Abstract:

We introduce a new model of a realization space of a polytope that arises as the positive part of a real variety. The variety is determined by the slack ideal of the polytope, a saturated determinantal ideal of a sparse generic matrix that encodes the combinatorics of the polytope. The slack ideal offers a uniform computational framework for several classical questions about polytopes such as rational realizability, projectively uniqueness, non-prescribability of faces, and realizability of combinatorial polytopes. The simplest slack ideals are toric. We identify the toric ideals that arise from projectively unique polytopes. New and classical examples illuminate the relationships between projective uniqueness and toric slack ideals.

The Slack Realization Space of a Polytope

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joint work with:

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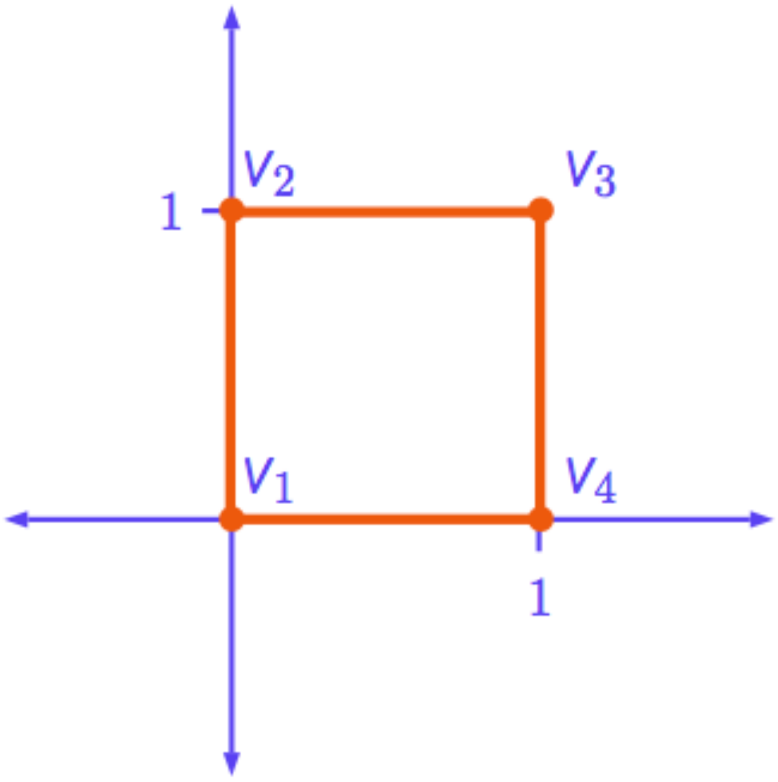
Antonio Macchia (U Bari)
Amy Wiebe (U Washington)

$P \subset \mathbf{R}^d \quad d$-polytope



combinatorics of P:

vertices: v_1, v_2, v_3, v_4

facets: $\{v_1, v_2\}, \{v_2, v_3\},$
$\{v_3, v_4\}, \{v_1, v_4\}$

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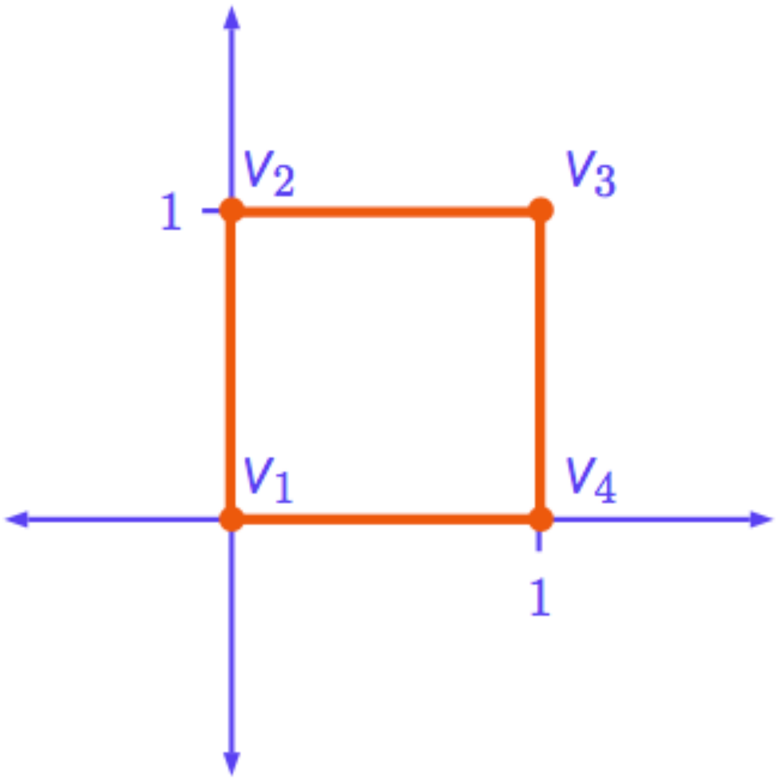

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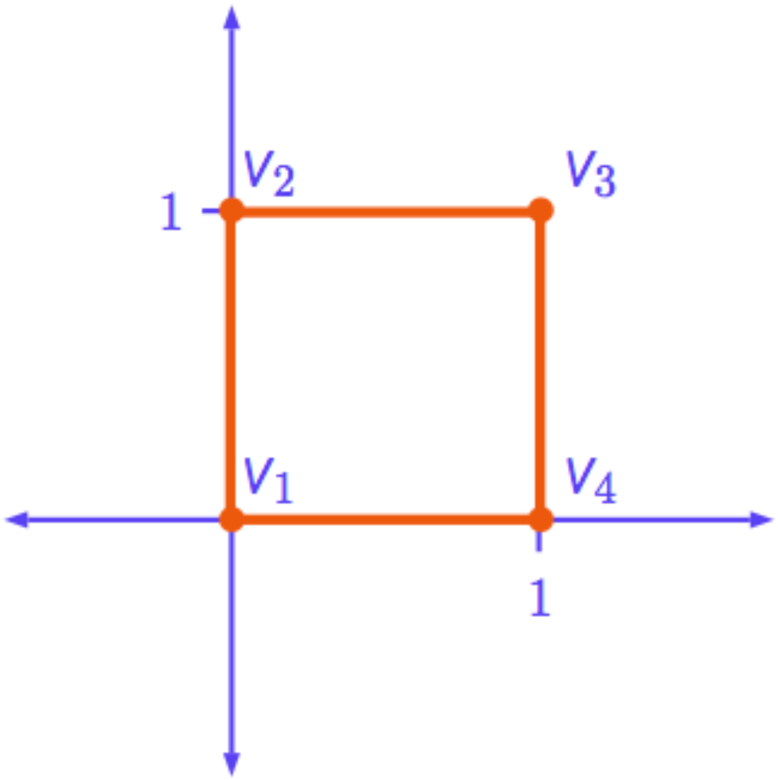
combinatorial equivalence:

$$Q \stackrel{c}{=} P \quad \leftrightarrow \quad P \,\&\, Q \text{ have the same vertex-facet incidences}$$

