Day 5 Talk 4

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"Uniform cover inequalities for the volume of coordinate sections and projections of convex bodies"

<u>Ref:</u> Arxiv paper by the same name.

<u>Background</u>

Classical inequalities

<u>Loomis-Whitney inequality (1949)</u>

$$|K|^{n-1} \le \prod_{i=1}^{n} |P_i K|$$

Fix $\{e_i\}$ orthonormal basis. $P_i K = P_{e_i^\perp} K$

projection

hyperplane orthogonal to e_i

"Dual" inequality

<u>Meyer (1988)</u>

$$\frac{n!}{n^n} \prod_{i=1}^{n} |K \cap e_i^\perp| \le |K|^{n-1}$$

If John's condition

if $I_n = \sum_{i=1}^{n} c_j \, u_j \otimes u_j \qquad u_j \in S^{n-1} \qquad c_j > 0$

identity

$$\frac{n!}{n^n} \prod_{j=1}^{m} |K \cap u_j^\perp|^{c_j} \leq |K|^{n-1} \leq \prod_{j=1}^{m} |P_{u_j^\perp} K|^{c_j}$$

~~Thm~~ Thm Bollobas-Thomason (95)

$\sigma \subseteq [n] = \{1 \ldots n\}, \qquad |\sigma| = d$

Def If $\sigma_1 \ldots \sigma_r \subseteq \sigma$, we say that $(\sigma_1 \ldots \sigma_r)$ is an s-uniform cover of σ, if each $j \in \sigma$ is contained in exactly s of σ_i.

Def $F_\sigma = \text{span}\{e_i \mid e_i \in \sigma\}$

$E_\sigma = F_\sigma^\perp$

If $(\sigma_1 \ldots \sigma_r)$ is an s-uniform cover of $[n]$, then

$$|K|^s \leq \prod_{i=1}^{r} |P_{F_{\sigma_i}}(K)|$$

$$|K|^{r-s} \leq \prod_{i=1}^{r} |P_{E_{\sigma_i}}(K)|$$

Proved by Hölder, like Loomis-Whitney

Questions

Q : (V. Milman)

Def $V_k(K) = V(\underbrace{K \ldots K}_{k\text{-times}}, \overbrace{B_2^n \ldots B_2^n}^{n-k\text{-times}})$

mixed volume, see talk by (Zvavitch or arxiv paper)

Is it true that

$$\boxed{\frac{V_k(K+L)}{V_{k-1}(K+L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)}} \ ? \quad k \geq 2$$

Why? Analogues true in other contexts.

If true, it implies $V_k(K+L)^{1/k} \geq V_k(K)^{1/k} + V_k(L)^{1/k}$

G - Hortzoulakis - Paouris For $L = [-\xi, \xi]$, $\xi \in S^{n-1}$ unit vector

it would give

$$(*) \quad \frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leq \frac{S(K)}{|K|}.$$

Proved inequality $(*)$ with different constant $\neq 1$

Fact

$$\frac{S(P_\xi(K))}{|P_\xi(K)|} \leq \frac{2(n-1)}{n} \frac{|S(K)|}{|K|}$$

- From a "local Loomis-Whitney inequality"

$$\frac{2(n-1)}{n} |P_i K| |P_j K| \geq |P_{ij} K| \cdot |K|,$$

where P_{ij} is the projection onto $\operatorname{span}\{e_i, e_j\}^\perp$

In fact $\frac{2(n-1)}{n}$ is optimal constant, $\exists$

Counter examples for constant $= 1$.

Some results from G-Fradelizi-Meyer

If $0 \leq p \leq k \leq n$ and $F \in G_{n,k}$ (grassmannian), then

$$\frac{V_{k-p}(P_F K)}{|P_F K|} \leq \binom{n-k+p}{n-k} \frac{V_{n-p}(K)}{|K|}$$

- Vitali's question has an affirmative answer iff $\ell = 1, k = 2$

Discussions with Alex Koldobsky:

proved that:

$$S(K) = c_n \int_{S^{n-1}} |P_{\xi^\perp} K| \, d\sigma(\xi) \quad c_n \sim \sqrt{n}$$

