Day 3 talk 2

Monika Ludwig

"Hessian Valuations"

Rk. supplementary to slides.

J. w. Colesanti & Mussnig

$\mathcal{F}(X) = \{ f : X \mapsto \mathbb{R} \}$ a space of $\mathbb{R}$-valued func.

f, g $\quad f \vee g = \max\{f, g\}$

$f \wedge g = \min\{f, g\}$

<u>Def</u> $Z : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is a valuation $\Leftrightarrow$

$$Z(f) + Z(g) = Z(f \vee g) + Z(f \wedge g)$$

$\forall f, g \in \mathcal{F}(X)$ s.t. $f \vee g$, $f \wedge g \in \mathcal{F}(X)$

Q: classification of interesting valuations on $\mathcal{F}(X)$

Motivation: • Erlangen program.

Classifying invariant Valuations

• Theory of / Case of valuations on convex bodies

• $\mathcal{K}^n$ convex bodies $\quad Z : \mathcal{K}^n \to \mathbb{R}$, +)

$$Z(K) + Z(L) = Z(K \cup L) + Z(K \cap L)$$

• Hilbert's third problem

<u>Thm</u> (Hadwiger '52)

Z continuous, rigid motion invariant valuation

$\Leftrightarrow \exists\, c_0 \dots c_n$

$$Z(K) = c_0 V_0(K) + \dots + c_n V_n(K)$$

V_i intrinsic volumes

- $V_i(K) = \frac{\binom{n}{i}}{n v_{n-i}} \int_{S^{n-1}} S_i(K,\xi)\,d\xi = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\partial K} H_{n-i-1}(K,x)\,dx$

- Steiner formula

$$V_n(K+tB^n) = \sum_{i=0}^{n} t^{n-i} v_{n-i} V_i(K)$$

↗ volume

- Crofton Formula: $V_i(K) = \int_{\mathrm{Graff}(n,i)} V_0(K \cap E)\,d\mu_i(E)$

$$= \int_{\mathrm{Gr}(n,i)} V_i(\mathrm{proj}_E K)\,d\nu_i(E))$$

- Principal Kinematic formula & proof sketch using Hadwiger

- Hadwiger very useful in convex geometry ⇝ would be nice to have something similar for function spaces

- Generalizements of Hadwiger much harder to prove.

Thm L. & Reitzner

Cauchy function: $\xi(x+y) = \xi(x) + \xi(y)$, $x,y \geq 0$

$Z: \mathcal{P}_0^n \to \mathbb{R}$ is $SL(n)$ invariant valuation

⟺

$\exists\ c_0, c_0' \in \mathbb{R}$ and a Cauchy function $\xi: [0,\infty) \to \mathbb{R}$

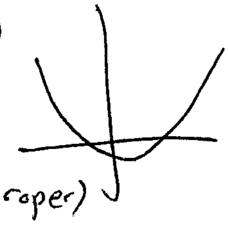
$$Z(P) = c_0 V_0(P) + c_0' \ldots$$

Thm Abstract Hadwiger theorem '07 Alexer GAFA

Results on many classes of functions: L^p, Orlicz, definable functions, Sobolev, BV, quasi-concave, log-concave ...

This talk: convex functions

Def Convex functions $\mathrm{Conv}(\mathbb{R}^n)$



- allow ∞-value, but not $\equiv \infty$ (proper)
- require coercive

$$\lim_{|x|\to\infty} v(x) = +\infty$$

· **Def** epi-convergent (Γ-convergence)

Thm Colesanti, L., Mussnig '17

$Z : \mathrm{Conv}(\mathbb{R}^n) \to [0,\infty)$ is continuous, $SL(n)$ & translation invariant valuation

$\Longleftrightarrow$

$\exists$ non-negative functions $\zeta_0 \in C(\mathbb{R})$ and $\zeta_n \in D^{n-1}(\mathbb{R})$ such that

$$Z(u) = \zeta_0(\min_{x\in\mathbb{R}^n} u(x)) + \int_{\mathrm{dom}(u)} \zeta_n(u(x))\,dx \quad \forall u \in \mathrm{Conv}(\mathbb{R}^n)$$

Sketch of proof: - consider piecewise affine functions

- use Blaschke to valuations on polytopes

$Z_s(P) = Z(I_P + s)$, P polytope

valuation on convex polytopes.