$P \subset \mathbf{R}^d$ $\quad d$-polytope



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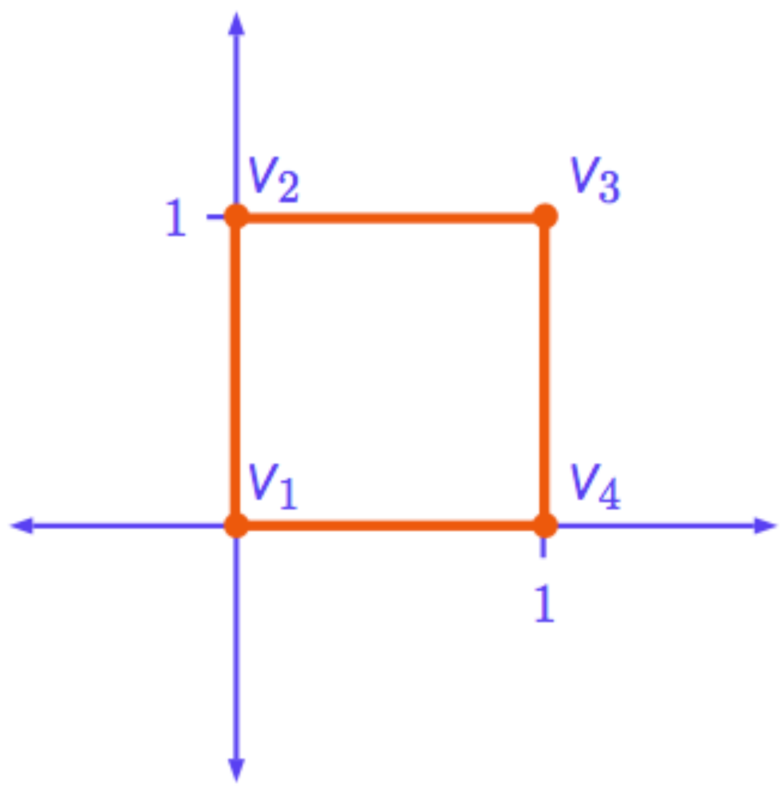
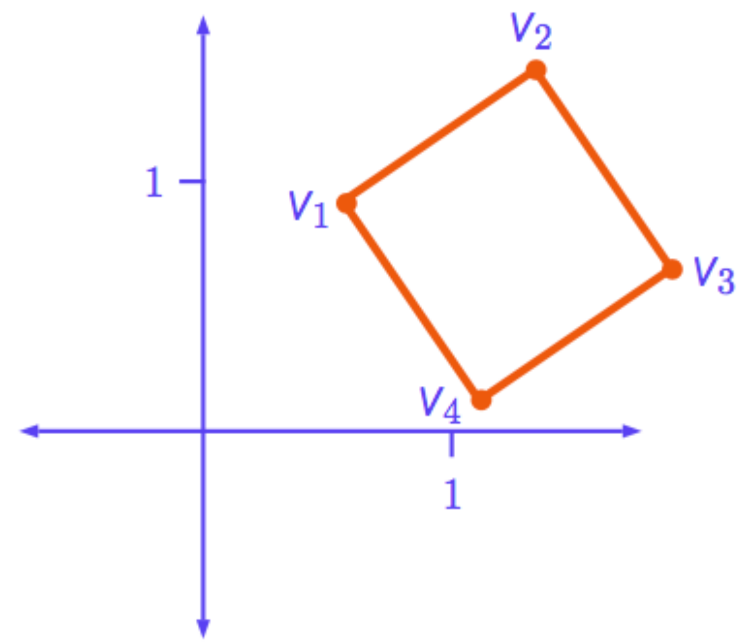
All quadrilaterals are combinatorially equivalent to a square

