Abstract:

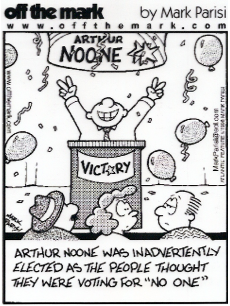
When does a candidate have the approval of a majority? How does the geometry of the political spectrum influence the outcome? When mathematical objects have a social interpretation, the associated results have social applications. We will show how generalizations of Helly's Theorem can be used to understand voting in "agreeable" societies. This talk also features research with undergraduates.

Voting in Agreeable Societies

Francis Edward Su

Harvey Mudd College
Claremont, California, USA

Geometric and Topological Combinatorics
MSRI Connections for Women 2017



thanks: NSF DMS-1002938

Featuring, among others, the work of...

Deborah Berg

Kathryn Nyman

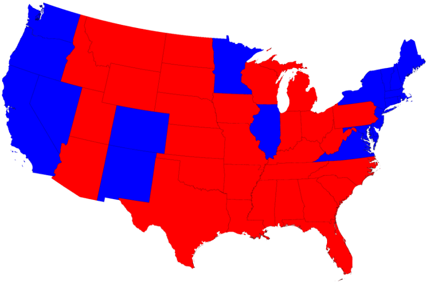
Maria Klawe

Overview

Sets Model Preferences in Some Space

2016 U.S. PRESIDENTIAL ELECTION

Space of states

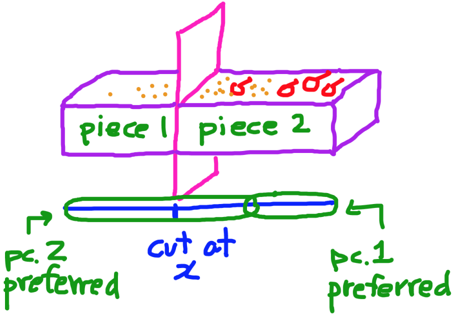


Clinton set, *Trump set*

Sets Model Preferences in Some Space

CAKE-CUTTING

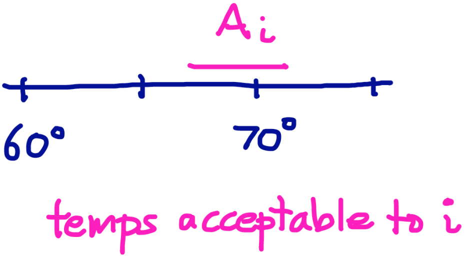
Space of divisions



Sets Model Preferences in Some Space

THERMOSTAT DISAGREEMENTS

Space of temperatures



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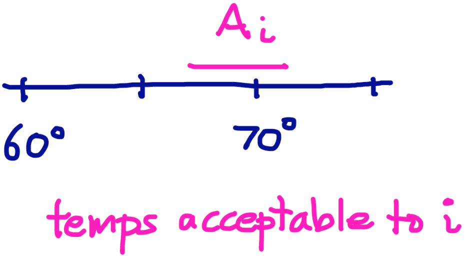

Important: intersections of these sets!

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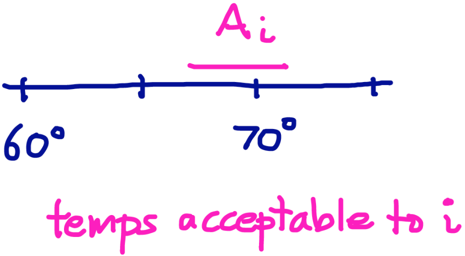

Important: intersections of these sets!

Space has geometry/topology, a notion of "closeness"

Restaurants along Shattuck Ave.

A B C D E F G H I J K L M N

- Where to go for dinner?
  How popular is the most popular restaurant?

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- I predict: one restaurant will get at least $\frac{1}{2}$ the votes!

