In this talk, I will discuss domino tilings of three dimensional manifolds. In particular, I will focus on the connected components of the space of tilings of such regions under local moves. Using topological techniques we introduce two parameters of tilings: the flux and the twist. Our main result characterizes when two tilings are connected by local moves in terms of these two parameters. (I will not assume any familiarity with the theory of tilings for the talk.)

On the connectivity of three-dimensional tilings 8/31/17

Carly Klivans 3:30pm

Theorem: (FKMS '16) t_0, t_1 tilings of R.

- t_0 ... t_1

$$\begin{array}{ccc} t_0 & & t_1 \\ \downarrow & & \downarrow \\ \text{Refine}(t_0) & \rightsquigarrow \text{flip \& trit} \rightsquigarrow & \text{Refine}(t_1) \end{array}$$

$$\Longleftrightarrow$$

$$\text{Flux}(t_0) = \text{Flux}(t_1)$$

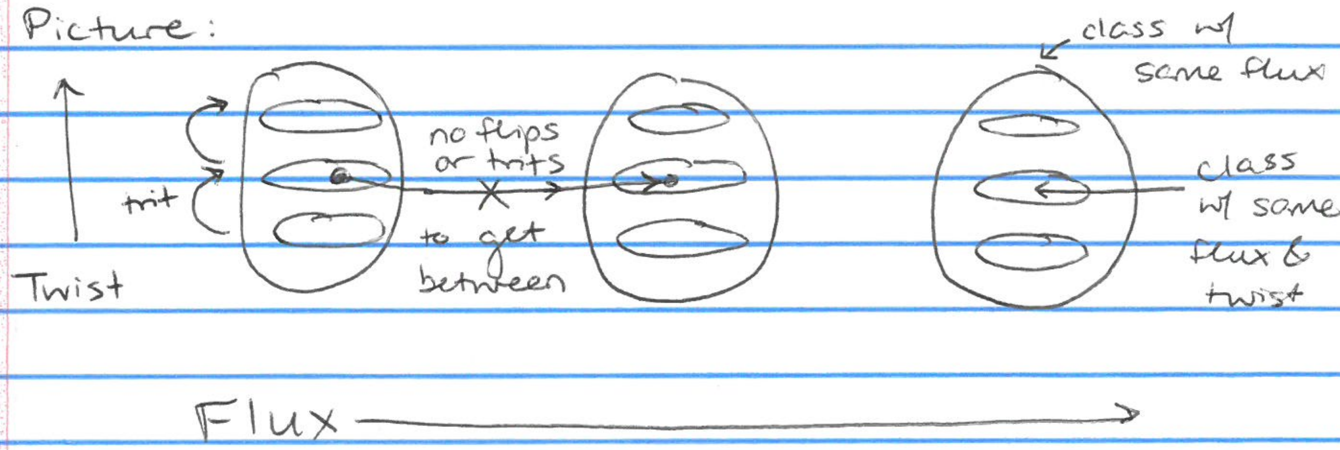
- t_0 ... t_1

$$\begin{array}{ccc} t_0 & & t_1 \\ \downarrow & & \downarrow \\ \text{Refine}(t_0) & \rightsquigarrow \text{flip} \rightsquigarrow & \text{Refine}(t_1) \end{array}$$

$$\Longleftrightarrow$$

$$\text{Flux}(t_0) = \text{Flux}(t_1) \ \& \ \text{Twist}(t_0) = \text{Twist}(t_1)$$

Picture:



On The Connectivity of Three Dimensional Tilings

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Brown University

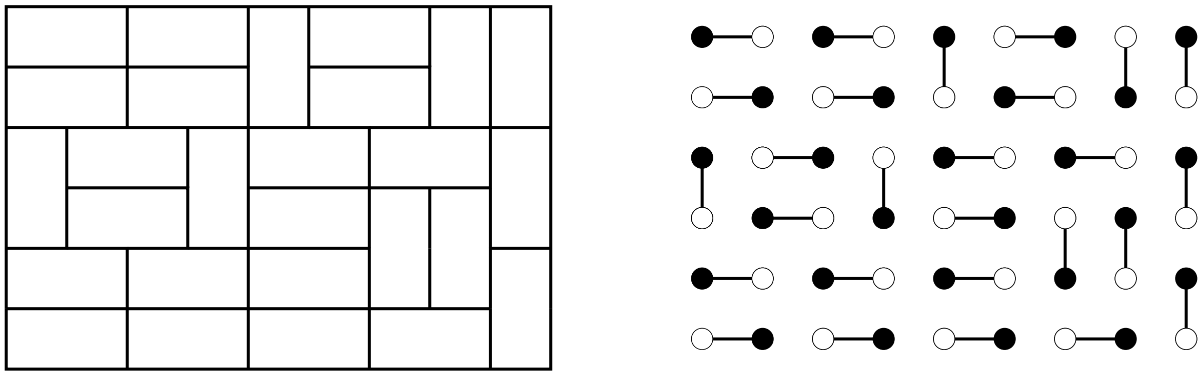
Joint work with Juliana Freire, Pedro Milet and Nicolau Saldanha
PUC-Rio