$P \subset \mathbf{R}^d \quad d$-polytope

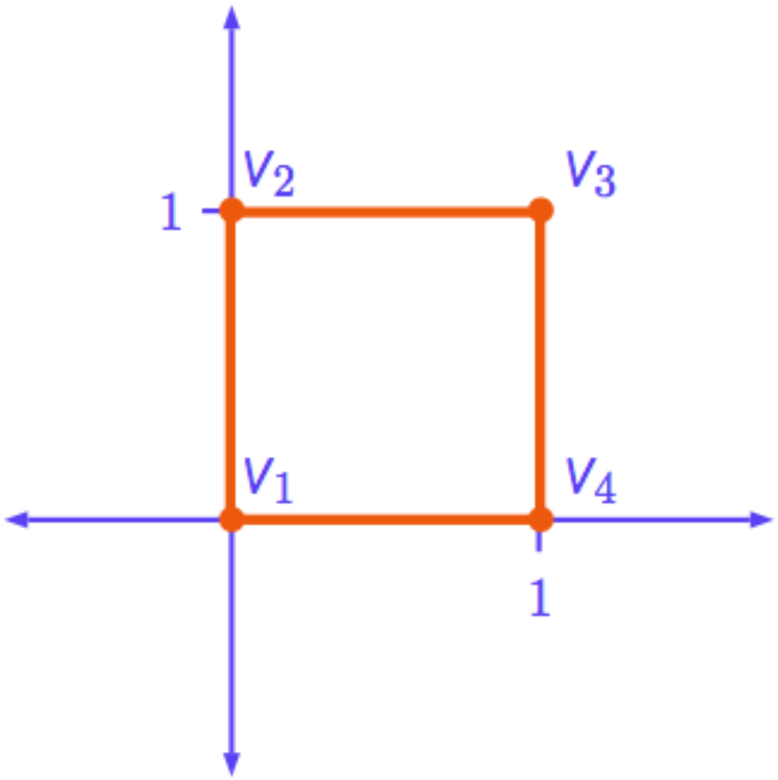


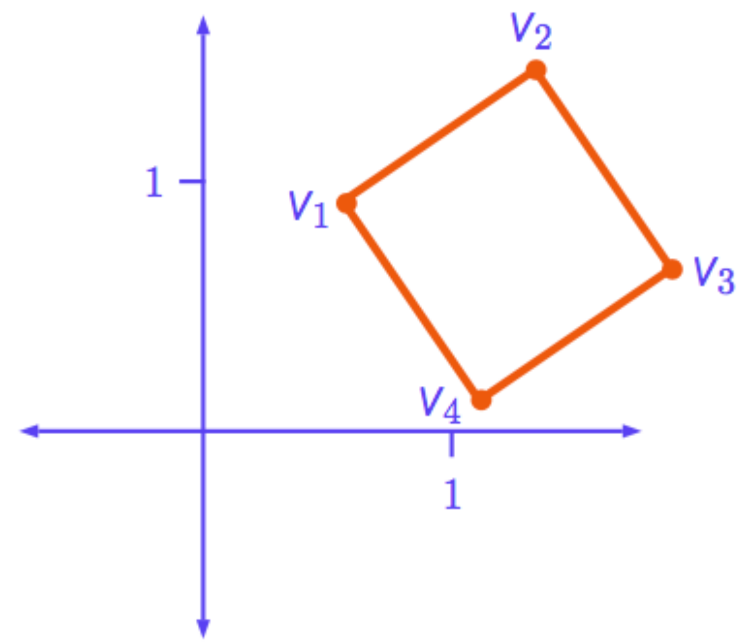


affine equivalence: $Q \stackrel{a}{=} P \iff Q = \psi(P), \quad \psi(x) = Ax + b$

preserves parallel lines,
e.g. scaling, rotation, reflection, translation

$P \subset \mathbf{R}^d \quad d\text{-polytope}$





affine equivalence: $Q \stackrel{a}{=} P \;\Leftrightarrow\; Q = \psi(P), \quad \psi(x) = Ax + b$

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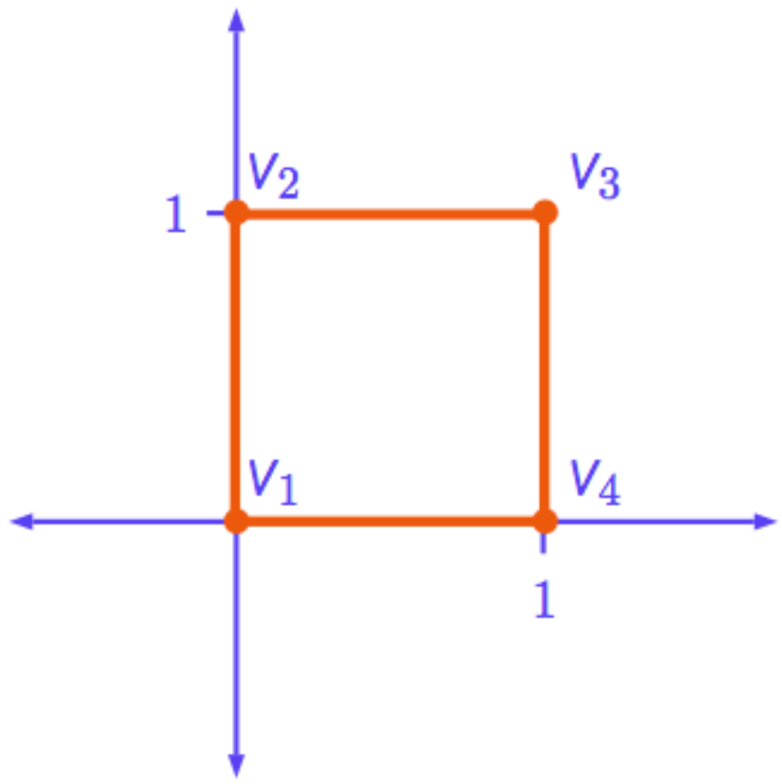
Parallelograms are affinely equivalent to a square

$P \subset \mathbf{R}^d \quad d$-polytope



combinatorics of P:

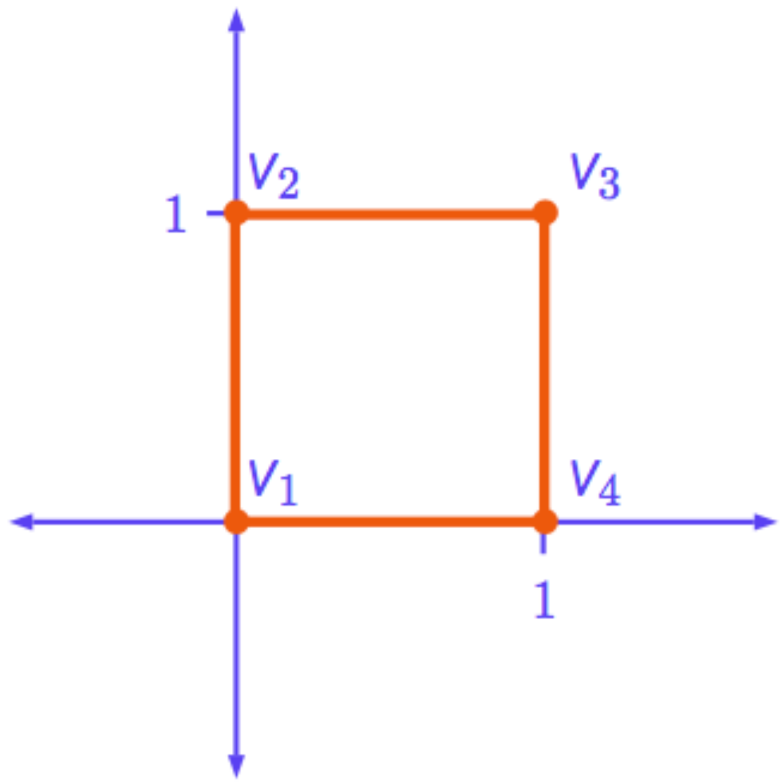
vertices: v_1, v_2, v_3, v_4

facets: $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}$

projective equivalence:

$$Q \overset{p}{=} P \iff Q = \phi(P), \quad \phi(x) = \frac{Ax + b}{c^\top x + \delta} \qquad \det \begin{bmatrix} A & b \\ c^\top & \delta \end{bmatrix} \neq 0$$