- use Hadwiger

$\tilde{Z}_s(P) = \psi_0(s) V_0(P) + \psi_n(s) V_n(P)$

$\shortparallel$

$Z(I_P + s)$

Q: Classification Rigid motion invariant classification of valuations on $\mathrm{Conv}(\mathbb{R}^n)$?

Thm (Alesker) Annals 1999

$Z : K^n \to \mathbb{R}$ is continuous, rotationally invariant, polynomial valuation

<=> polynomial

classification

Def: polynomial when restricted to translations.

Thm (Colesanti, L., Mussnig '17+)

$Z_{\varphi_i}(v) = \int_{\mathbb{R}^{2n}} J(v(x), x, y_i) \, d\,\Theta_i(v, (x,y))$

is a continuous valuation on $Conv(\mathbb{R}^n)$

- $\Theta_i(v, \cdot)$ Heassian measure of v.

Def Hessian measure see slides

Hessian valuations, not rotationally invariant, but translationally invariant.

large class of rigid motion invariant valuations with different J

Are these all?
Speaker doesn't expect these to be all examples, but could be dense. Not known

Qs: Classification of continuous, rigid motion invariant valuations on K_0

... more on slides

Hessian Valuations

Monika Ludwig
joint work with Andrea Colesanti and Fabian Mussnig

Technische Universität Wien & MSRI Berkeley

MSRI, November 2017

Valuations on Function Spaces

- $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ space of real valued functions on X
- $f, g \in \mathcal{F}(X)$: $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$

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- $\mathrm{Z} : \mathcal{F}(X) \to \mathbb{R}$ is a **valuation** $\iff$

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Examples

- L^1 norm: $\mathrm{Z} : \begin{cases} L^1(\mathbb{R}^n) \to \mathbb{R} \\ f \mapsto \|f\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx \end{cases}$
- Dirichlet energy: $\mathrm{Z} : \begin{cases} W^{1,2}(\mathbb{R}^n) \to \mathbb{R} \\ f \mapsto \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx \end{cases}$

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Question:

- Classification of interesting valuations on $\mathcal{F}(X)$

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

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Groups G acting on $\mathbb{R}^n$

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- Group of rigid motions $\mathrm{O}(n) \ltimes \mathbb{R}^n$: $x \mapsto Ux + b$
  where U is an orthogonal $n \times n$ matrix and $b \in \mathbb{R}^n$

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- Special linear group $\mathrm{SL}(n)$: $x \mapsto Ax$
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Invariance: $\mathrm{Z}(f \circ \phi^{-1}) = \mathrm{Z}(f)$ for all $f \in \mathcal{F}(\mathbb{R}^n)$ and $\phi \in G$

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  Dehn 1902, Sydler 1965, Jessen & Thorup 1978, . . .

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  Dehn 1902, Sydler 1965, Jessen & Thorup 1978, ...
- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Bernig 2006, Fu 2006, Hug 2005, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Parapatits 2011, Faifman 2013, Böröczky 2015, Li 2016, Ma 2016, Mussnig 2017 ...

Valuations on Convex Bodies

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

- $V_0(K), \dots, V_n(K)$ — intrinsic volumes of K
- V_n — n-dimensional volume
- $2\, V_{n-1}(K) = S(K)$ — surface area
- $V_0(K)$ — Euler characteristic

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Corollary

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, i-homogeneous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$i \in \{0, \ldots, n\}$ *and* $\exists\, c \in \mathbb{R}$ *:*

$$\mathrm{Z}(K) = c\, V_i(K)$$

for every $K \in \mathcal{K}^n$.