Tilings

Domino tilings of cubiculated Regions R:

- cubical complexes embedded as a finite polyhedron in $\mathbb{R}^N$
- connected oriented topological manifolds

Dimer covers of the dual graph R^*



A two dimensional tiling

Theory of Two-Dimensional Tilings

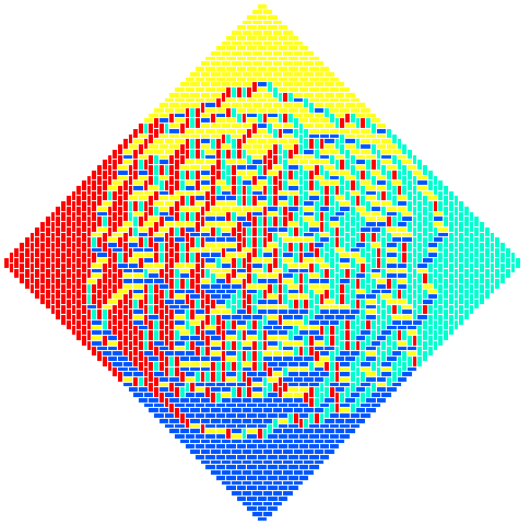
Combinatorics, Probability, Statistical Physics

Q: How many tilings are there of a given region?

Kasteleyn '61, Temperley and Fisher '61 Pfaffians, determinants, transfer matrix-method

Q: What does a random tiling look like?

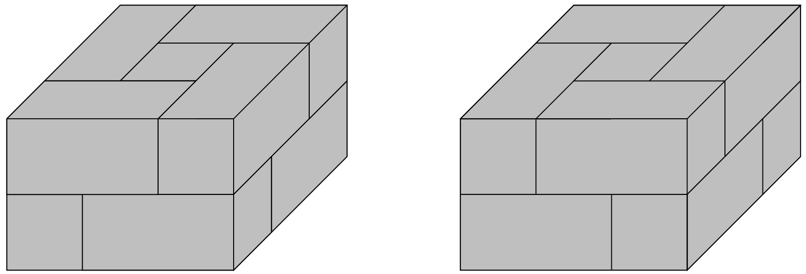
Jockush, Propp, Shor '95 Arctic circle theorem



Q: Can we move from one tiling to another?

Thurston '90 Flip moves and height functions

Three-Dimensional Tilings



Dominoes: $2 \times 1 \times 1$ bricks of two adjacent cubes

Goal: Understand the space of tilings. What does a typical 3-dimensional tiling look like? How many tilings are there?

Focus: Connectivity by local moves.

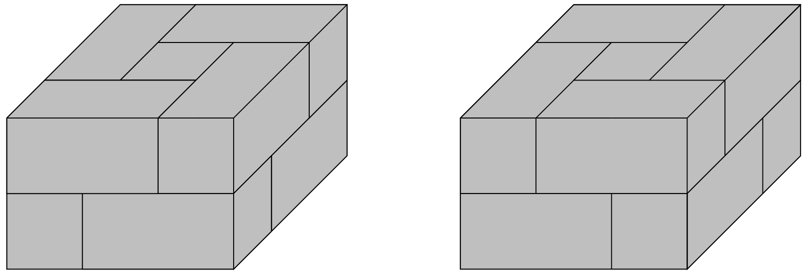
How and when can we move from one tiling to another?

Connectivity - Local Moves

Flip: Remove two adjacent parallel dominoes and place them back rotated within a $2 \times 2 \times 1$ block.

> **Theorem:** (Thurston) In two dimensions, any two tilings of a simply connected region are flip connected.

Not the case in 3d:



Two tilings of the $3 \times 3 \times 2$ box with no flips.

