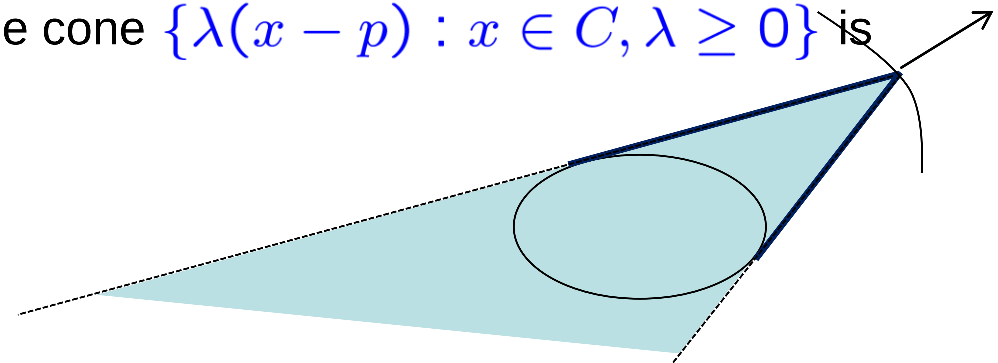
Some elements of Gruber's proof carry over easily

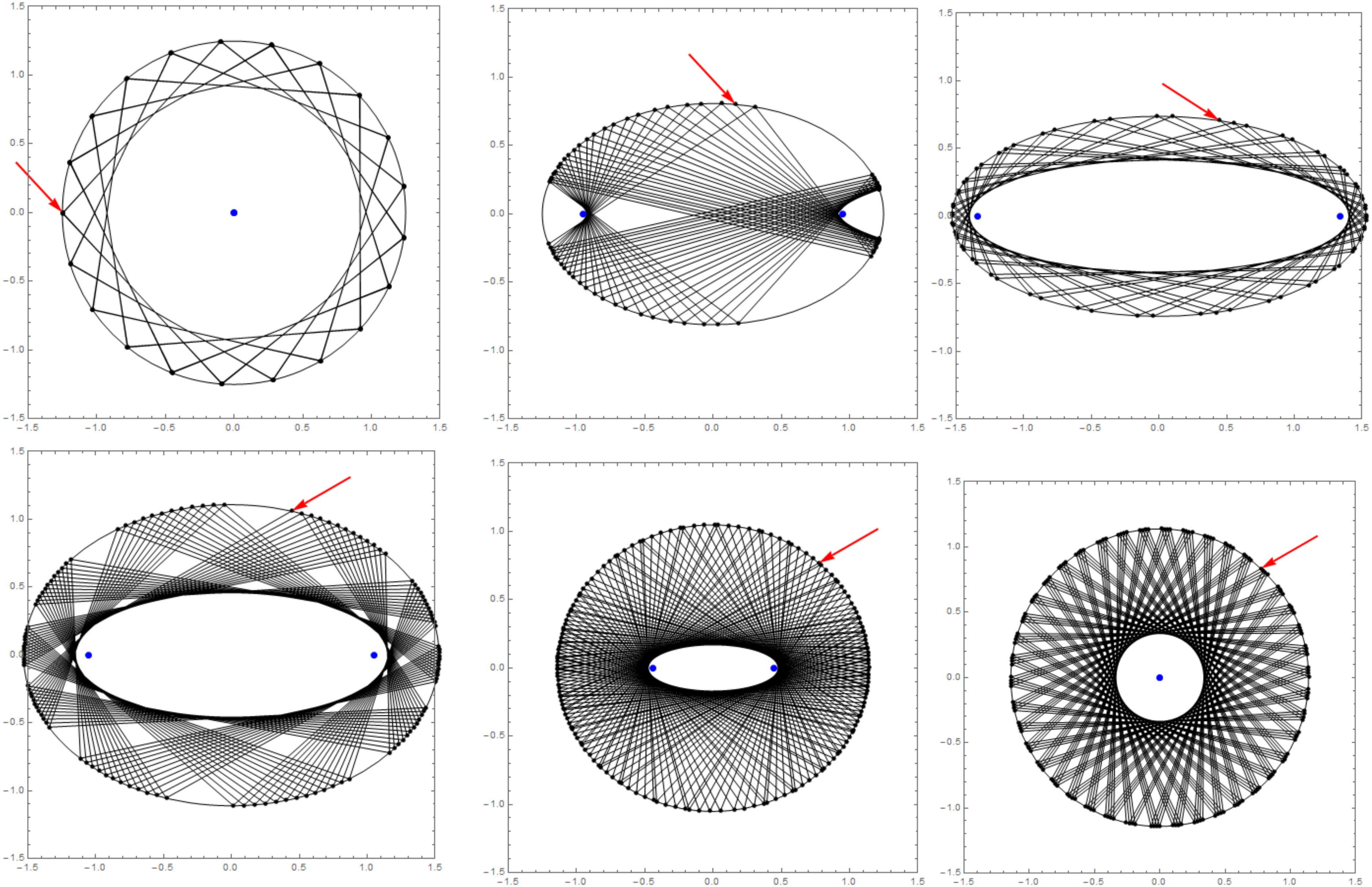
Lemma: For a table $K \subset \mathbb{R}^n$ the convex body $C \subset K$ is a caustic if and only if for any subspace of a fixed dimension, $P_E C \subset P_E K$ is a caustic of $P_E K$

The only adaptation needed when talking about non-Euclidean is that the geometry corresponds to the quotient space (thus a caustic for a (K,T) billiard implies a caustic for the $(P_E K, T \cap E)$ billiard.

Others are somewhat different in the Minkowski case, for example

Lemma (Euclidean): For a table $K \subset \mathbb{R}^n$ the convex body $C \subset K$ is a caustic if and only if for any point $p \in \partial K$ the cone $\{\lambda(x-p) : x \in C, \lambda \geq 0\}$ is symmetric with respect to $n_K(p)$




1.5
1.0
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−1.5
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5

More fun: invariant measure, mean free path

Back to the Birkhoff point of view, look at the phase space, which can be though of as part of $\partial K \times \partial T \subset \mathbb{R}^{2n}$

Namely those vectors for which $\{(q,p) : \langle n_K(q), n_T(p) \rangle > 0\}$

On this phase space there is a billiard map.

In the Euclidean (2D, but can be generalized) case, this space is sometimes identified with $\partial K \times (0, \pi)$, and the measure $|\sin(\alpha)| d\alpha d\sigma_K$ is preserved by the billliard map.

The "mean free path" is the average length of a billiard chord with respect to this measure, and it is

$$\pi A/L$$

For Minkowski billiards, the measure preserved is

$$\langle n_K(q), n_T(p) \rangle d\sigma_K(q) d\sigma_T(p)$$

The mean free path length is a ratio of Holmes Thompson volume.

N.B. two particles oscillate on [0,1], rebounding at the endpoints and colliding elastically (retaining kinetic energy and momentum) is modeled by a billiard in a right angled triangle: