Day 4 Talk 5

Van Handel

"the dimension-free structure of non-homogeneous random matrices."

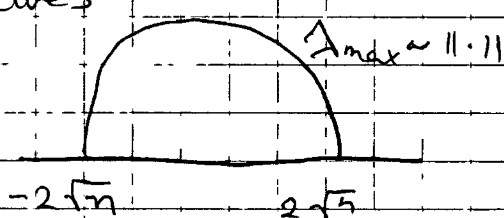
Simple setting: $\cdot\ X$ $n \times n$ symmetric

$\cdot\ X_{ij} \sim N(0, b_{ij}^2)$ indep.

Q. What can we say about spectrum?

Classical Wigner-matrix $b_{ij} = 1 \ \forall i,j$

distribution of eigenvalues $\longrightarrow$



- What if b_{ij} very different?
- maximum / minimum

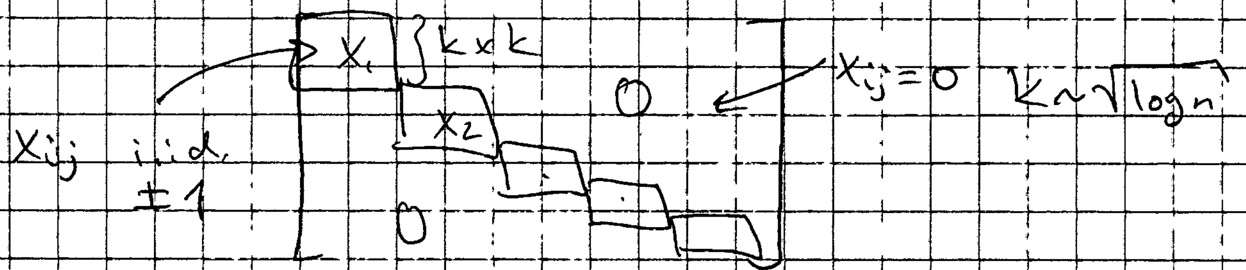
E.g. When does X $(\infty \times \infty)$ random matrix define a bdd operator ℓ_2?
(if $b_{ij} = 1$ oper norm $\to \infty$)

Conjecture (Latała ~ 15 years ago)

$$\mathbb{E}\|X\| \asymp \mathbb{E}\max_i \sqrt{\sum_j x_{ij}^2}$$

- Would answer previous problems.
- What if x_{ij} not Gaussian

- Can not be true for non-Gaussian random matrices: Counter-example (Seginer)



Now:

- $\mathbb{E}\max_i \sqrt{\sum_j X_{ij}^2} = \sqrt{k} = (\log n)^{\frac{1}{4}}$

- $\mathbb{E}\|X\| = \mathbb{E}\max_i \underbrace{\|X_i\|}_{\text{Wigner}}$

  By universality X_i behave like Gaussians.

  $\sim \sqrt{\log n}$

Q Is it true for Gaussian? Why?

Thm (L v H.) Yes

$$\mathbb{E}\|X\| \asymp \mathbb{E}\max_i \sqrt{\sum_j x_{ij}^2} \asymp \max_i \sqrt{\sum_j b_{ij}^2} + \max_{ij} b_{ij}^* \sqrt{\log i}$$

decreasing rearrangement of $\max_j b_{ij}$

Similar methods give

In fact can do more

$$\mathbb{E}\,\|X\|_{S_p} \overset{\star}{\asymp} \mathbb{E}\Big[\Big(\sum_i\Big(\sum_j X_{ij}^2\Big)^{p/2}\Big)^{\frac{1}{p}} \asymp \text{explicit expression}$$

$\star$ constant indep of p

$\|X\|_{S_p}$: ℓ_p-norm of eigenvalues

In fact

$$\mathbb{E}\,\Phi(\|X\|_{\ell_p(\ell_2)}) \leq \mathbb{E}\,\Phi(\|X\|_{S_p})$$
$$\leq \mathbb{E}\,\Phi(C\,\|X\|_{\ell_p(\ell_2)})$$
$\forall\ 2 \leq p \leq \infty \quad \forall$ incr. convex Φ

Rk.

Focus of this talk: 1st estimate.

Some history:

- "Wigner bound" $\mathbb{E}\|X\| \lesssim \max_{ij} b_{ij}\sqrt{n}$ ← increasing variance to be maximal
- Non-commutative Khintchine (Lust, Piquard, Pisier)
  "Matrix concentration"
  (also called Matrix chernoff)
  could get $\rightsquigarrow$ $\mathbb{E}\|X\| \lesssim \sqrt{\log n}\,\max_i \sqrt{\sum_j b_{ij}^2}$
- Latala: $\mathbb{E}\|X\| \lesssim \max_i \sqrt{\sum_j b_{ij}^2} + \|(b_{ij})\|_4$

$\mathbb{P}$

Thm. 1 (Bandeira-Rvtt)

$$\mathbb{E}\|X\| \lesssim \max_i \sqrt{\sum_j b_{ij}^2} + \max_{ij} b_{ij} \sqrt{\log n}$$