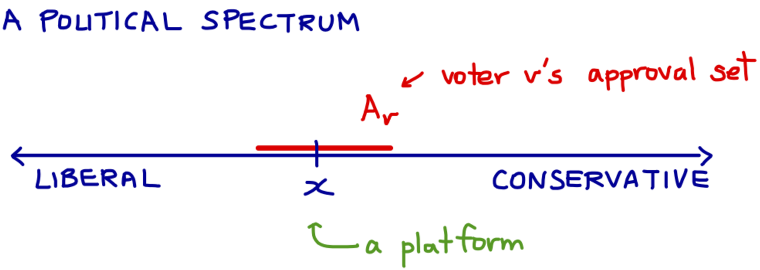
Third parties in American Politics

graphic: *The Simpsons*

Approval Voting

Voters vote for as many options they "approve"

A Model


A POLITICAL SPECTRUM
voter v's approval set
A_v
LIBERAL
x
CONSERVATIVE
a platform

Society of Voters

A *society* of voters consists of:

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Q. How popular is the most popular platform? Or candidate?

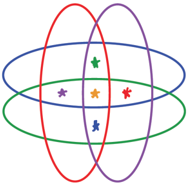
Intersection Theorems

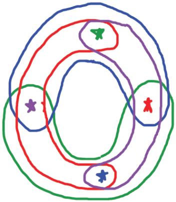
If approval sets are intervals.... they're convex sets.

Intersection Theorems

Helly's Theorem, 1913

In a finite family of convex sets in d dimensions, if every $d+1$ sets intersect mutually, then all sets have a point in common.





Note: true for infinite collections if sets are also compact.

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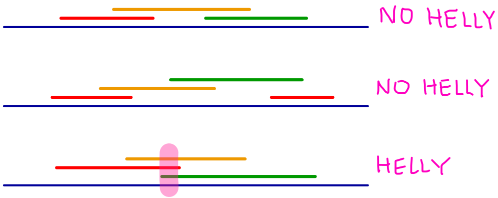

Helly-on-a-line

Any pairwise-intersecting family of intervals
must have a point in all sets.

Setting Thermostats

Helly-on-a-line rephrased:

If any 2 people can agree on a temperature,
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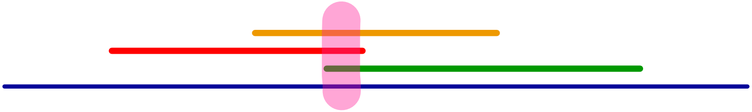

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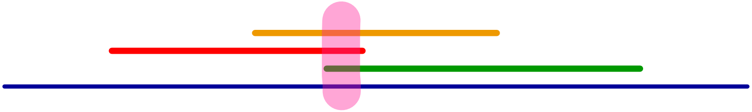

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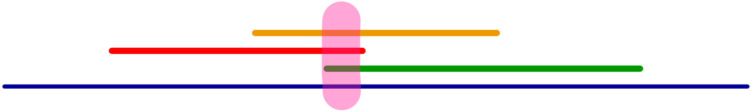
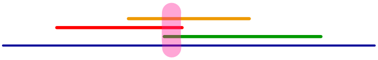


- Proof: each person i has a low and hi temp: L_i, H_i.
- Since each pair A_i, A_j overlap, $L_i \leq H_j$.
- So $\max_i L_i \leq \min_j H_j$, any temp in between works. $\square$

Super-agreeable Societies

- A *linear* society: spectrum is a line.



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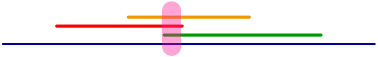

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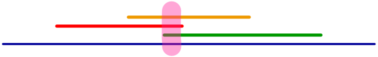

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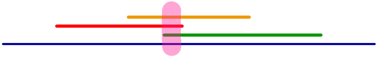

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Hypothesis too strong?

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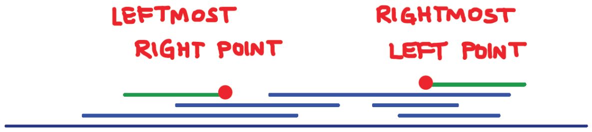

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We can say more...

