In this talk we will concentrate on finite simplicial complexes (that is, points, line segments, triangles, and higher-dimensional simplices nicely glued together) that triangulate manifolds. A (d-1)-dimensional complex is called balanced if its graph is d-colorable in the usual graph-theoretic sense. After reviewing what is known about the face numbers of triangulated manifolds without the balancedness assumption, we will discuss several very recent balanced analogs of these results. One of them is a lower bound on the number of edges of a balanced triangulation of a manifold M in terms of the number of vertices and the 1st homology of M. The most recent results are joint work with Martina Juhnke-Kubitzke, Satoshi Murai, and Connor Sawaske.

9/1/17
9:30am

Lower Bound Theorems for Manifolds and Balanced Manifolds

(joint work with Juhnke-Kubitzke, Murai and Sawaske)

I. Simplicial complexes

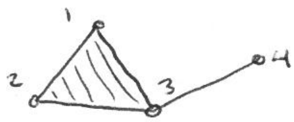
V - vertex set

Δ - simplicial complex is a collection of subsets of V that is closed under inclusion

ie. $F \in \Delta,\ G \subseteq F \Rightarrow G \in \Delta$

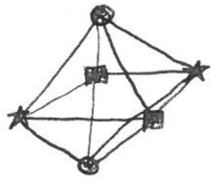
Example: $V = \{1, 2, 3, 4\}$

$$\Delta = \left\{ \begin{array}{l} \{1,2,3\} \quad \{3,4\}, \quad \{1,2\}, \{1,3\}, \\ \{2,3\}, \quad \{1\}, \{2\}, \{3\}, \{4\}, \emptyset \end{array} \right\}$$



$\dim \Delta = 2$

Example:



Elements of Δ are faces

$F \in \Delta \longrightarrow \dim F = |F| - 1$

$\dim \Delta = \max\{\dim F \mid F \in \Delta\}$

vertices - 0-faces

edges - 1-faces

facets - maximal faces under inclusion

Examples: • Boundary complex of simplicial

$\cap$| polytopes

  ◦ simplicial spheres - simplicial complexes whose geometric realization is a

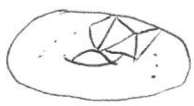
$\cap$| sphere $\quad \|\Delta\| \cong S^{d-1}$

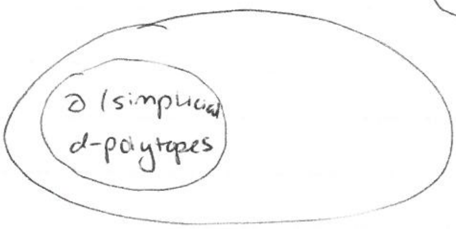
★ not the same in dimension 4 and higher!

  ◦ simplicial manifolds

   $\|\Delta\| \cong$ closed manifold

   ie. triangulation of a torus, etc.





(d-1)-spheres for $d \geq 4$

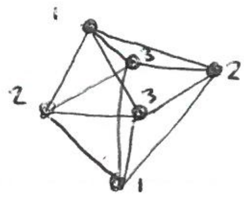
Balanced complexes

• $(d-1)$-simplicial complex Δ is <u>balanced</u> if there exists a coloring

$$K : V \longrightarrow [d]$$

s.t. $K(v) \neq K(w) \quad \forall \{v, w\} \in \Delta$

Example: Octahedron

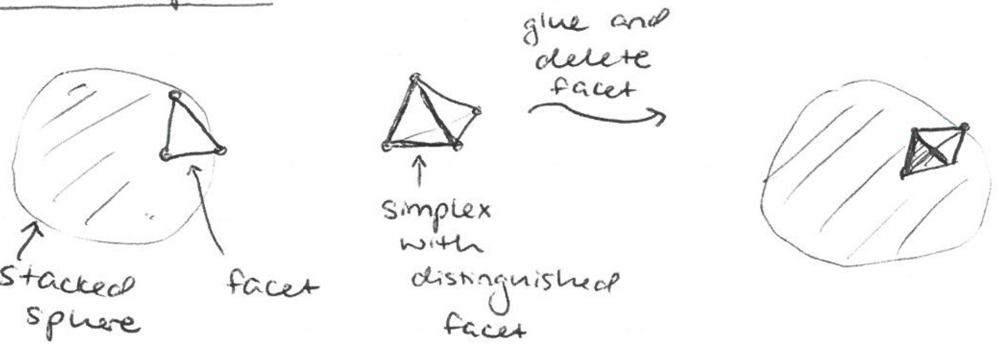


Example: Cross-polytope $\quad \text{conv}\{\vec{e}_1, \dots, \vec{e}_d, -\vec{e}_1, \dots, -\vec{e}_d\}$

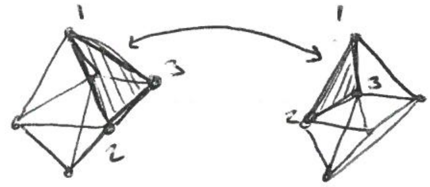
$$K(\vec{e}_i) = K(-\vec{e}_i) = i$$

Since $\vec{e}_i$ and $-\vec{e}_i$ are never adjacent

Stacked Spheres



Stacked Cross-polytopal Sphere



glue mantaining coloring

P - graded poset

$\Delta(P)$ $\quad K(P) = \mathrm{rank}(P)$ is a coloring.