Connectivity - Local Moves

$3 \times 3 \times 2$ box:

Number of tilings: 229

Connected components: 3

Sizes: $227, 1, 1$

$4 \times 4 \times 4$ box:

Number of tilings: $5,051,532,105$

Connected components: 93

Sizes: $4,412,646,453$

$2 \times 310,185,960$

$2 \times 8,237,514$

$2 \times 718,308$

$2 \times 283,044$

$6 \times 2,576$

24×618

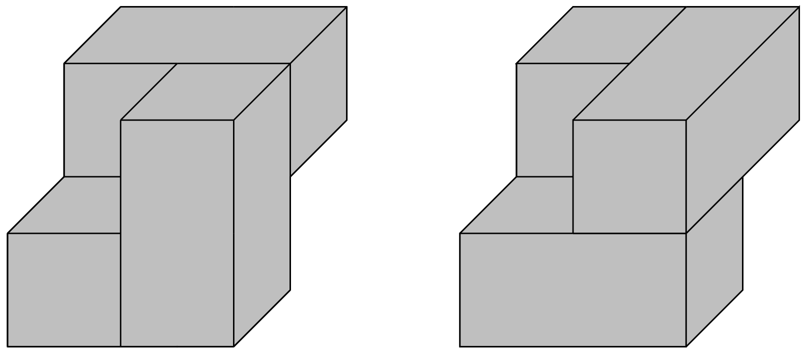
24×236

6×4

24×1

Connectivity - Local Moves

Trit: Remove and replace 3 dominoes, one parallel to each axis inside a $2 \times 2 \times 2$ box.



A positive trit

Question: Are tilings of three-dimensional regions connected by flips and trits?

Connectivity - Local Moves

Question: Are all tilings of three-dimensional regions connected by flips and trits?

In general, **no**.

Examples include tilings of:

- Cylinders: $\mathcal{D} \times [0, n]$
- Tori: $\mathbb{R}^3/\mathcal{L}$, $\mathcal{L} = 8\mathbb{Z}^3$

Open:

- Boxes: $[0, L] \times [0, M] \times [0, N]$

Connectivity - Local Moves

Need two topological invariants:

Flux: "Flow across surfaces"
Twist: "Knottedness by trits"

And need the notion of **Refinement**:

- Decompose each cube into $5 \times 5 \times 5$ smaller cubes.
- Decompose each domino into $5 \times 5 \times 5$ smaller dominoes, each parallel to the original.

Connectivity - Local Moves

Theorem (FKMS '16)

For two tilings t_0 and t_1 of R:

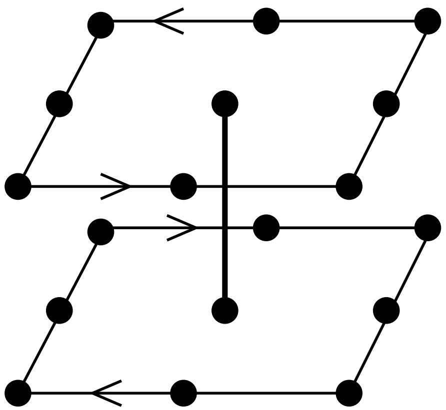
- There exists a sequence of flips and trits connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1)$.

Difference of Tilings

For two tilings t_0, t_1

- $t_1 - t_0 :=$ union of tiles (with orientation of t_0 reversed).



Yields a system of cycles. (Ignore trivial 2-cycle.)

Homologically: $t_1 - t_0 \in Z_1(R^*; \mathbb{Z})$

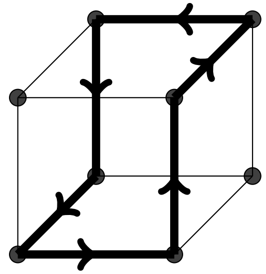
Topological Invariant I - Flux

Fix a base tiling $t_\oplus$

$$\mathrm{Flux}(t) := [t - t_\oplus] \in H_1(R^*; \mathbb{Z})$$

flip $\rightsquigarrow$ boundary of a square

trit $\rightsquigarrow$ boundary of 3 squares



Proposition

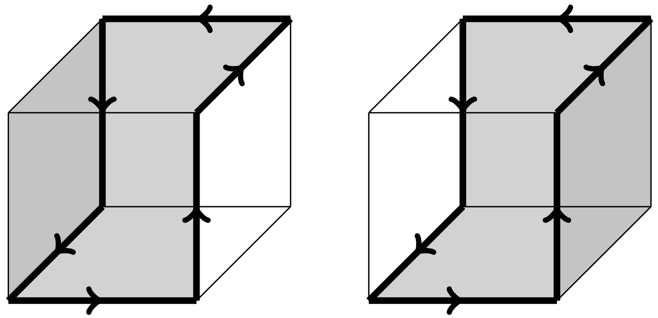
If t_0 and t_1 differ by flips and trits then $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

If t' is a refinement of t, then $\mathrm{Flux}(t') = \mathrm{Flux}(t)$.

