Day 5 Talk 1

Alexander Litvak

"The smallest singular value of a d-regular random square matrix"

Motivation

$M = \{\mu_{ij}\}_{n\times n}$ adjacency graph of random d-regular directed graph with n-vertices (uniform probability)

$3 \leq d \leq n/2$

Conjecture Vu '08, Frieze, Vu ICM-talk '14:

M non-singular with high probability

$\mathbb{P} \xrightarrow[n\to\infty]{} 1$.

Rk if $d=1$ M is a permutation matrix

$\rightsquigarrow$ invertible

$d=2$ conjecture fails

$d=d_0$, $d=n-d_0$ are essentially the same

Thm Cook '14 (arxiv date)

Conjecture **true** for $d \gg \ln^2 n$ with probability $1 - \frac{1}{d^c}$

Rk $d \gg \ln^2 n$ means $d/\ln^2 n \to \infty$ as $n \to \infty$

Thm (LLTTP, '15)

Conjecture $C \leq d \leq n/\ln^2 n$ with ~~probabi~~ probability $1 - C \ln^3 d/\sqrt{d}$

Q: Quantitative estimates for smallest singular value $s_n(M)$?

- Thm above says $s_n(M) \neq 0$ w.h.probability
- Bound $s_n(M)$ from below?

Some bounds for $s_n(M)$ by BR '15, for Erdös-Rényi

Theorem Cook '16

$$\mathbb{P}\left(s_n(M) \geq n^{-C_0 \ln n / \ln d}\right) \geq 1 - \ln^c n/\sqrt{d}$$

Thm LLTTY '17

$C < d < n/\log^2 n$. Then

$$\mathbb{P}\left(s_n(M) \geq 1/n^6\right) \geq 1 - \ln^2 d/\sqrt{d}$$

Proof ideas: – Remove "bad events" with small probability

(switch to blackboard)

$s_n(M) > \lambda_0 \iff \forall x \in S^{n-1} \quad \|Mx\| \geq \lambda_0$

⇝ splitting sphere

Rk: Goal estimate Mx $\forall x$, treat different x differently

Main idea: splitting sphere → goes back Kashin + Komlos

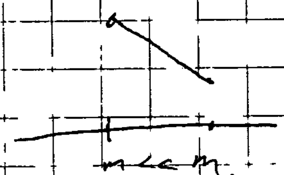
↳ Schechtman, L.P.R.T. → R.V.

split sphere to
- sparse + close
- other vectors

Introduce
- Almost constant vectors (flat)
- Steep vectors
- sloping vectors

Intuitive:

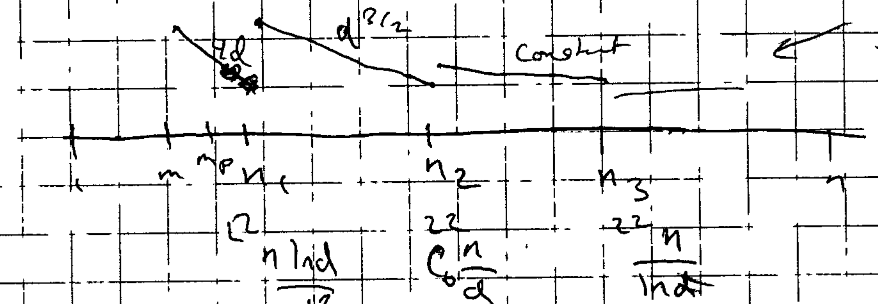
Steep: $x \in S^{n-1}$ $\{x_i^*\}$:



← big changes

↑ decreasing rearrangement of co-ordinates

Sloping: → others



← jumps $\quad \rho = \sqrt{\frac{\alpha}{\ln d}}$

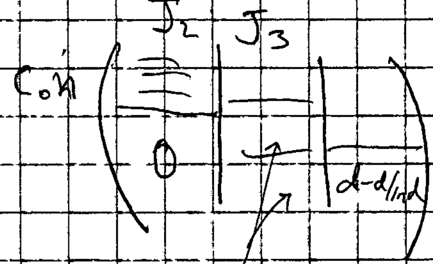
Steep :