Intrinsic Volumes

- K convex body with smooth boundary

$$V_i(K) = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\mathbb{S}^{n-1}} s_i(K,\xi)\, d\xi = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\partial K} H_{n-i-1}(K,x)\, dx$$

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Application: Principal kinematic formula

For $K, L \in \mathcal{K}^n$,

$$\int_{\phi \in \mathrm{O}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi = \sum_{i=0}^n \frac{v_i\, v_{n-i}}{\binom{n}{i}\, v_n}\, V_i(K)\, V_{n-i}(L)$$

- $d\phi$ normalized Haar measure on $\mathrm{O}(n) \ltimes \mathbb{R}^n$
- Blaschke, Chern, Hadwiger, Santaló, ...

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- **Proof.** $\mathrm{Z}(K, L) = \int_{\phi \in \mathrm{O}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ continuous valuations on $\mathcal{K}^n$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ rigid motion invariant

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$$\Rightarrow \quad \mathrm{Z}(K, L) = \sum_{i=0}^n c_i(L)\, V_i(K) = \sum_{i,j=0}^n c_{ij}\, V_i(K)\, V_j(L)$$

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Determine c_{ij} by choosing suitable bodies! □

Valuations on Convex Bodies

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

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for every $K \in \mathcal{K}^n$.

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for every $K \in \mathcal{K}^n$*.*

Theorem (Blaschke 1937)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

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$\exists\, c_0, c_n \in \mathbb{R}$*:*

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Theorem (L. & Reitzner 2017)

$\mathrm{Z} : \mathcal{P}_0^n \to \mathbb{R}$ *is an* $\mathrm{SL}(n)$ *invariant valuation*

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$\exists\ c_0, c_0' \in \mathbb{R}$ *and a Cauchy function* $\zeta : [0, \infty) \to \mathbb{R}$*:*

$$\mathrm{Z}(P) = c_0\, V_0(P) + c_0'\, (-1)^{\dim P}\, \mathbb{1}_{\mathrm{relint}\, P}(0) + \zeta\big(V_n(P)\big)$$

for every $P \in \mathcal{P}_0^n$.

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Corollary (Hadwiger 1970)

$\mathrm{Z} : \mathcal{P}_0^n \to \mathbb{R}$ *is a continuous and* $\mathrm{SL}(n)$ *invariant valuation*

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Abstract Hadwiger Theorem

Theorem (Alesker: GAFA 2007)

For a compact subgroup G of $\mathrm{SO}(n)$, the space of continuous, G and translation invariant valuations on $\mathcal{K}^n$ is finite dimensional.

$$\iff$$

G acts transitively on $\mathbb{S}^{n-1}$.

Abstract Hadwiger Theorem

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For a compact subgroup G of $\mathrm{SO}(n)$, the space of continuous, G and translation invariant valuations on $\mathcal{K}^n$ is finite dimensional.

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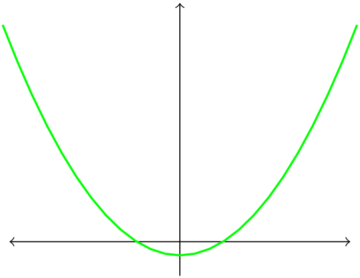
G acts transitively on $\mathbb{S}^{n-1}$.

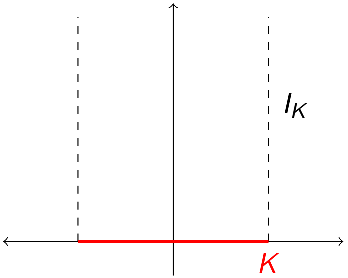
- $\mathrm{U}(n)$ invariance (Alesker: GAFA 2001, Fu: JDG 2006, Bernig & Fu: Annals 2001, Wannerer: JDG 2014, AiM 2014)
- $\mathrm{SU}(n)$ invariance (Bernig: GAFA 2009)
- G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$ invariance (Bernig: Israel J. 2011, Bernig & Voide: Israel J. 2016)
- $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot\mathrm{U}(1)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ invariance (Bernig & Solanes: JFA 2014, PLMS 2017+)