(Converse is false, examples in $\mathbb{R}^3/8\mathbb{Z}^3$)

Seifert Surfaces

A (discrete) Seifert surface for a pair t_0, t_1 is a connected embedded oriented topological surface S (mapped continuously and injectively into R^*) with boundary $t_0 - t_1$.



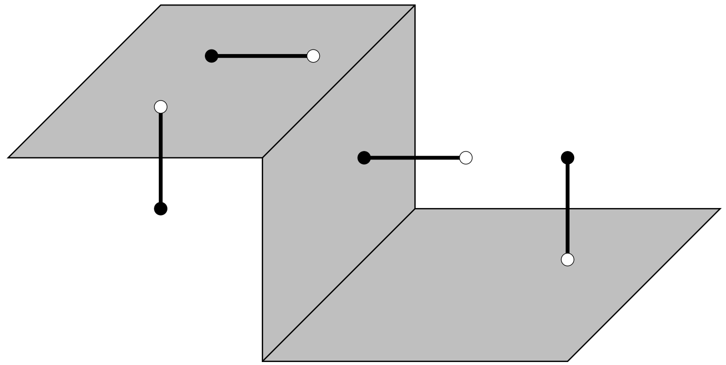
Lemma

If $\text{Flux}(t_0) = \text{Flux}(t_1)$ then for sufficiently large k there exists a discrete Seifert surface for the pair after k refinements.

Flow Through a Surface

$$\varphi(v;t;S) = \mathrm{c}(v) \cdot \begin{cases} +1, & \text{end above } S \\ 0, & \text{end on } S \\ -1, & \text{end below } S \end{cases} \qquad \mathrm{flow}(t;S) = \sum_v \varphi(v;t;S)$$

$\mathrm{c}(v)$ is $+1$ if v is a black tile and -1 if v is a white tile.



Theorem

For S an embedded discrete surface without boundary, if $S = \partial(\text{manifold})$ then $\mathrm{flow}(t;S) = 0$.

Flux vs. flow

- $\text{flow}(t; S)$ really only depends on the homology class of the surface.

Proposition

If $\text{Flux}(t_0) = \text{Flux}(t_1)$ then
$\text{flow}(t_0; a) = \text{flow}(t_1; a)$ for all $a \in H_2(R; \mathbb{Z})$.

Define the *modulus* of a tiling:

$$m := \mu(\text{Flux}(t)) := \gcd_{a \in H_2} \text{flow}(t; a)$$

(**Twist** is well-defined up to the modulus.)

Twist

Fix a base tiling $t_\oplus$.

$$\text{Twist}(t) := \text{flow}(t; t - t_\oplus) \ \in \mathbb{Z}/m\mathbb{Z}$$

Proposition

If $t_0 \rightsquigarrow \text{trit} \rightsquigarrow t_1$ then

$\text{Flux}(t_0) = \text{Flux}(t_1)$ and $\text{Twist}(t_0) = \text{Twist}(t_1) \pm 1$

- Intuitively, the twist records how "twisted" a tiling is by trits.
- If $\text{Flux}(t) = 0$ then $\text{Twist}(t) \in \mathbb{Z}$. (e.g. Boxes)

  (Combinatorial formulation of twist for boxes.)

- Discrete form of helicity.