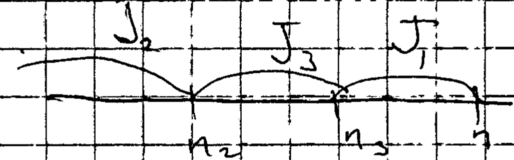
$B(\rho)$ = almost const $= \{x \in S^{n-1} \mid \exists \lambda \text{ s.t. } |\{i : |x_i - \lambda| < \rho\}| > n - n_3\}$

Two steps in proof

① Almost constant $B(p)$ → steep / → sloping

② other vectors

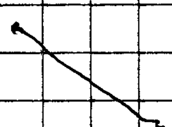


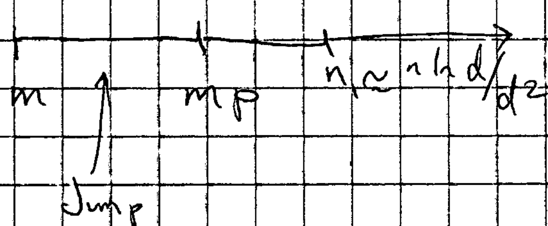


# of rows with many 1's not large

$M_x \to \langle R_i, x \rangle$ — big

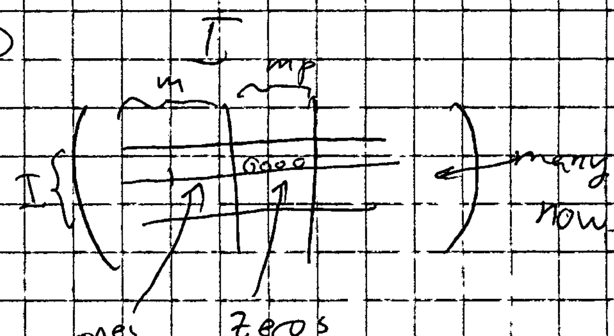
⇝ gives bound for case ①.





in most matrices can find

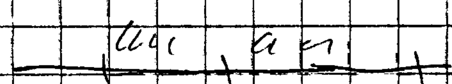
⇝



$$(R_i, x) = x_{j_0} + \sum_{k=1}^{d-1} x_{jk}$$

⇝ many rows where (R_i, x) separated from zero

For

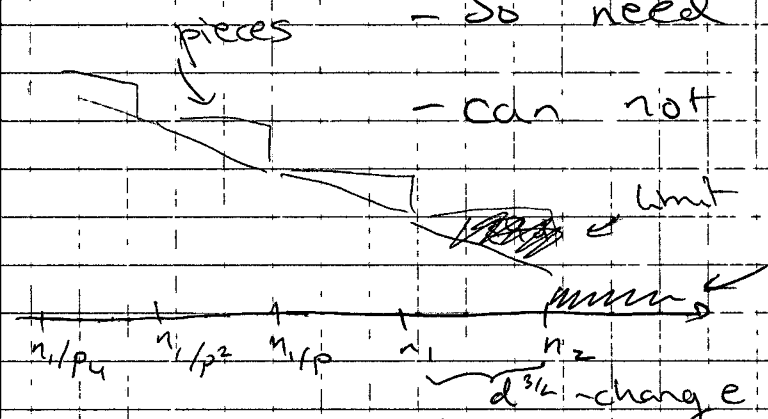


Apply union bound

$\|Mx\|_2, x_0$

Need bound for all x, but infinite.