$P \subset \mathbf{R}^d \quad d\text{-polytope}$



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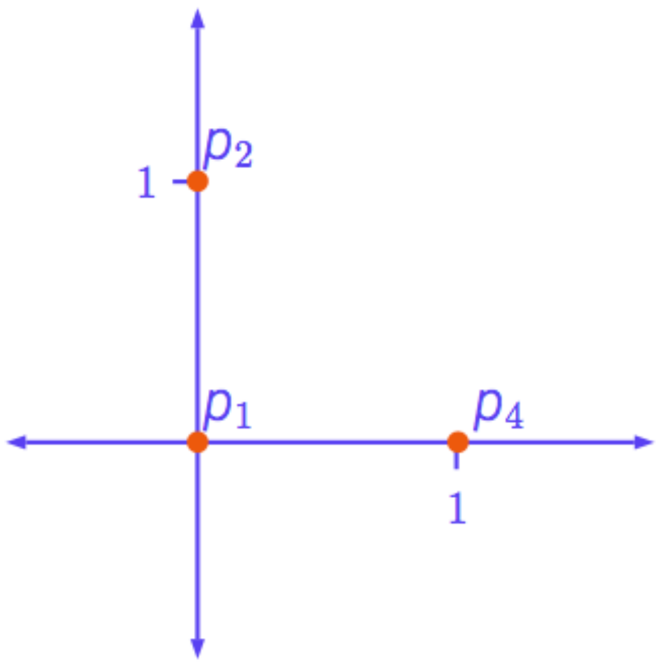
All quadrilaterals are projectively equivalent to a square.
A square is projectively unique.

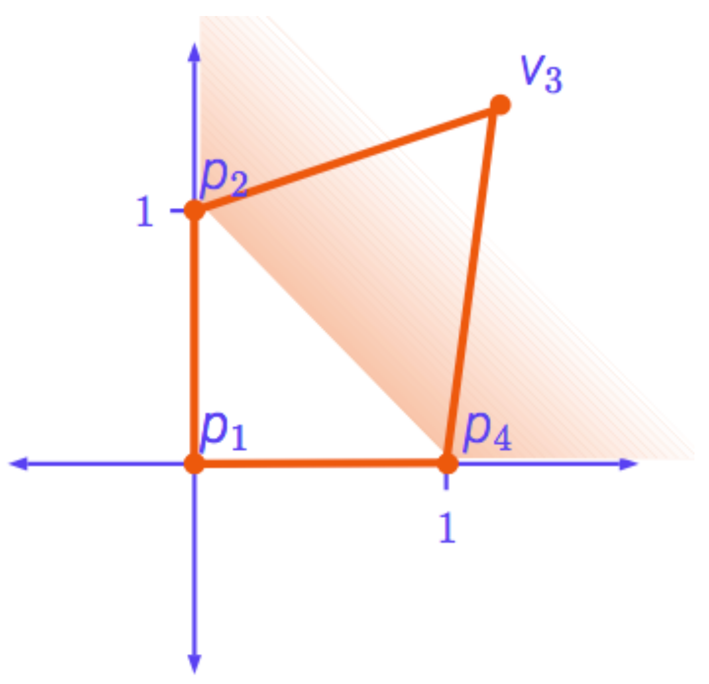
Realization Spaces

set of all realizations of polytopes combinatorially equivalent to P

Mod out affine transformations:

fix an affine basis of $d+1$ common vertices





$$\left\{ \begin{array}{ll} (x,y) : & x>0,\ y>0 \\ & x+y>1 \end{array} \right\}$$

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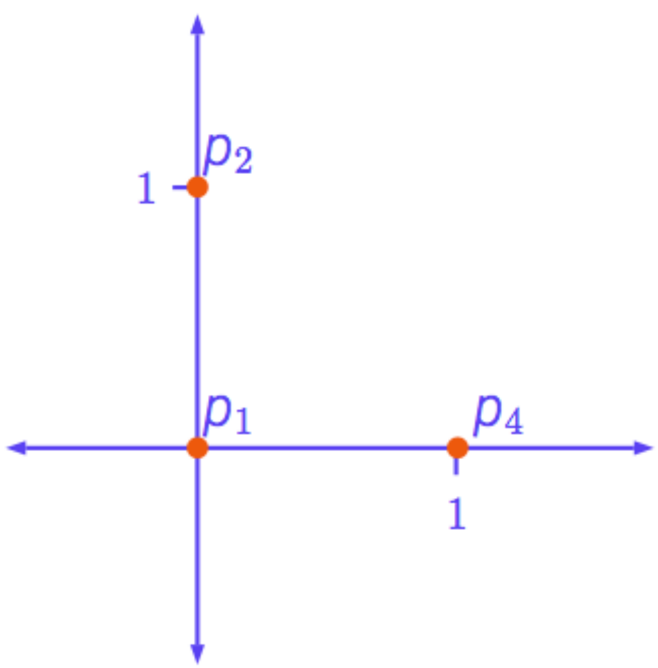