Main Theorem

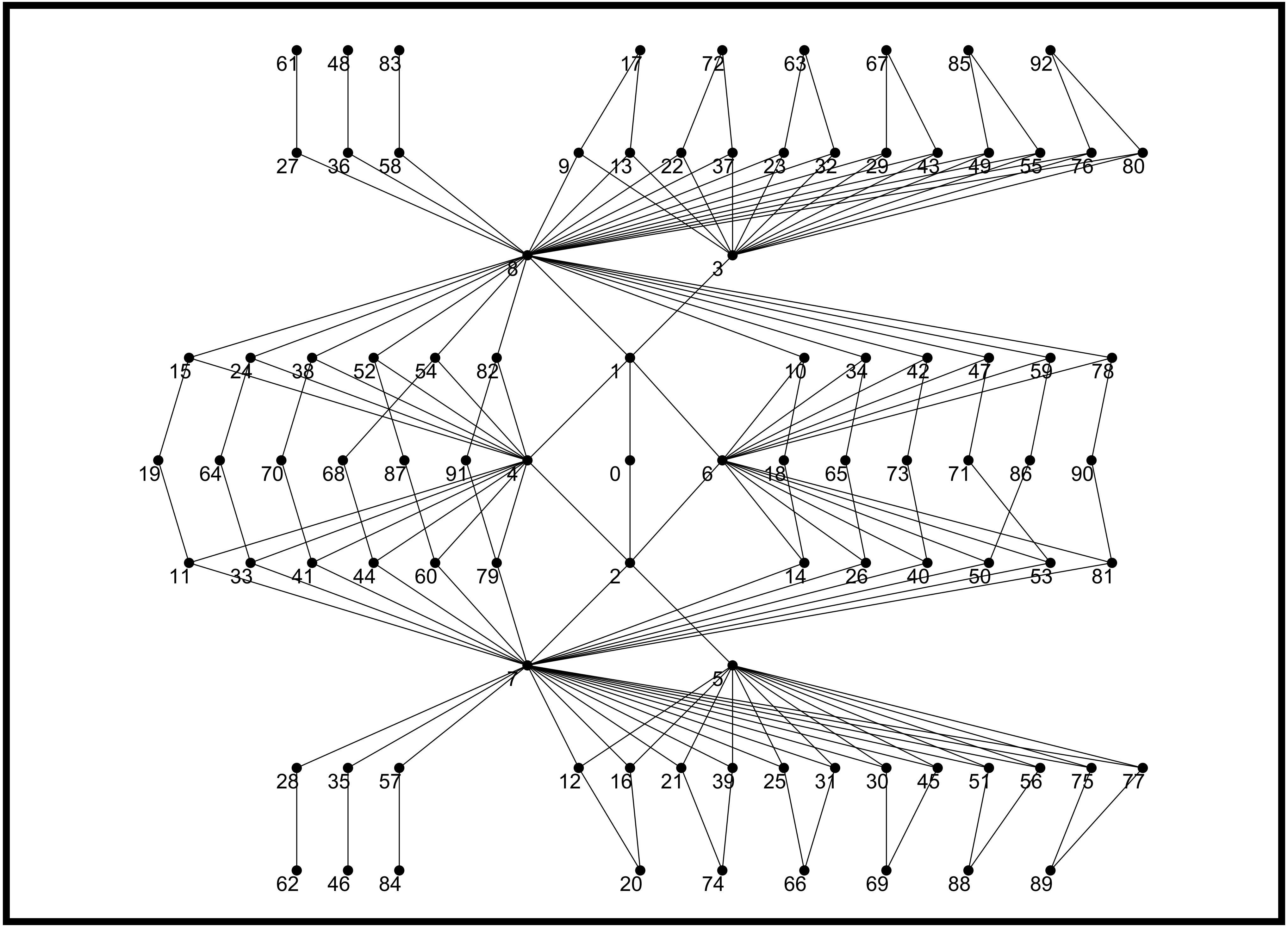
Theorem FKMS '16

For two tilings t_0 and t_1 of R:

- There exists a sequence of flips and trits connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1)$.

Proof: height forms, winding forms.


61 48 83 17 72 63 67 85 92
27 36 58 9 13 22 37 23 32 29 43 49 55 76 80
8 3
15 24 38 52 54 82 1 10 34 42 47 59 78
19 64 70 68 87 91 4 0 6 18 65 73 71 86 90
11 33 41 44 60 79 2 14 26 40 50 53 81
7 5
28 35 57 12 16 21 39 25 31 30 45 51 56 75 77
62 46 84 20 74 66 69 88 89

Questions

Q: How often are refinements necessary?

Open: Are they necessary at all for boxes?

Q: How many refinements are necessary?

Q: How many tilings are there?

Alternating sum based on parity of twist.

Q: How is the twist distributed?

Normally distributed? Giant component?

Q: What does a random tiling look like?