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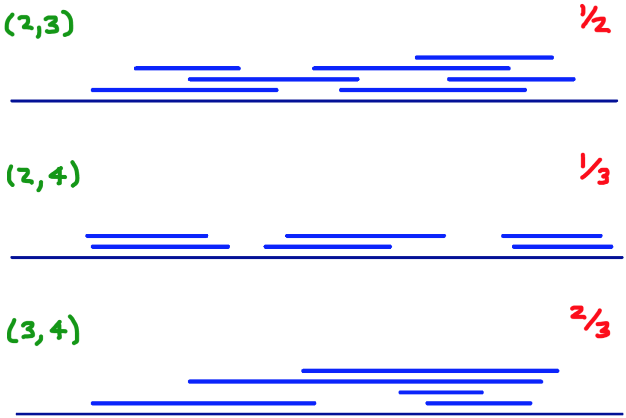
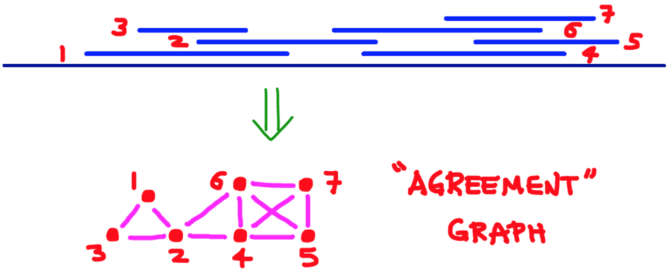


Proof Sketch: Agreeable Society Theorem

- Construct an agreement graph G:



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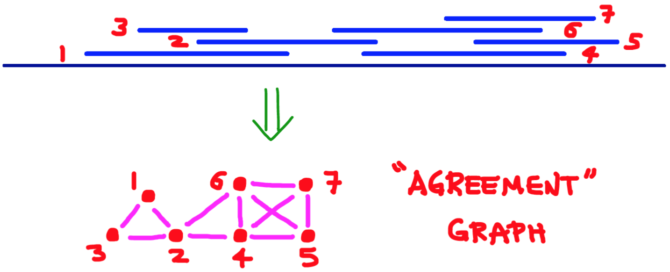

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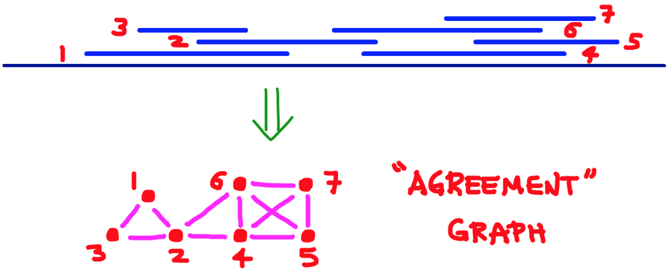

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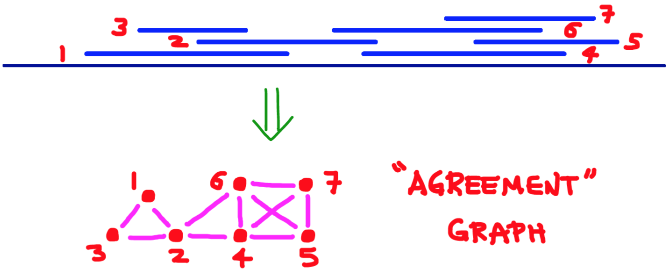

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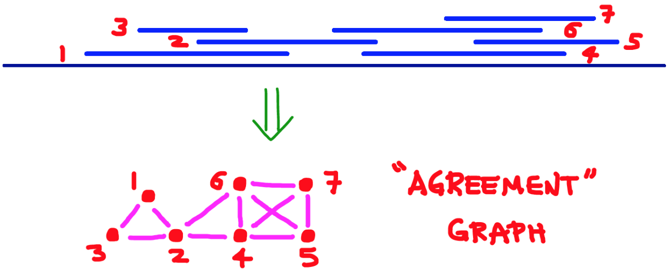

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- Color overlapping voters different colors.
- For linear societies, G is perfect: (chromatic# = clique#)
- (k, m)-condition $\implies$ color classes aren't too big, so there have to be many colors! □

Related

Hadwiger-Debrunner, 1957

Let $\mathcal{A}$ be a collection of compact intervals on the line such that among any m intervals, some k mutually intersect. Then the sets can be *pierced* by a family of $m - k + 1$ points, i.e., every set contains one point in the family.