Valuations on Function Spaces

- Valuations on definable functions:
  Baryshnikov, Ghrist & Wright: AIM 2013
- Valuations on L_p and Orlicz functions:
  Tsang: IMRN 2010, TAMS 2012; L.: AAM 2013;
  Ober: JMAA 2014; Kone: AAM 2014; Li & Ma: JFA 2017
- Valuations on Sobolev and BV functions:
  L.: AIM 2011, AJM 2012; Wang: IUMJ 2014; Ma: SCM 2016
- **Valuations on convex functions:**
  Cavallina & Colesanti: AGMS 2015;
  Colesanti, L. & Mussnig: IMRN 2017 & CVPDE 2017 & 2017+
- Valuations on quasi-concave functions:
  Colesanti & Lombardi: 2017; Colesanti, Lombardi & Parapatits: 2017
- Valuations on log-concave functions:
  Colesanti, L. & Mussnig: CVPDE 2017; Mussnig: 2017+

Valuations on Convex Functions





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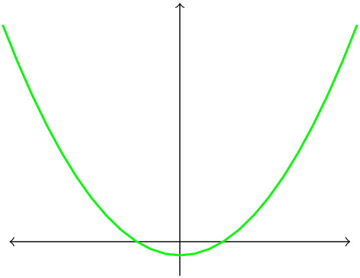
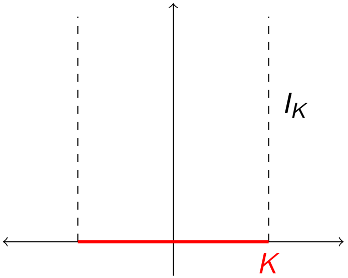
  $= \{u : \mathbb{R}^n \to (-\infty, \infty] : u$ convex, l.s.c., proper, coercive$\}$
- $u : \mathbb{R}^n \to (-\infty, \infty]$ is coercive $\Leftrightarrow \lim_{|x|\to+\infty} u(x) = +\infty$

Valuations on Convex Functions





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  $= \{u : \mathbb{R}^n \to (-\infty, \infty] : u \text{ convex, l.s.c., proper, coercive}\}$
- $u : \mathbb{R}^n \to (-\infty, \infty]$ is coercive $\Leftrightarrow \lim_{|x| \to +\infty} u(x) = +\infty$
- u_k is epi-convergent to u in $\mathrm{Conv}(\mathbb{R}^n) \Leftrightarrow$
  - $u(x) \leqslant \liminf_{k\to\infty} u_k(x_k)$ for every (x_k) with $x_k \to x$
  - $\forall x, \exists (x_k)$ with $x_k \to x$ such that $u(x) = \liminf_{k\to\infty} u_k(x_k)$

Valuations on Convex Functions

Theorem (Colesanti, L. & Mussnig 2017)

$\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0, \infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists$ *non-negative functions* $\zeta_0 \in C(\mathbb{R})$ *and* $\zeta_n \in D^{n-1}(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min\nolimits_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n\big(u(x)\big)\, dx$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

- $\mathrm{dom}\, u = \{x \in \mathbb{R}^n : u(x) < \infty\}$
- $D^{n-1}(\mathbb{R}) = \Big\{\zeta \in C(\mathbb{R}) : \zeta \text{ is decreasing}, \displaystyle\int_0^\infty t^{n-1}\zeta(t)\, dt < \infty\Big\}.$

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Theorem (Blaschke 1937)

$\mathrm{Z}: \mathcal{K}^n \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\Longleftrightarrow$$

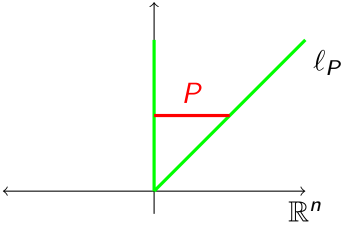
$\exists\, c_0, c_n \in \mathbb{R}$:

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Sketch of Proof

- $\text{Conv}_{\text{p.a.}}(\mathbb{R}^n) = \{u \in \text{Conv}(\mathbb{R}^n) : u \text{ piecewise affine, finite}\}$