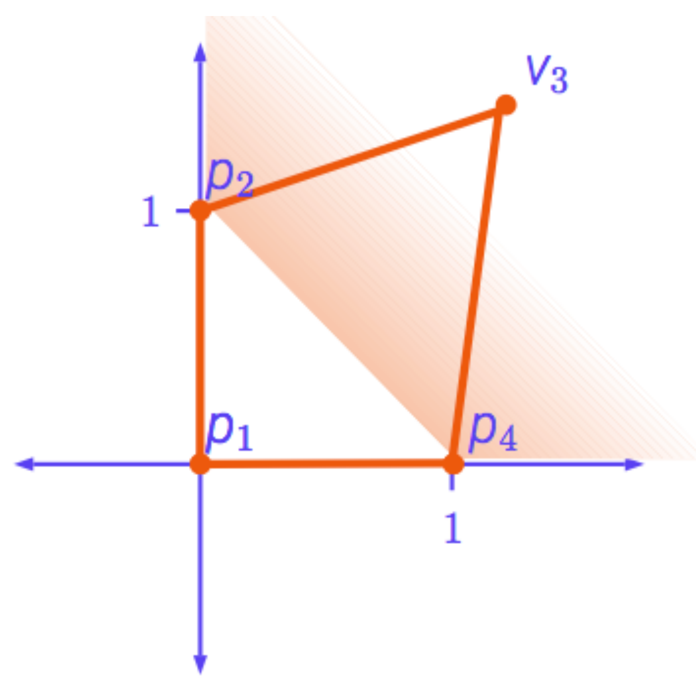

$$\left\{ (x,y) \;:\; \begin{array}{c} x>0,\; y>0 \\ x+y>1 \end{array} \right\}$$

$$\mathcal{R}(P,B) = \left\{ \begin{array}{l} Q = \operatorname{conv}(q_1,\ldots,q_v) \subset \mathbf{R}^d \;:\; q_i = p_i \;\forall i \in B \\ Q \overset{c}{=} P \end{array} \right\}$$

Realization Spaces

set of all realizations of polytopes combinatorially equivalent to P

Mod out projective transformations:
fix a projective basis of $d+2$ common vertices

Reduced realization space of P:

$$\mathcal{R}_{\mathrm{red}}(P, B') = \left\{ \begin{array}{l} Q = \mathrm{conv}(q_1, \ldots, q_v) \subset \mathbf{R}^d \ : \ q_i = p_i \ \forall i \in B' \\ Q \stackrel{c}{=} P \end{array} \right\}$$

The reduced realization space of a square is a single point.
Same for all projectively unique polytopes.

Main Results

- A new model for realization spaces of polytopes that arises as the positive part of an algebraic variety.
- Naturally mods out affine equivalence, so no choice of basis is needed. Nice way to study projective equivalence.
- The ideal defining the variety is a computational engine for questions concerning realizations such as rational realizability, convex realizability, freeness of faces, projective uniqueness.
- The ideal suggests a new way to classify polytopes.

$P \subset \mathbf{R}^d$ d-polytope

vertices: $\{p_1, \ldots, p_v\}$

facet inequalities: $\beta_j - a_j^T x \geq 0, \;\; j = 1, \ldots, f$

Slack matrix of P:

$$S_P = \begin{pmatrix} & \vdots & \\ \cdots & \beta_j - a_j^\top p_i & \cdots \\ & \vdots & \end{pmatrix}_{v \times f}$$

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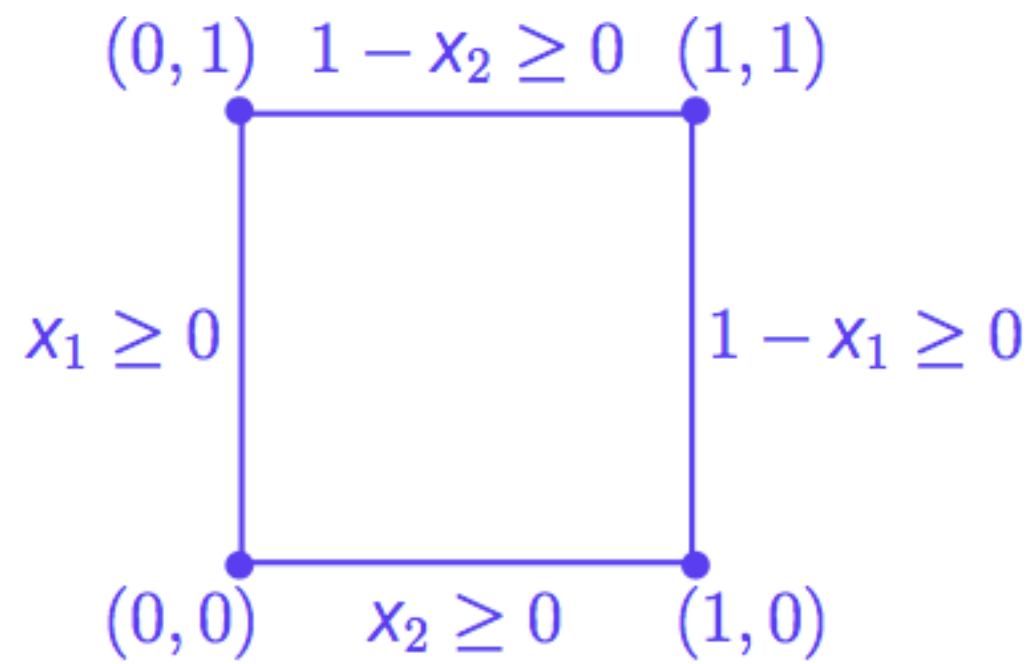

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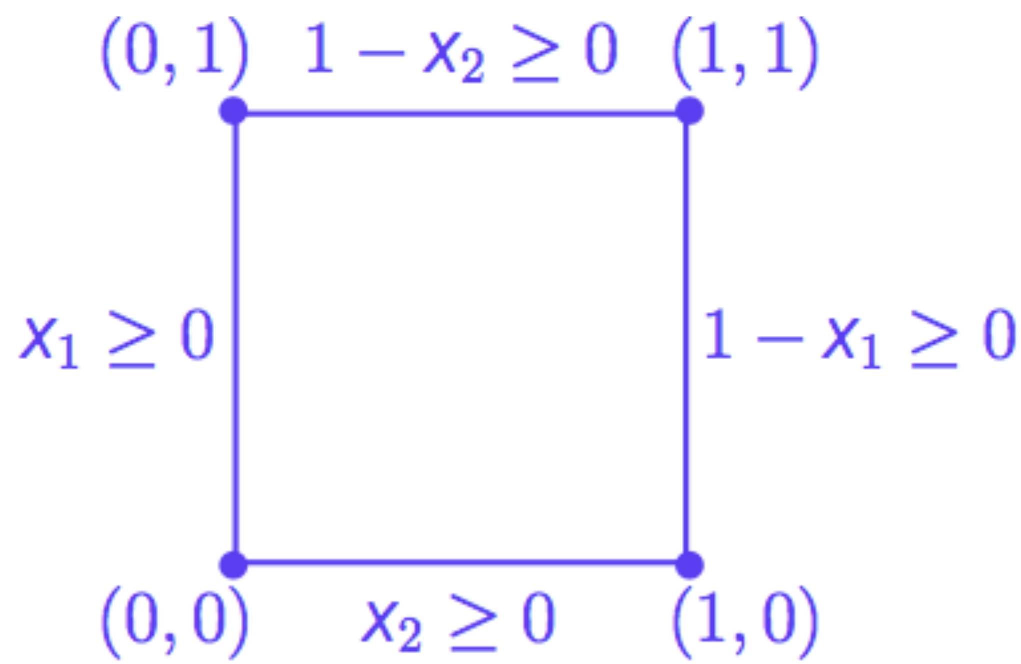