Related

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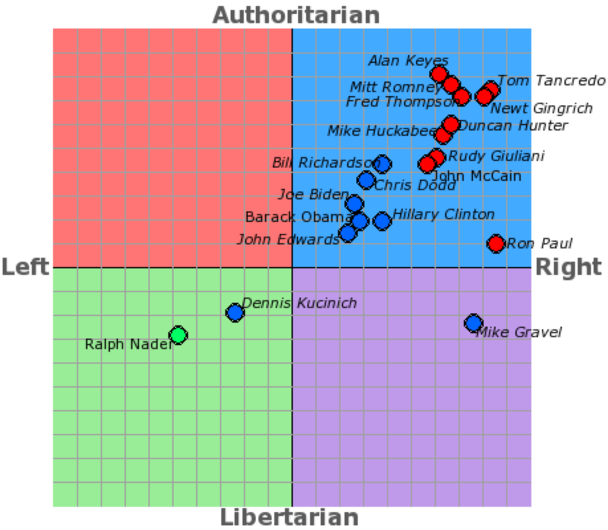
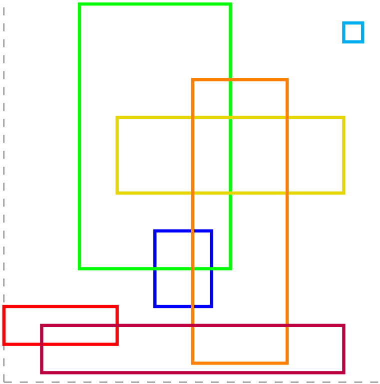
Note: *piercing* $\# =$ min'l size of group representing all voters.
Q. What new questions does this interpretation suggest?

Generalization: change the spaces



Box societies in $\mathbb{R}^d$

- Political prefs: convex vs. boxes vs. box-convex
- Boxes have Helly property: If boxes pairwise intersect, then there's a point in all boxes



Box societies in $\mathbb{R}^d$

Theorem. (Berg et. al. 2010)

In an $\mathbb{R}^d$-convex (k, m)-agreeable society, the agreement proportion β satisfies
$$\beta \geq 1 - \left(1 - \frac{\binom{k}{d+1}}{\binom{m}{d+1}}\right)^{1/(d+1)}.$$

Uses Fractional Helly Theorem.

Ex. $\mathbb{R}^2$-convex $(3, 4)$-agreeable society: $\beta \geq 0.0914$.

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If $k \leq m \leq 2k - 2$, in an n-voter $\mathbb{R}^d$-box (k, m)-agreeable society, $\beta \geq \frac{n-m+k}{n}$. (best possible)

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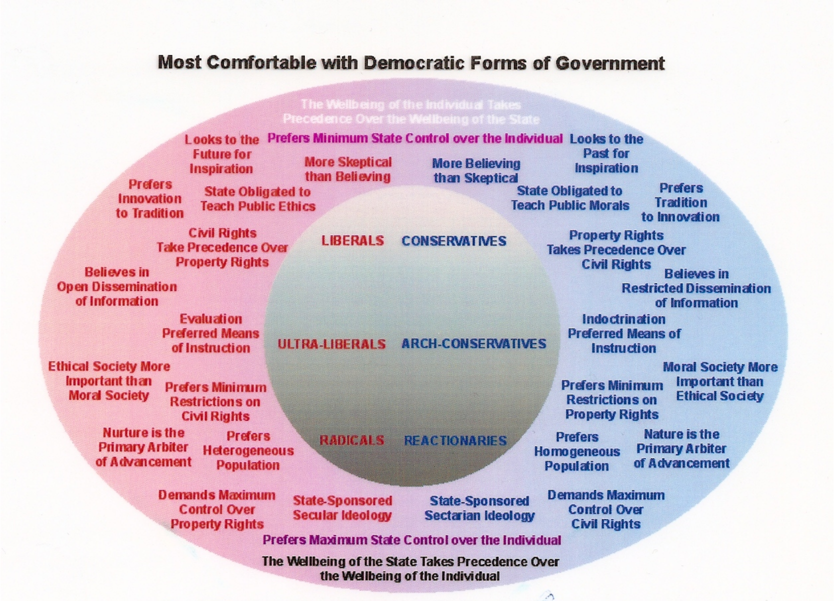
Open Q: results for *box-convex* sets?