- if Z is a zonoid, then

Zonoid = limit of zonotope

$\underbrace{\text{zonotope}}_{}$ = Minkowski-sum of intervals

$$|Z|^{1/n} \min_{\xi} S(P_{\xi^\perp} K) \leq b_n S(Z)$$

$b_n \sim 1$, sharp for ball

- If Z is a zonoid, which is also isotropic, then

$$|Z|^{1/n} \max_{\xi} S(P_{\xi^\perp} K) \geq \frac{c}{(\log n)^2} S(Z)$$

Question: To estimate the best constant $\gamma_{n,k} > 0$:

$$\forall K \quad |K|^{\frac{n-k}{n}} \min_{E \in G_{n,k}} S(P_E(K)) \leq \gamma_{n,k} S(K)$$

G J.W. Koldobsky - Valettas:

from before

$$|K| \, S(P_{\xi^\perp} K) \leq 2 \frac{n-1}{n} S(K) \, |P_{\xi^\perp} K|$$

$$|K| \min_{\xi} S(P_{\xi^\perp} K) \leq 2 \frac{n-1}{n} S(K) \min_{\xi} |P_{\xi^\perp} K|$$

$$r(\Pi K) \leq \left(\frac{|\widetilde{\Pi K}|}{\omega_n} \right)^{1/n} \leq w(\Pi K) = \int_{S^{n-1}}$$

$\longrightarrow$

$$= \int_{S^{n-1}} |P_{\xi^\perp} \tilde{K}| \, d\sigma = \frac{1}{c_n} S(\tilde{K})$$

ΠK - projection body of K

$$= \frac{1}{c_n} \partial_K |K|^{\frac{n-1}{n}}$$

$$\partial_K = \min\left\{ \frac{S(TK)}{|TK|^{\frac{n-1}{n}}} : T \in SL(n) \right\}$$

$$\Rightarrow |K|^{1/n} \min_{\xi} S(P_\xi K) \leq^{\frac{2(n-1)}{n}} \frac{\partial_K}{c_n} \sim \frac{\partial_K}{\sqrt{n}}$$

$$\underbrace{\sqrt{n}}_{\text{ball}} \lesssim \partial_K \lesssim n$$

cube/simplex

<u>Sections</u>

$$as(K) = \int_{S^{n-1}} |K \cap \xi^\perp| \, d\sigma(\xi)$$

$$as_r(K) = \int_{G_{n,n-r}} |K \cap E| \, d\nu_{n,n-r}(E)$$

<u>Koldobsky</u>: If K is an <u>intersection body</u>, then

$$as(K) \leq b_n |K|^{1/2} \max_{\xi} as(K \cap \xi^\perp)$$

($b_n \sim 1$, sharp for ball)

Proof using Fourier analysis ~~methods~~ methods.

Question: To estimate the best $\gamma_{n,k} \geq 0$:

$$\forall K, \forall k$$

$$as(K) \leq \gamma_{n,k}^{k} |K|^{\frac{k}{n}} \max_{E \in G_{n,n-k}} as(K \cap E)$$

G - ~~K~~ - Dann - ~~Breizikas~~ Breizikkos

- $\gamma_{n,k} \leq CL_K$
- $\gamma_{n,k} \leq C\sqrt{\frac{n}{k}} \left(\log \frac{en}{k}\right)^{3/2}$
- $\gamma_{n,k} \lesssim \gamma_{n,1} \simeq L_n = \max_{K \subset \mathbb{R}^n} L_K$

We ask for

$$|K \cap e_i^{\perp}| \, |K \cap e_j^{\perp}| \leq C |K \cap \langle e_i, e_j \rangle^{\perp}| \, |K|$$

true, discovered in the course of these studies.

This talk main results:

Theorem 1 If $\sigma \subseteq [n]$ and $(\sigma_1, \dots, \sigma_r)$ is an s-uniform cover of σ, then

$$\prod_{i=1}^{r} |P_{E_{\sigma_i}}(K)| \geq \gamma(n,d,r,s)\, |P_{E_\sigma}(K)|^{s} |K|^{r-s}$$

and $\gamma(n,d,r,s) = \binom{n}{d}^{r-s} \binom{n - \frac{ds}{r}}{n-d}^{-r}$

probably not optimal.

