Matroids and Tropical Geometry

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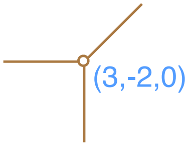
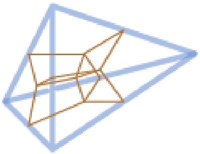
Introductory Workshop: Geometric and Topological Combinatorics
MSRI, September 5, 2017





Preface.

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- This is the Introductory Workshop.
- Focus on accessibility for grad students and junior faculty.
- # (questions by students + postdocs) $\geq$ # (questions by others)

Summary.

- Matroids are everywhere.
- Many matroid sequences are (conj.) unimodal, log-concave.
- Geometry helps matroids.
- Tropical geometry helps matroids and needs matroids.
- (If time) Some new constructions and results.

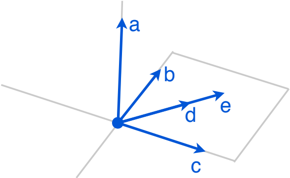
Joint with Carly Klivans (06), Graham Denham+June Huh (17).



Matroids

Goal: Capture the combinatorial essence of independence.

E= set of vectors spanning $\mathbb{R}^d$.
$\mathcal{B}$ = collection of subsets of E which are bases of $\mathbb{R}^d$.



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$\mathcal{B} = \{abc, abd, abe, acd, ace\}$

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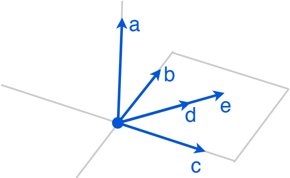

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Properties:
(B1) $\mathcal{B} \neq \emptyset$
(B2) If $A, B \in \mathcal{B}$ and $a \in A - B$,
then there exists $b \in B - A$
such that $(A - a) \cup b \in \mathcal{B}$.

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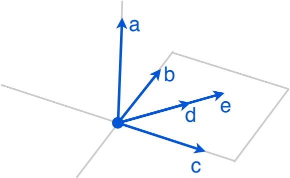

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Definition. A set E and a collection $\mathcal{B}$ of subsets of E are a **matroid** if they satisfies properties (B1) and (B2).

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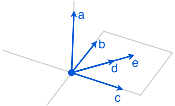
Many matroids in “nature":



1. Linear matroids
E= set of vectors spanning $\mathbb{R}^d$.
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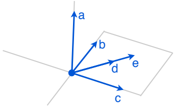

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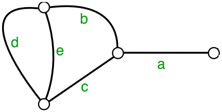

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E= edges of a connected graph G.
$\mathcal{B}$ = spanning trees of G.

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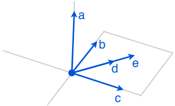

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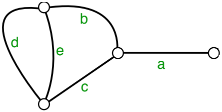

3. Algebraic matroids
E = set of elements in a field extension L/K.
$\mathcal{B}$ = transcendence bases for L/K in E

$a = z^3,\ b = x + y,\ c = x - y$
$d = xy,\ e = x^2y^2.$

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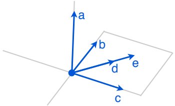

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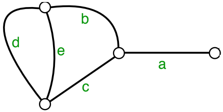

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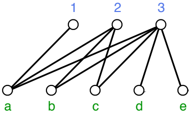
4. Transversal matroids
E = “bottom" vertices of a bipartite graph.
$\mathcal{B}$ = maxl sets that can be matched to the top.



Theorem for matroids $\mapsto$ Theorems for vectors, graphs, field exts, matchings,. . .

Many points of view.

1. Bases

$\mathcal{B} = \{abc, abd, abe, acd, ace\}$

2. Independent sets

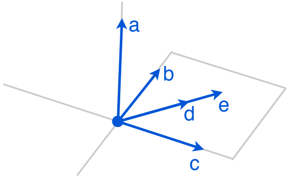
$\mathcal{I} = \{abc, abd, abe, acd, ace,$
$ab, ac, ad, ae, bc, bd, be, cd, ce,$
$a, b, c, d, e,$
$\emptyset\}$

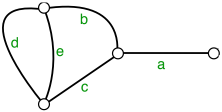
3. Circuits (dependences.)

$\mathcal{C} = \{de, bcd, bce\}$

4. Flats (spanned sets.)

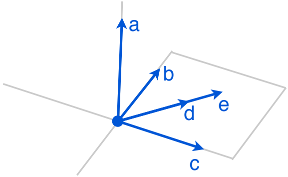
$\mathcal{F} = \{abcde$
$ab, ac, ade, bcde,$
$a, b, c, de,$
$\emptyset\}$

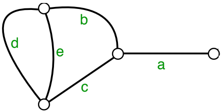




Many points of view.

1. Bases (polytope)
2. Independents (simplicial complex)
3. Circuits (monomial ideal)
4. Flats (poset)





It is as if one were to condense all trends of present day mathematics onto a single finite structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist.