Circular Societies



Set Intersections on a Circle

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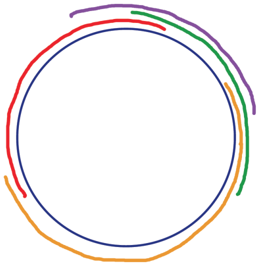

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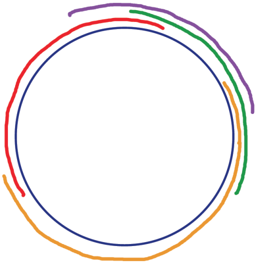

A. No, but there is a point in a strict majority of the sets!

Circular Societies

Circular Society Theorem (Niedermaier-Rizzolo-S. 2014)

In a circular society, if any 2 voters agree on a candidate, then some candidate will win strict majority approval.

Uses a KKM theorem for trees and cycles.

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Theorem (Hardin 2010)

In a (k, m)-agreeable circular society, there is a candidate that has the approval of at least $\frac{k-1}{m}$ of the voters.

Generalization: change the hypotheses

Consider *fractional pairwise agreeability*: suppose α of all voter pairs can agree on some candidate.

What is the agreement proportion (of the most popular candidate)?

Pairwise Agreeability

Meta-Theorem

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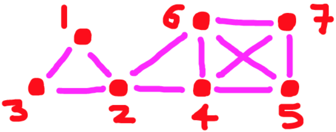
Think about the agreement graph G.

If there are sufficiently many edges,
then there must be a large clique.



Pairwise Agreeability

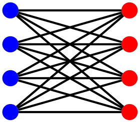
Turán's Theorem 1941

Any graph G with n vertices that does not have an $(r+1)$-clique as a subgraph has at most $(1-\frac{1}{r})\frac{n^2}{2}$ edges.

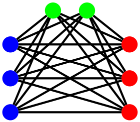
Examples for $n = 8$:

Bound achieved by complete r-partite graph, where size of each partite set varies by at most 1.

$r = 2$



$r = 3$



Interval Societies - fractional agreement

Theorem (Abbott-Katchalski 1979, re-interpreted)

If an interval graph has pairwise agreement α,
then the agreement proportion is at least β, where

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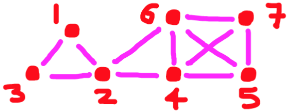
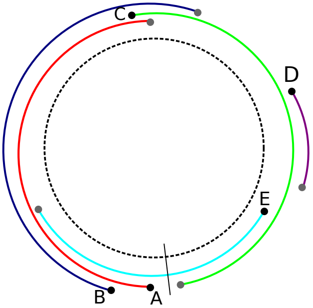
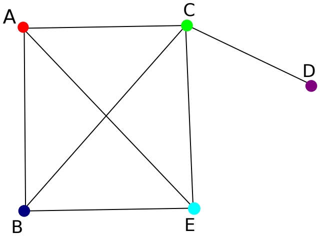
Ex: So pairwise agreement of $\frac{3}{4} = \frac{1}{2}(2 - \frac{1}{2})$ of all voter pairs ensures an agreement proportion of at least $\frac{1}{2}$.



Circular Societies - fractional agreement

Big problem here: clique number k does not imply agreement number k.