non-negative matrix whose zero pattern captures the combinatorics of P

$$S_P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

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- *P has infinitely many slack matrices:* $\{S_P D_f\}$
- *affinely equivalent polytopes have the same slack matrices*

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Lemma (GGKPRT 2013): S slack matrix of $P \Rightarrow$ $\text{conv}(\text{rows}(S))$ is affinely equivalent to P

S_P *is a realization of P (representative of affine eq. class of P)*

$P \subset \mathbf{R}^d$ d-polytope

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Slack matrix of P:

$$S_P = \begin{pmatrix} & \vdots & \\ \cdots & \beta_j - a_j^\top p_i & \cdots \\ & \vdots & \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & p_1^\top \\ \vdots & \vdots \\ 1 & p_i^\top \\ \vdots & \vdots \\ 1 & p_v^\top \end{pmatrix}}_{v \times (d+1)} \underbrace{\begin{pmatrix} \beta_1 & \cdots & \beta_j & \cdots & \beta_f \\ -a_1 & \cdots & -a_j & \cdots & -a_f \end{pmatrix}}_{(d+1) \times f}$$

$$\operatorname{rank}(S_P) = d + 1$$

The all-ones vector $\mathbf{1}$ is in the column span of S_P.

combinatorial equivalence:

Theorem follows from (GGKPRT 2013):

A nonnegative S is a slack matrix of a realization of P

$\Leftrightarrow$ (1) $\text{support}(S) = \text{support}(S_P)$

(2) $\text{rank}(S) = d + 1$

(3) $\mathbf{1} \in \text{column span of } S$

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projective equivalence:

Theorem (GPRT 2017):

$$Q \overset{p}{=} P \ \Leftrightarrow \ S_Q = D_v S_P D_f \text{ for some } D_v, D_f$$

Symbolic slack matrix of P:

$S_P(x) =$ replace every positive entry in S_P by a variable x_i

$$S_P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow S_P(x) = \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & x_3 & x_4 \\ x_5 & 0 & 0 & x_6 \\ x_7 & x_8 & 0 & 0 \end{pmatrix}$$

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$$I_P = \langle x_2x_4x_5x_8 - x_1x_3x_6x_7 \rangle$$

Theorems (GMTW 2017):

$$\mathcal{V}_+(I_P) = \{\text{nonnegative matrices } S \text{ satisfying (1) and (2)}\}$$

$$(1) \quad \text{support}(S) = \text{support}(S_P)$$

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$$\mathcal{V}_+(I_P)/(\mathbf{R}^v_{>0} \times \mathbf{R}^f_{>0}) \leftrightarrow \text{proj. eq. classes with the comb. of } P$$

$$\mathcal{V}_+(I_P)/(\mathbf{R}^v_{>0} \times \mathbf{R}^f_{>0}) \text{ rationally equivalent to } \mathcal{R}_{\text{red}}(P, B')$$

slack realization space of P

Is there a convex realization?

(Altshuler & Steinberg 1985):
a non-polytopal 3-sphere on 8 vertices

$$12345, 12346, 12578, 12678, 14568, 34578, 2357, 2367, 3467, 4678$$

$$S_P(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\ 0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 & x_{13} \\ 0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\ 0 & x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} \\ x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\ x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\ x_{30} & x_{31} & 0 & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 \end{bmatrix}$$

$I_P = \langle 1 \rangle \quad \Rightarrow$ there is no rank 5 matrix with this support

There isn't even a point/hyperplane configuration in $\mathbf{C}^4$ with the above incidences.

Can a face be freely prescribed?

Lemma: F face of $P \;\Rightarrow\; S_F$ submatrix of S_P & $I_F \subset I_P \cap \mathbf{C}[\mathbf{x}_F]$

F prescribable in $P \;\Leftrightarrow\; \mathcal{V}_+(I_F) = \mathcal{V}_+(I_P \cap \mathbf{C}[\mathbf{x}_F])$

(Barnette 1987): 4-d prism over a square pyramid with a non-prescribable cubical facet

$$S_P(\mathbf{x}) = \begin{bmatrix} x_1 & 0 & 0 & 0 & x_2 & x_3 & 0 \\ x_4 & 0 & 0 & 0 & 0 & x_5 & x_6 \\ x_7 & 0 & 0 & x_8 & 0 & 0 & x_9 \\ x_{10} & 0 & 0 & x_{11} & x_{12} & 0 & 0 \\ x_{13} & 0 & x_{14} & 0 & 0 & 0 & 0 \\ 0 & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 \\ 0 & x_{18} & 0 & 0 & 0 & x_{19} & x_{20} \\ 0 & x_{21} & 0 & x_{22} & 0 & 0 & x_{23} \\ 0 & x_{24} & 0 & x_{25} & x_{26} & 0 & 0 \\ 0 & x_{27} & x_{28} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