Almost tight $\longrightarrow$)(

$$\left(\mathbb{E}\|X\|^{\log n}\right)^{\frac{1}{\log n}}$$

Proof of lower bound:

$$\left[\mathbb{E}\|X\|^{\log n}\right]^{\frac{1}{\log n}} \geq \left[\mathbb{E}\|X\|^2\right]^{1/2}$$

$$\geq \|\mathbb{E} X^2\|^{1/2} = \max_i \sqrt{\sum_j b_{ij}^2}$$

$$\left[\mathbb{E}\|X\|^{\log n}\right]^{\frac{1}{\log n}} \geq \left[\mathbb{E}|X_{ij}|^{\log n}\right]^{\frac{1}{\log n}} \simeq \sqrt{\log n}\, b_{ij}.$$

Moment method:

- $\|X\| = \max_i |\lambda_i|$
- $\mathrm{Tr}\left[X^{2p}\right]^{1/2p} = \left[\sum_i |\lambda_i|^{2p}\right]^{\frac{1}{2p}}$
- $\|X\| \leq \mathrm{Tr}\left[X^{2p}\right]^{1/2p} \leq n^{\frac{1}{2p}} \|X\|$, $n^{\frac{1}{2p}} \simeq 1 \Rightarrow p \sim \log n$

$$\left[\mathbb{E}\|X\|^p\right]^{1/p} \simeq \left(\mathbb{E}\,\mathrm{Tr}[X^{2p}]\right)^{1/2p} \qquad p \sim \lg n$$

- Compare to a much smaller Wigner matrix

- switch to a combinatorial quantity

- Must use $p \sim \log n$ - moment

Discard this approach $\longrightarrow$ Try something different:

$$\|X\| = \sup_{v \in S^{n-1}} |\underbrace{\langle v, Xv \rangle}|$$

- Gaussian variable on the sphere

- Many tools: Majorizing measures?

Thm 2 (RvH '17)

$$\mathbb{E}\|X\| \lesssim \max_i \sqrt{\sum_j b_{ij}^2} + \max_{ij} b_{ij}{}^* \underbrace{\log i}$$

Can't get beyond $\sqrt{\log i}$ to with this method

Now:

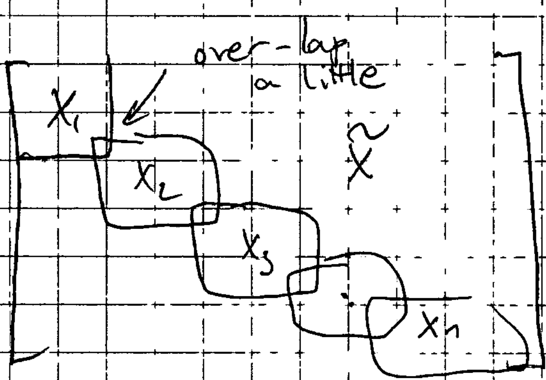
<u>Combine</u> the approaches of the two prior sub-optimal results.

<u>Give</u> names

$$a = \max_i \sqrt{\sum_j b_{ij}^2} \qquad b = \max_{ij} b_{ij}^* \sqrt{\log i}$$

To show: $\mathbb{E}\|X\| \lesssim a + b$

<u>Lemma</u> every matrix looks like this (up to rearr rows/columns)



$n \times n$ (n may be ∞)

$$X' = \bigcup X_i$$

* $\dim(X_i) \sim 2^{2^i}$
* $\max_j b_{ij} 1_{ij \in X'} \lesssim \frac{b}{\sqrt{\log(i+1)}}$ technically...
* $\max_j b_{ij} 1_{i,j \in \tilde{X}} \lesssim \frac{a}{\sqrt{i}}$

"nearly block-diagonal matrix"

Sketch of pf of Thm (using lemma)

In the core X':

by Thm 1 (dimension dep bound)

$$\mathbb{E}\|X_i\| \lesssim \underbrace{\max_i \sqrt{\textstyle\sum_j b_{ij}^2}}_{a} + \cancel{\sqrt{\log(2^{2^i})}} \cdot \frac{b}{\cancel{\sqrt{\log 2^{2^{i-1}}}}}$$

Bound that I want!

Combinatorial methods

$$\mathbb{E}\|X'\| \simeq \mathbb{E} \max_i \|X_i\|$$

$$\lesssim \max_i \mathbb{E}\|X_i\| + \sum_i \underbrace{\mathrm{Var}(\|X_i\|)^{1/2}}_{\lesssim \frac{b}{\sqrt{\log 2^{2^{i-1}}}}} .$$

by Gaussian concentration

+

$$\lesssim a + b \quad + b \quad \lesssim a + b$$

Outside the core: $\tilde{X}$ use Thm 2 (dimension free bound)

Gaussian process methods

$$\mathbb{E}\|\tilde{X}\| \lesssim \max_i \sqrt{\textstyle\sum_j b_{ij}^2} + \max_{ij} \frac{a}{\sqrt{i}} \log i$$

$$\lesssim a$$

∎

Need both methods