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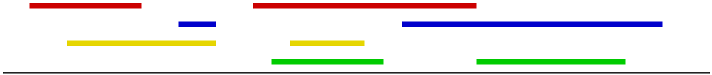
Ex: In a circular society, if every candidate has at least $\gamma = 20\%$ of the votes of the most popular candidate, and if there is $\alpha = 46\%$ pairwise agreement, then the most popular candidate will have at least $\beta = 25\%$ approval.

Generalization: Change the set geometry

Double-interval societies

with Kathryn Nyman, Maria Klawe, Jacob Scott:



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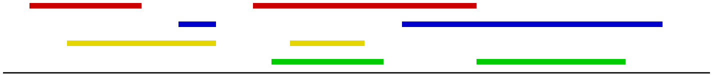

Theorem. (Klawe-Nyman-Scott-S. 2014)

The approval ratio of any n-voter pairwise-intersecting, double-interval society is at least

$$0.268 + \frac{0.789}{n} - \frac{1.732}{24n^2}.$$

Conjecture: tight bound is $1/3$.

Double-N Strings: Smallest Agreement Proportion?

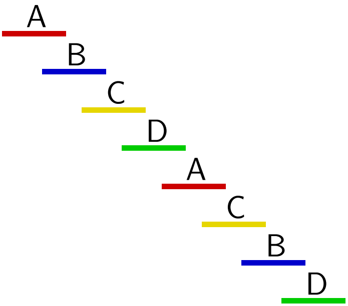

Figure: The double-4 string: ABCDACBD.
Has diam=1, since any 2 letters are 1 apart.

Double-N Strings: Smallest Agreement Proportion?

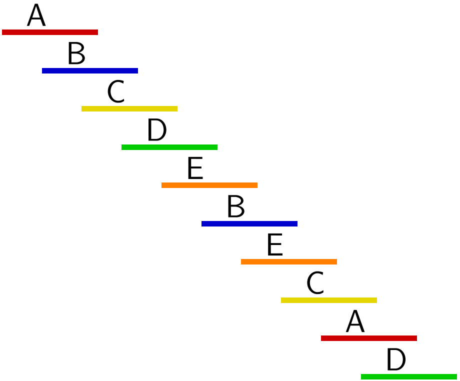

Figure: Double 5-string: ABCDEBECAD has diam=2.

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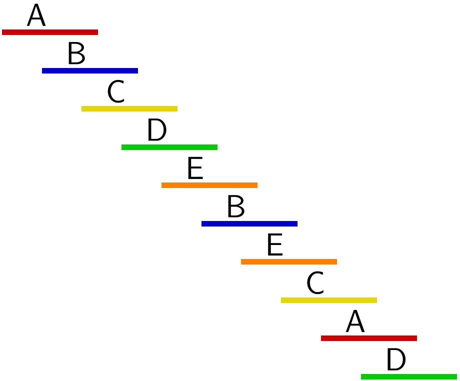

Figure: Double 5-string: ABCDEBECAD has diam=2.

Surprise: asymptotic agreement prop. for double n-strings $\geq 8/23 > 1/3$.

Double Interval Societies

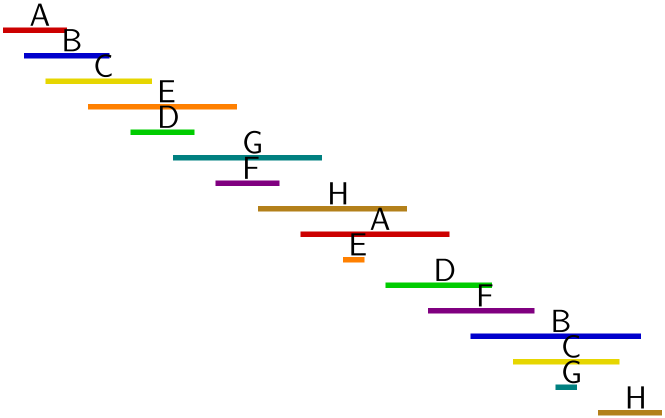

Figure: A society of size 8 with approval number 3.
Derived from double 8-string ABCDEFGHEADFCGBH.