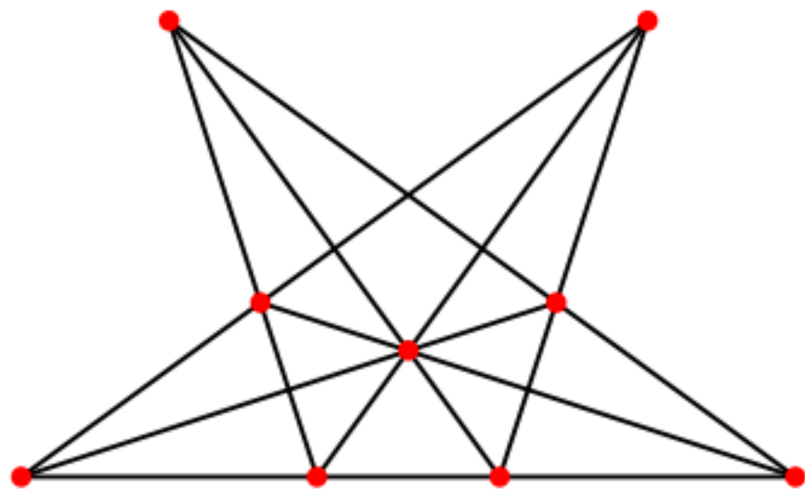
$\dim(I_F) = 16$

$\dim(I_P \cap \mathbf{C}[\mathbf{x}_F]) = 15$

with a bit more work conclude that F cannot be freely prescribed in P

Is there a rational realization?

Lemma: P is rational $\Leftrightarrow$ $\mathcal{V}_+(I_P)$ has a rational point



(Ziegler 2008)

$$S(\mathbf{x}) = \begin{bmatrix} x_1 & 0 & x_2 & 0 & x_3 & x_4 & x_5 & x_6 & 0 \\ x_7 & x_8 & x_9 & 0 & x_{10} & 0 & 0 & x_{11} & x_{12} \\ x_{13} & x_{14} & 0 & x_{15} & x_{16} & x_{17} & x_{18} & 0 & 0 \\ x_{19} & x_{20} & 0 & x_{21} & 0 & 0 & x_{22} & x_{23} & x_{24} \\ x_{25} & 0 & x_{26} & x_{27} & 0 & x_{28} & 0 & 0 & x_{29} \\ 0 & 0 & x_{30} & x_{31} & x_{32} & 0 & x_{33} & x_{34} & x_{35} \\ 0 & x_{36} & 0 & x_{37} & x_{38} & x_{39} & 0 & x_{40} & x_{41} \\ 0 & x_{42} & x_{43} & 0 & x_{44} & x_{45} & x_{46} & 0 & x_{47} \\ 0 & x_{48} & x_{49} & x_{50} & 0 & x_{51} & x_{52} & x_{53} & 0 \end{bmatrix}.$$

Scaling rows and columns allows one to set several variables to one.

Then $x_{46}^2 + x_{46} - 1$ is in the slack ideal.

Therefore, $x_{46} = \dfrac{-1 \pm \sqrt{5}}{2}$ and there is no rational realization.

I_P has no monomials

I_P can be binomial though typically it is not

Which are the binomial ideals, toric ideals?

A four-dimensional example

$P = \text{conv}\{0, 2e_1, 2e_2, 2e_3, e_{12} - e_3, e_4, e_{34}\}$ f-vector $(7, 17, 17, 7)$

$$S_P(x) = \begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & x_2 & 0 \\ x_3 & 0 & 0 & 0 & 0 & x_4 & 0 \\ x_5 & 0 & x_6 & 0 & 0 & 0 & x_7 \\ 0 & x_8 & x_9 & 0 & 0 & 0 & x_{10} \\ 0 & 0 & 0 & 0 & x_{11} & 0 & x_{12} \\ 0 & 0 & 0 & x_{13} & x_{14} & x_{15} & 0 \\ 0 & 0 & x_{16} & x_{17} & 0 & 0 & 0 \end{bmatrix}.$$

$\exists$ 49 6-minors of $S_P(x)$ all binomials except 4 like:

$$x_4x_5x_{10}x_{11}x_{13}x_{16} - x_4x_5x_9x_{12}x_{14}x_{17} + x_3x_7x_9x_{11}x_{15}x_{17} - x_3x_6x_{10}x_{11}x_{15}x_{17}$$

After saturating the ideal of minors, get the slack ideal:

$$I_P = \left\langle \begin{matrix} x_7x_9 - x_6x_{10} & x_{10}x_{11}x_{13}x_{16} - x_9x_{12}x_{14}x_{17} \\ x_7x_{11}x_{13}x_{16} - x_6x_{12}x_{14}x_{17} & x_2x_8x_{13}x_{16} - x_1x_9x_{15}x_{17} \\ x_4x_5x_{13}x_{16} - x_3x_6x_{15}x_{17} & x_2x_8x_{12}x_{14} - x_1x_{10}x_{11}x_{15}, \\ x_4x_5x_{12}x_{14} - x_3x_7x_{11}x_{15} & x_2x_3x_7x_8 - x_1x_4x_5x_{10} \\ x_2x_3x_6x_8 - x_1x_4x_5x_9 & \end{matrix} \right\rangle$$

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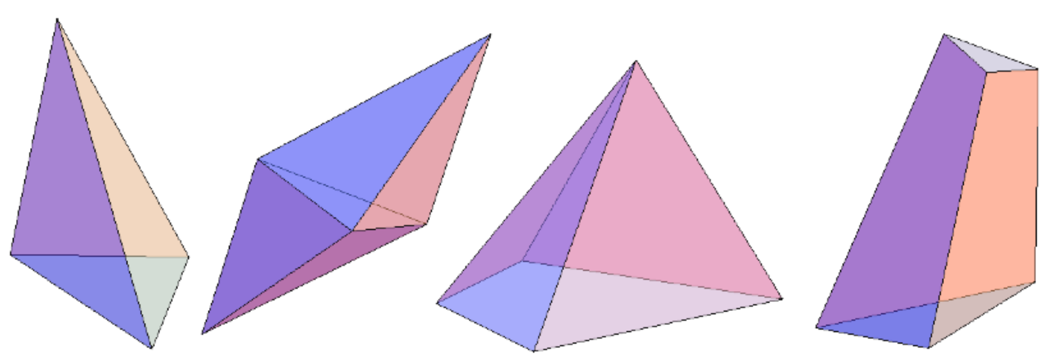
$d = 2:$