↑ order complex

So the order complex of P is balanced

II. Face Enumeration

Δ $(d-1)$-simplicial complex

$f_i = \#\, i$-faces of Δ

$f_{-1} = 1$

$f_0 = \#$ vertices

$f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ f-vector

$h(\Delta) = (h_0, h_1, \ldots, h_d)$ h-vector

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x-1)^{d-i}$$

$h_0 = f_{-1} = 1$

$h_1 = [x^{d-1}] = f_0 - d f_{-1} = f_0 - d$

$h_2 = f_1 - (d-1) f_0 + \binom{d}{2}$

$h_d = \pm \tilde{\chi}(\Delta)$ ← reduced Euler characteristic (up to a sign)

g-vector: $g_2 = h_2 - h_1$
$\qquad = f_1 - d f_0 + \binom{d+1}{2}$

Example:
$f(\partial(d\text{-simplex})) = \left(1, \binom{d+1}{1}, \binom{d+1}{2}, \ldots, \binom{d+1}{d}\right)$
$\Rightarrow h = (1, 1, \ldots, 1)$

Example: $f_i(\partial(d\text{-crosspoly})) = 2^i \binom{d}{i}$

$\Rightarrow h = \left(1, \binom{d}{1}, \ldots, \binom{d}{d}\right)$

<u>Topology</u>

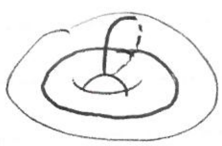
$\tilde{\beta}_i(\Delta) = \dim \tilde{H}_i(\Delta)$

$\tilde{\beta}_i$ counts the number of "i-dimensional holes" for $i > 0$

$\tilde{\beta}_0 = \#\text{connected components} - 1$

Example: $\tilde{\beta}_i(S^{d-1}) = \begin{cases} 1, & i = d-1 \\ 0, & \text{else} \end{cases}$

Example: $\tilde{\beta}_i(S^1 \times S^1) = \begin{cases} 1, & i = 2 \\ 2, & i = 1 \\ 0, & i = 0 \end{cases}$



$m(\Delta)$ = minimal # generators of $\pi_1(\Delta)$

$m(\Delta) \geq \tilde{\beta}_1(\Delta)$

III. Old and New Results

Lower Bound Theorem: • Let Δ be a triangulation of $(d-1)$-dim. connected normal pseudomanifold, $d-1 \geqslant 2$, then $h_2(\Delta) \geq h_1(\Delta)$.

- if $d-1 \geq 3$, equality holds iff Δ is a stacked sphere

(Barnette '73, Kalai '87, Fogelsanger '88, Tay '95)

- If Δ is also balanced, then $h_2/\binom{d}{2} \geqslant h_1/\binom{d}{1}$
- If Δ is balanced and $d-1 \geqslant 3$, then equality holds iff Δ is a stacked polytopal sphere.

(Klee-N., 2016)

Theorem: • Δ - a triangulation of a connected $(d-1)$-dimensional normal pseudomanifold with $d-1 \geq 3$. Then $h_2(\Delta) \geq h_1(\Delta) + \binom{d+1}{1}\tilde{\beta}_1$. (*)

In fact, $h_2(\Delta) \geq h_1(\Delta) + \binom{d+1}{2} m(\Delta)$. (**)

- Equality holds iff Δ is a stacked manifold (for $d-1 \geq 4$)

allowed to add a "handle" by adding a simplex along two facets that are sufficiently far away

(*) (N, Swartz '09, Murai '15)

(**) (Murai, N. '16)

Theorem: (Juhnke-Kubitzke, Murai, N, Sawaske, '16+)

- Δ - a balanced ~~[illegible]~~ simplicial manifold of dim $d-1 \geq 3$. Then
$h_2(\Delta)/\binom{d}{2} \geq h_1(\Delta)/\binom{d}{1} + 2\tilde{\beta}_1(\Delta)$

- if $d-1 \geq 4$, equality holds iff Δ is a "balanced stacked manifold"
{ can add "handles" w/ ~~[illegible]~~ crosspolytopes and must preserve coloring

Higher dimensional faces

Generalized Lower Bound Theorem (Stanley '80):
P a d-dimensional simplicial polytope. Then
$h_0(\partial P) \leq h_1(\partial P) \leq \ldots \leq h_{\lfloor d/2 \rfloor}(\partial P)$

(h-vector symmetric $\Rightarrow$ unimodal sequence)

Balanced analogue: if P is also balanced then $\frac{h_0(\partial P)}{\binom{d}{0}} \leq \frac{h_1(\partial P)}{\binom{d}{1}} \leq \ldots \leq \frac{h_{\lfloor d/2 \rfloor}(\partial P)}{\binom{d}{\lfloor d/2 \rfloor}}$

Theorem: Let Δ be a $(d-1)$-dimensional balanced manifold. If all vertex links are polytopal spheres, then

$$\frac{h_i(\Delta)}{\binom{d}{i}} \geq \frac{h_{i-1}(\Delta)}{\binom{d}{i-1}} + 2\left(\tilde{\beta}_{i-1} - \tilde{\beta}_{i-2} + \ldots \pm \tilde{\beta}_0\right)$$