$\ell_P \in \text{Conv}(\mathbb{R}^n)$

$P \in \mathcal{P}_0^n$

- $\text{Conv}_{\text{cone}}(\mathbb{R}^n) = \{u \in \text{Conv}(\mathbb{R}^n) : u \text{ translate of } s + \ell_P, P \in \mathcal{P}_0^n, s \in \mathbb{R}\}$

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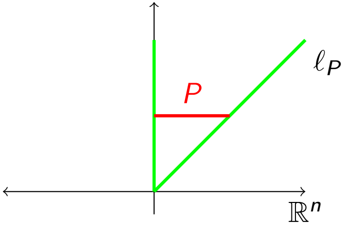

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Lemma

Let $\mathrm{Z}_1, \mathrm{Z}_2$ *be continuous valuations on* $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$.

$\mathrm{Z}_1 = \mathrm{Z}_2$ *on* $\mathrm{Conv}_{\mathrm{cone}}(\mathbb{R}^n) \Rightarrow \mathrm{Z}_1 = \mathrm{Z}_2$ *on* $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$

Sketch of Proof

- Define $\mathrm{Z}_s : \mathcal{P}^n \to [0, \infty)$ by $\mathrm{Z}_s(P) = \mathrm{Z}(l_P + s)$

  $\Longrightarrow \mathrm{Z}_s$ is a continuous, $\mathrm{SL}(n)$ and translation valuation

  $\overset{Blaschke}{\Longrightarrow} \exists\, \zeta_0, \zeta_n : \mathbb{R} \to [0, \infty) : \mathrm{Z}_s(P) = \zeta_0(s)\, V_0(P) + \zeta_n(s)\, V_n(P)$

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- $\zeta_0(s) = \psi_0(s)$ and $\zeta_n(s) = \dfrac{(-1)^n}{n!} \dfrac{d^n}{ds^n} \psi_n(s) \implies \cdots$

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- Lemma $\Longrightarrow$ Theorem □

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for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Question:

- Classification of rigid motion invariant valuations on $\mathrm{Conv}(\mathbb{R}^n)$: Hadwiger theorem on $\mathrm{Conv}(\mathbb{R}^n)$

Remark (Level-set based valuations)

Let $\omega \in C(\mathbb{R})$ have compact support. For $k \in \{0, \dots, n\}$, the functional

$$u \mapsto \int_{\mathbb{R}} \omega(t)\, V_k(\{u \leqslant t\})\, dt$$

is a continuous, rigid motion invariant valuation on $\text{Conv}(\mathbb{R}^n)$.

- Bobkov, Colesanti & Fragalà: Manuscripta 2014
- Cavallina & Colesanti: AGMS 2015
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$\mathrm{SL}(n)$ and translation invariant valuations:

- $u \mapsto \zeta_0(\min_{\mathbb{R}^n} u(x)) = \int_{\mathbb{R}} \omega(t)\, V_0(\{u \leqslant t\})\, dt$ with $\zeta_0(t) = v_n \int_t^\infty \omega(s)\, ds$

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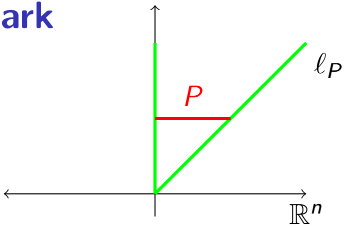

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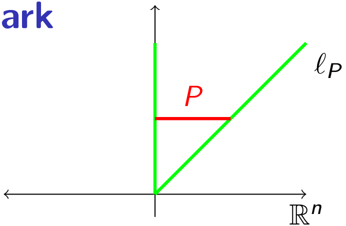

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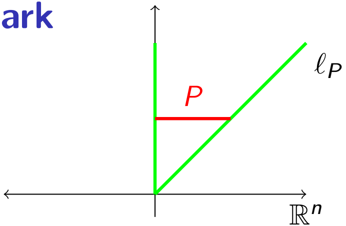