Tantalizing

DOUBLE-INTERVAL SOCIETIES

$a(S)$	n	Approval Ratio	Observed n	Observed Approval Ratio
2	≤ 4	≥ 0.500	4	0.500
3	≤ 8	≥ 0.375	8	0.375
4	≤ 12	≥ 0.333	12	0.333
5	≤ 15	≥ 0.333	15	0.333
6	≤ 19	≥ 0.316	18	0.333
7	≤ 23	≥ 0.304	21	0.333
8	≤ 26	≥ 0.308	24	0.333
9	≤ 30	≥ 0.300	27	0.333
10	≤ 34	≥ 0.294	30	0.333
11	≤ 38	≥ 0.289	33	0.333
12	≤ 41	≥ 0.293	36	0.333

TABLE 1. On the left, this table shows for a given approval number the largest n that is given by inequality (4.3) as well as the resulting bound on the approval ratio derived from inequality (4.1). On the right, this table shows, for a given approval number, known examples of the largest n that has this approval number and the observed approval ratio in that case, obtained by a modification of a double-n string construction.

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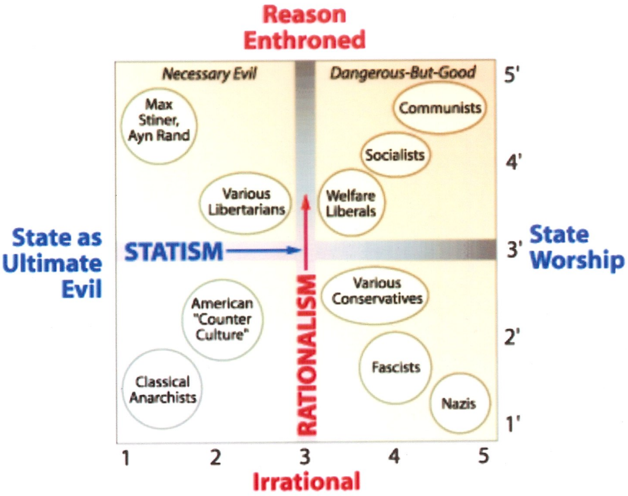
Questions: For double-N strings, is there systematic way to construct strings of smallest diameter?
Beyond double-N strings, is there general construction yielding societies with lowest agreement ratios?

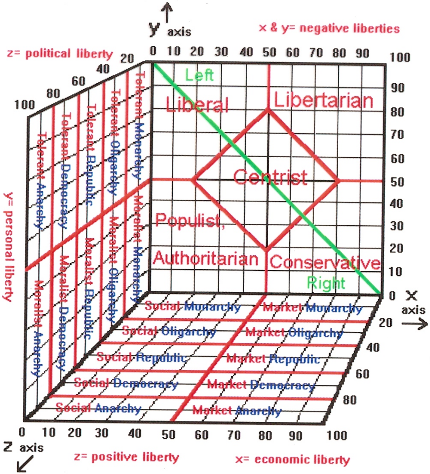
Further questions...

- Agreement proportion for other spaces? How does topology affect agreement?
- Non-convex approval sets?
- Probabilistic voting results?
- Statistics: estimates from samples?
- Other "social" applications?

Two Dimensional Space of Platforms



Three Dimensional Space of Platforms



29-Dimensional Approval Sets?

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eHarmony matches singles based on a deeper level of compatibility, not likes and dislikes, but true compatibility. Do you and your potential mate resolve conflict in a similar fashion? Are you both romantics at heart? And we are the only online dating web site that matches singles based on these 29 Dimensions®. To help you better understand these dimensions, we've grouped them into Core Traits and Vital Attributes.

Core Traits are defining aspects of who you are that remain largely unchanged throughout your adult life. Vital Attributes are based on learning experience, and are more likely to change based on life events and decisions you make as an adult.

Take Aways

- Math can model/answer questions in the social sciences
- Social problems can motivate new mathematical questions