- So need a good net, union bound. ← Not enough
- can not have too large a net



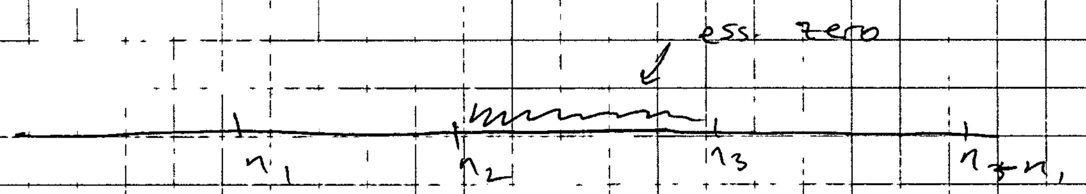
Normalize $x^*_{n_1} = 1$

Construct net piece by piece

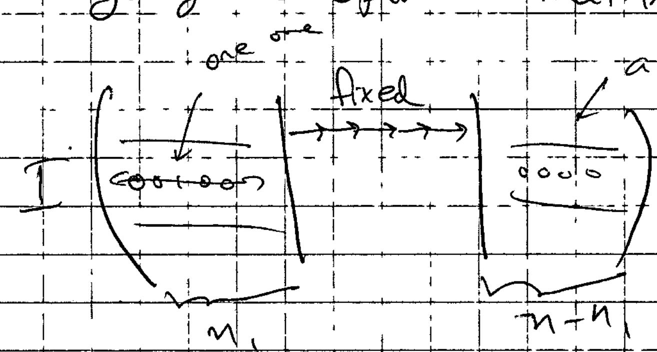
$\exists$ good net: $d^{3/2}$-net in ℓ_∞-norm

$$|\mathcal{N}| \le e^{10 n_1 \ln d} \le e^{n_2 d/4}$$

↑ net



⇝ going to split matrix



if $\|Mx\|$ small

Estimate $\langle R_i, x\rangle > \frac{1}{4}$ ⇝ Estimate probability that many $\langle R_i, x\rangle < 1/4$

$\mathbb{P}(\cap \Omega_i) \longrightarrow e^{-n_2 d/2}$ ↙ kills 10k with ε-net size.

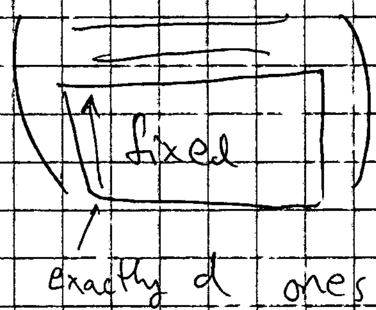
What if vectors x is not almost constant

~~Just~~ Just main idea!

x - not almost constant case.

bound for

$s_n(M) \longleftrightarrow \langle R_1, x \rangle \geq \frac{1}{4}$



$\binom{2d}{d}$

A, B - large

$|x_i - x_j| > \rho/16$

$i \in A$, $j \in B$

The smallest singular value of a d-regular random square matrix

Alexander Litvak

University of Alberta

based on a joint work with

A. Lytova, K. Tikhomirov, N. Tomczak-Jaegermann, and P. Youssef

MSRI, Berkeley, 2017

Motivation

We consider adjacency matrices of random d-regular directed graphs (digraphs) on n vertices, which can be also seen as bipartive graphs (bigraphs).

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Remark 2. If $d = 2$ the conjecture fails (see e.g., Vu, Cook).

Remark 3. The cases $d = d_0$ and $d = n - d_0$ are essentially the same (by interchanging zeros and ones).

Results

Theorem (Cook, 2014)

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Theorem (BR, 2015)

In Erdos–Renyi model with $p_n = d/n$ one has

$$\mathbb{P}\left(s_n(M) \geq \varepsilon c_{n,d}\,\frac{\sqrt{d}}{n}\right) \geq 1 - \varepsilon - e^{-d},$$

where $c_{n,d} = \exp\left(-\frac{c\ln(n/d)}{\ln d}\right)$.

Theorem (Cook, 2016)

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(Of course, **2** is a partial case of **3** with $k = 2$.)