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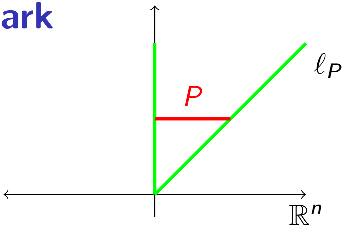

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Question:

- Classification of continuous, rotation invariant valuations on $\mathcal{K}_0^n$

Remark

Theorem (Alesker: Annals 1999)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rotation invariant, polynomial valuation*

$$\Longleftrightarrow$$

$\exists$ *polynomials* $p_0, \dots, p_{n-1}$ *in two real variables such that*

$$\mathrm{Z}(K) = \sum_{j=0}^{n-1} \int\limits_{\mathbb{R}^n \times S^{n-1}} p_j(x \cdot y, |x|^2)\, d\Phi_j(K; (x, y))$$

for every $K \in \mathcal{K}^n$.

- $\Phi_j(K, \cdot) : \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to [0, \infty)$
  jth generalized curvature measure of K
  (Federer: TAMS 1959)

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  (Federer: TAMS 1959)
- Polynomial valuations lie dense in smooth valuations.
  (Alesker: Israel J. 2006)

Hessian Valuations

Theorem (Colesanti, L. & Mussnig 2017+)

Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support w.r.t. 2nd and 3rd variable. For $i \in \{0, 1, \ldots, n\}$, the functional $\mathrm{Z}_{\zeta,i} \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^{2n}} \zeta(u(x), x, y)\, d\Theta_i(u, (x, y)),$$

is a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$. If $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x))\, [\mathrm{D}^2 u(x)]_{n-i}\, dx.$$

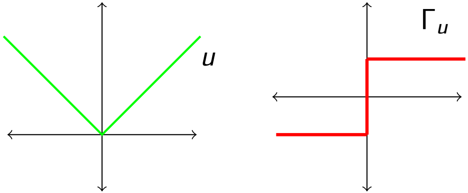
- $\Theta_i(u, \cdot)$ Hessian measure of u
- $\mathrm{D}^2 u$ Hessian matrix of u
- $[\mathrm{D}^2 u]_j$ jth elementary symmetric function of the eigenvalues of $\mathrm{D}^2 u$

Hessian Measures

- On smooth functions: $[\mathrm{D}^2 u(x)]_j \, dx$
  Caffarelli, Nirenberg, Spruck: Acta 1985;
  Trudinger, Wang: Annals 1999; ...
- On Monge-Ampère functions (currents)
  Fu: IUMJ 1989
- On (semi-)convex, finite functions:
  Colesanti, Hug: TAMS 2000
- Extension to $\mathrm{Conv}(\mathbb{R}^n)$ using Lipschitz regularization:
  Colesanti, L. & Mussnig 2017+

Hessian Measures

$u \in \mathrm{Conv}(\mathbb{R}^n), \eta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n), s \geqslant 0$



Graph of the subdifferential map of u:

$$\Gamma_u = \{(x,y) \colon x \in \mathrm{dom}(u),\ y \in \partial u(x)\}.$$

Parallel set:

$$P_s(u,\eta) = \{x + sy \colon (x,y) \in \eta \cap \Gamma_u\}$$

Steiner formula:

$$\mathcal{H}^n(P_s(u,\eta)) = \sum_{i=0}^{n} \binom{n}{i} s^i\, \Theta_{n-i}(u,\eta)$$

Hessian Measures

- Smooth functions: $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, $\beta \in \mathcal{B}(\mathbb{R}^n)$,

$$\Gamma_u = \{(x, \nabla u(x)) \colon x \in \mathbb{R}^n\}$$

and

$$P_s(u, \beta \times \mathbb{R}^n) = \{x + s\, \nabla u(x) \colon x \in \beta \times \mathbb{R}^n\}$$

- Volume of parallel set

$$\mathcal{H}^n(P_s(u, \beta \times \mathbb{R}^n)) = \int_{P_s(u,\beta\times\mathbb{R}^n)} dz = \int_\beta \det(I_n + s\, \mathrm{D}^2 u(x))\, dx$$

by the change of variables $z = x + s\, \nabla u(x)$

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- $\det(I_n + s\,\mathrm{D}^2 u(x)) = \sum_{j=0}^{n} \binom{n}{j} s^j\,[\mathrm{D}^2 u(x)]_j.$
- $\Theta_i(u, \beta \times \mathbb{R}^n) = \int_\beta [\mathrm{D}^2 u(x)]_{n-i}\,dx$

Hessian Valuations

Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support and $i \in \{0, 1, \dots, n\}$.