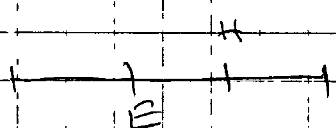
Theorem 2 With the same assumptions, and

K centered

$$\prod_{i=1}^{r} |K \cap E_{\sigma_i}| \leq \underbrace{\frac{(cd)^{ds}}{d_1^{d_1} \cdots d_r^{d_r}}}_{\leq \left(\frac{er}{s}\right)^{ds}} |K \cap E_\sigma|^{s} |K|^{r-s}$$

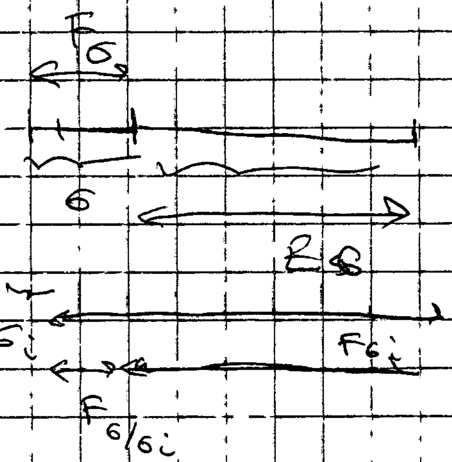
It can be improved

Paper by: Alonso-Gutierrez, Arstein-Avidan, González, ..., Vilà



$$|P_{E\cap H} K|\,|K| \leq \text{opt\_const}\, |P_E K|\,|P_H K|$$

Proof ideas for Thm 1



For every $y \in P_{E_\sigma}(K)$ we define

$$K(y) = \{t \in F_\sigma : y + t \in K\}$$

- $$|K| = \int_{P_{E_\sigma}(K)} |K(y)|\,dy$$

- For every $i \le r$ define

$$K_i(y) = \{t \in F_{\sigma/\sigma_i} : y + t \in E_{\sigma_i}\}$$

- $$|P_{E_{\sigma_i}}(K)| = \int_{P_{E_\sigma}(K)} |K_i(y)|$$

- $\forall i \;\; K_i(y) = P_{F_{\sigma/\sigma_i}}(K(y))$ and $(\sigma/\sigma_1, \dots, \sigma/\sigma_r)$ is an r-s-uniform cover of σ

- By the BT-inequality

$$|K(y)|^{r-s} \le \prod_{i=1}^{r} |K_i(y)|$$

$$\prod_{i=1}^{r} |P_{E_{6_i}}(K)| = \prod_{i=1}^{r} \int_{P_{E_6}(K)} |K_i(y)|$$

$$\geq \left[\int_{P_{E_6}} \left(|K_1(y)| \dots |K_r(y)| \right)^{1/r} dy \right]^r$$

$$\geq \left(\int_{P_{E_6}} |K(y)|^{\frac{r-s}{r}} \right)^r \geq \dots |P_{E_6}(K)|^{\dots} \Big(\underbrace{\int_{P_{E_6}(K)} |K(y)|}_{|K|} \Big)^{\dots}$$

Useful inequalities:

Berwald: $A \subseteq \mathbb{R}^n$ convex, $\phi: A \to \mathbb{R}^+$ concave, then

$$q \longmapsto \left[\binom{m+q}{m} \frac{1}{|A|} \int_A |\phi(y)|^q dy \right]^{1/q} \text{ is}$$

decreasing on $(0, \infty)$. By BM

$$|\phi(y)| = |K(y)|^{1/d} \text{ is concave.}$$

Berwald 2: If $f_1 \dots f_r : A \to \mathbb{R}^+$ (A convex) concave, nonzero and ~~α_i~~ $\alpha_1 \dots \alpha_r > 0$

$$\frac{1}{|A|} \int_A \prod_{i=1}^{r} f_i^{\alpha_i}(x)\, dx \leq \frac{\binom{\alpha_1 + m}{m} \dots \binom{\alpha_r + m}{m}}{\binom{\alpha_1 + \dots + \alpha_r + m}{m}} \prod_{i=1}^{r} \frac{1}{|A|} \int_A f_i^{\alpha_i} dx$$

K convex, centered of volume 1

$$|K \cap E_{6_i}|^{\frac{1}{d_i}} \sim \frac{1}{|P_{F_{6_i}}(Z_{d_i}(K))|^{1/d_i}} \text{ — Paouris}$$

$$|P_{E_{6_i}}(Z_d(K))|^{1/d_i}$$

$$h_{Z_d}(\xi) = \left(\int_K |\langle x, \xi \rangle|^d \, dx \right)^{1/d}$$

If $p < q$ then $Z_q(K) \subseteq c \frac{q}{p} Z_p(K)$