$d = 3:$



$d = 4:$ *(GPRT): 11 combinatorial classes including all products of simplices*

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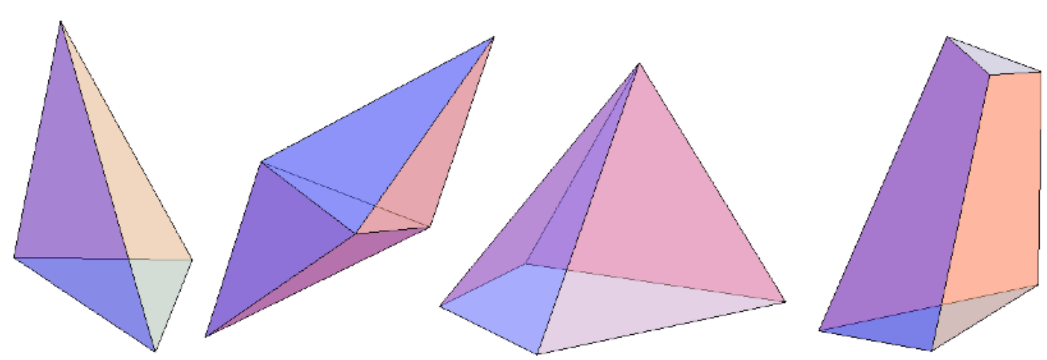


$d = 4:$ (GPRT): 11 combinatorial classes including all products of simplices

all toric & projectively unique!!

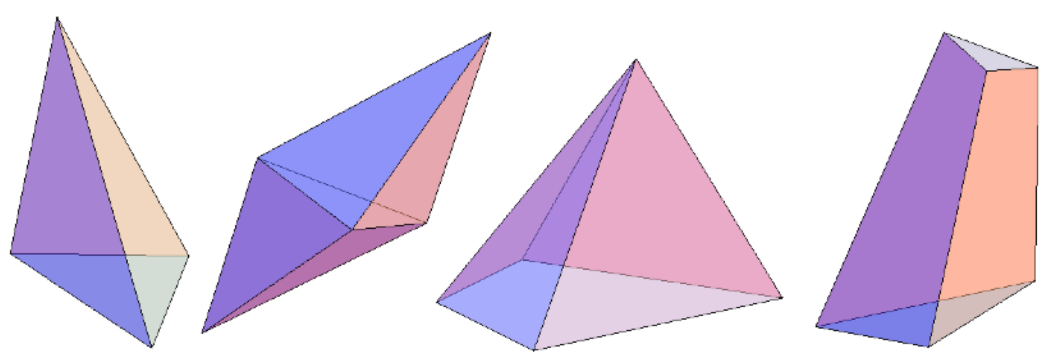
Could I_P not be the defining ideal of $\mathcal{V}_+(I_P)$?

If I_P is toric, then it is.

Which slack ideals are toric ideals?

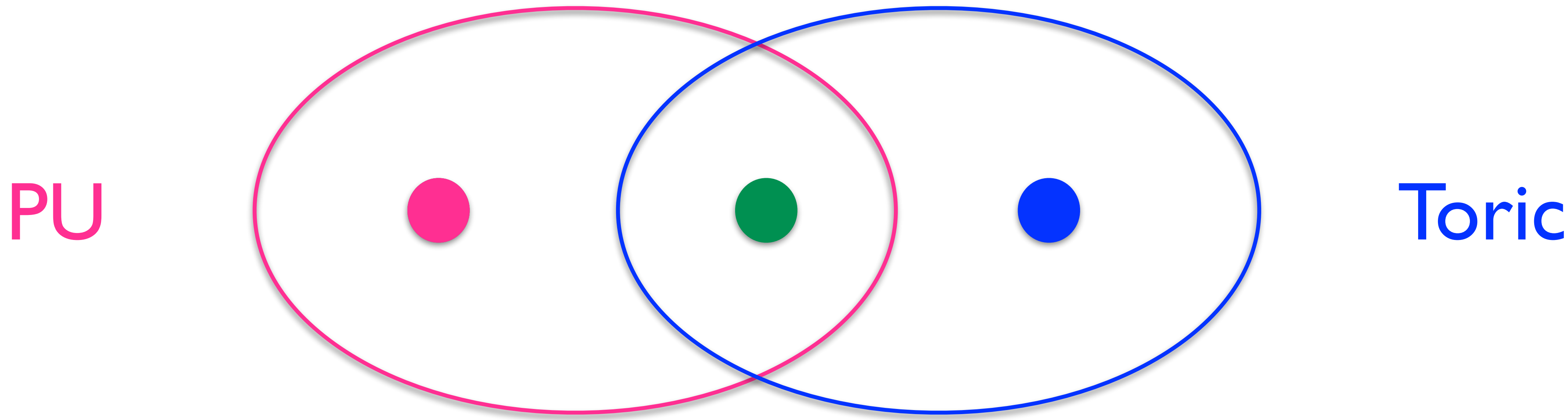
$d = 2:$

$d = 3:$



all toric & projectively unique!!

$d = 4:$ *(GPRT): 11 combinatorial classes including all products of simplices*



- *There are infinitely many projectively unique polytopes in high enough dimension whose slack ideals are not toric.* (Adiprasito-Ziegler 2015 & GPRT)

- *In dimension 5, there are non-projectively unique polytopes whose slack ideals are toric.* (Gouveia-Macchia-T-Wiebe)

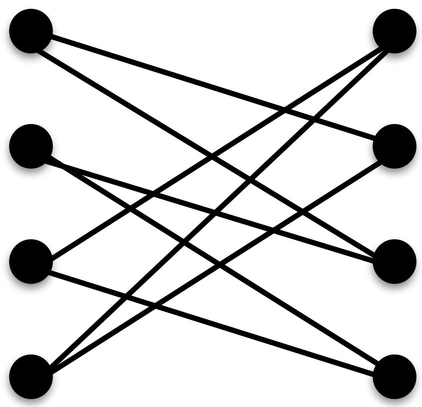
- *Which polytopes are PU & have toric slack ideals?*

  e.g. v or $f = d+2$ (products of simplices), example from before

Graphic slack ideals

K_P- bipartite graph supporting S_P

$$S_P = \begin{bmatrix} 0 & 3 & 6 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 0 & 0 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

$K_P =$ 

A_P - vertex-edge incidence matrix of K_P

T_P - toric ideal of A_P

Definition: I_P is graphic if $I_P = T_P$

(GMTW 2017): I_P is graphic if and only if it is toric and P is projectively unique.

Thank You