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\Gamma_u} \zeta(u(x), x, y)\, d\Theta_i(u, (x, y))$$

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- $\mathrm{Z}_{\zeta,i}$ is a valuation on $\mathrm{Conv}(\mathbb{R}^n)$.
- $\mathrm{Z}_{\zeta,i}$ is continuous on $\mathrm{Conv}(\mathbb{R}^n)$.

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Some special cases:

- $\mathrm{Z}_{\zeta,n}(u) = \int_{\mathrm{dom}(u)} \zeta(u(x), x, \nabla u(x))\, dx$

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- $\mathrm{Z}_{\zeta,n}(u) = \displaystyle\int_{\mathrm{dom}(u)} \zeta(u(x), x, \nabla u(x)) \, dx$
- u^* Legendre transform of u and $u^* \in \mathrm{Conv}(\mathbb{R}^n)$:
  $\mathrm{Z}_{\zeta,i}(u^*) = \displaystyle\int_{\mathbb{R}^{2n}} \zeta(\langle x, y \rangle - u(x), y, x) \, d\Theta_{n-i}(u, (y, x))$

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- $\mathrm{Z}_{\zeta,0}(u) = \displaystyle\int_{\mathrm{dom}(u^*)} \zeta(\langle \nabla u^*(y), y\rangle - u^*(y), \nabla u^*(y), y)\, dy$

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Theorem (Colesanti, L. & Mussnig 2017+)

Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n)$ *have compact support. For every* $i \in \{0, \ldots, n\}$, *the functional* $\mathrm{Z}_{\zeta,i} \colon \operatorname{Conv}(\mathbb{R}^n) \to \mathbb{R}$, *defined by*

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^{2n}} \zeta(u(x), y)\, d\Theta_i(u, (x,y)),$$

is a continuous, **translation invariant** *valuation. If* $u \in \operatorname{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, *then*

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n} \zeta(u(x), \nabla u(x))\, [\mathrm{D}^2 u(x)]_{n-i}\, dx.$$

Hessian Valuations

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- Alesker 2017: Valuations defined by Monge-Ampère measures
$$u \mapsto \int_{\mathbb{R}^n} \xi(x) \det(\mathrm{D}^2 u(x), \ldots, \mathrm{D}^2 u(x), A_{k+1}(x), \ldots, A_n(x)) \, dx$$

Hessian Valuations

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- In general, not level-set based: only for $i = 0$ and $i = n$.

Hessian Valuations

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is a continuous, rigid motion invariant valuation.

Theorem (Colesanti, L. & Mussnig 2017)

$\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0, \infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\iff$$

$\exists$ *non-negative functions* $\zeta_0 \in C(\mathbb{R})$ *and* $\zeta_n \in D^{n-1}(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x \in \mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n\big(u(x)\big) \, dx$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Questions

- Classification of continuous, rotation invariant valuations on $\mathcal{K}_0^n$ and on $\mathcal{K}^n$
- Do polynomial valuations lie dense in all continuous, rotation invariant valuations on $\mathcal{K}^n$?

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- Classification of continuous, rotation invariant valuations on $\mathcal{K}_0^n$ and on $\mathcal{K}^n$
- Do polynomial valuations lie dense in all continuous, rotation invariant valuations on $\mathcal{K}^n$?
- Classification of continuous, rigid motion invariant valuations on $\text{Conv}(\mathbb{R}^n)$
- Do Hessian valuations lie dense in all continuous, rigid motion invariant valuations on $\text{Conv}(\mathbb{R}^n)$?

Thank you!