Gian-Carlo Rota

Log concavity for graphs: Read + Hoggar

Proper coloring of a graph $G = (V, E)$:
color each vertex so that no two neighbors have the same color

Chromatic polynomial of G:

$$\chi_G(q) = \#\text{ of proper colorings of } V \text{ with } q \text{ colors}$$
$$\chi_G(q)/q = w_{v-1}q^{v-1} - w_{v-2}q^{v-2} + \cdots \pm w_1$$

Conjecture. (Read 1968, Hoggar 1974) (non-0 part of) the sequence $w_1, \ldots, w_v$ is unimodal $\Leftarrow$ log-concave:

$$w_1 \leq \cdots w_{k-1} \leq w_k \geq w_{k+1} \geq \cdots \geq w_{v-1}$$

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Why care? Log-concavity is easy or quite hard. Progress seems to require new ideas, constructions, connections.

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Note: log-concavity implies unimodality.

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Note: log-concavity implies unimodality.

Theorem. (Huh 2012) This is true.

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Broken circuit: $C - \max_< C$ for a circuit C

Two simplicial complexes from M:

$$\begin{aligned} IN(M) &= \{\text{independent sets}\} \\ \overline{BC}_<(M) &= \{\text{independent sets containing no broken circuit}\} \end{aligned}$$

***f*-vector**: $f_i(\Delta)$ = # of faces $F \in \Delta$ with $|F| = i+1$.

Fact 2. $f_i(\overline{BC}_<(M)) =$ coeffs of char. polynomial χ_M. Check!

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Conjectures. (Welsh 71 Mason 72, Rota 71 Heron 72 Welsh 76)
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Theorem. (Adiprasito–Huh–Katz 2015) These are strictly? true.

Log concavity for matroids, 2: Dawson + Huh

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$$h_0 x^v + h_1 x^{v-1} + \cdots + h_v x^0 = f_0 (x+1)^v + f_1 (x+1)^{v-1} + \cdots + f_v (x+1)^0$$

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>90% Theorem. (A.–Denham–Huh 17) This is true for any M.

Unimodality and log-concavity: relations

Two simple but useful observations (Brylawski, Lenz):

1. $IN(M) = \overline{BC}_{<}(M \times p)$ for $M \times p =$ free dual extension of M
2. $h(\Delta)$ log-concave $\Rightarrow f(\Delta)$ strictly log-concave .

Log-concavity implications:

$$\begin{array}{ccc} h_i(\overline{BC}_{<}(M)) & \longrightarrow & f_i(\overline{BC}_{<}(M)) \\ \downarrow & & \downarrow \\ h_i(IN(M)) & \longrightarrow & f_i(IN(M)) \end{array}$$

The log-concavity of $h_i(\overline{BC}_{<}(M))$ implies all the others.

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Juhnke-Kubitzke, Le, 2016: It also implies Swartz's conjecture:

$h_i(\overline{BC}_<(M))$ is **flawless** : $h_i \leq h_{s-i}$ for $i \leq s/2$.

Chari 97 and Swartz 03 proved this for $h_i(IN(M))$ only.

~~matroids~~ $\rightarrow$ geometries

When I wrote my book on 'matroids, I changed the name. I called it "Combinatorial Geometries" – but it didn't take. They said "that's really matroids, isn't it?"

Gian-Carlo Rota, Combinatorial Theory, Fall 1998. (Thanks to John Guidi.)

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(linear matroids) vs. (all matroids):

- Almost any matroid we come up with is linear (geometric).
- (Nelson, 16) Almost all matroids are not linear.

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- Almost any matroid we come up with is linear (geometric).
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- “Missing axiom" for linear matroids? No. (Mayhew et al, 14)
- This is not a flaw. Matroids are natural geometric objects.

Strategy: geometric models of matroids

To prove log-concavity of invariants of a **linear** matroid M:

1. Build an algebro-geometric model $X(M)$ for M.
2. (Combin invariants of M) = (Geom invariants of $X(M)$).
3. Algebraic-geometric inequalities for geometric invariants.

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Two algebro-geometric models.

$f_i(\overline{BC}_<(M))$: wonderful compactification $DP(\mathcal{A})$.

De Concini–Procesi 95

$h_i(\overline{BC}_<(M))$: critical set variety $\mathfrak{X}(\mathcal{A})$.

Varchenko 95, Orlik–Terao 95, Denham–Garrousian–Schulze 12

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Good news: This strategy works! (Huh, 2012, 15)

Bad news: ...only when M is a linear matroid.

Strategy of proofs: tropical geometric models

To prove log-concavity of invariants of **any** matroid M:

1. Build a tropical geometric model $X(M)$ for M.
2. (Combin invariants of M) = (Trop geom invariants of $X(M)$).
3. Algebro-geom inequalities for tropical geometric invariants.

Two tropical geometric models.

$f_i(\overline{BC}_<(M))$: tropical linear space $\mathrm{Trop}(M)$.

Sturmfels 02, A.–Klivans 03

$h_i(\overline{BC}_<(M))$: combinatorial critical set variety $\mathrm{Crit}(M)$.

A.–Denham–Huh 17

Good news: This works even when M is not realizable!

(Good or bad) news: We have to work harder for our inequalities.

Tropical geometry: a general philosophy

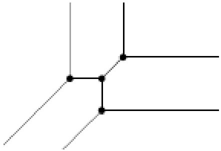
Tropicalization is a very useful general technique:

$$\begin{array}{rcl} \text{algebraic variety} & \mapsto & \text{tropical variety} \\ V & \mapsto & \mathrm{Trop}(V). \end{array}$$

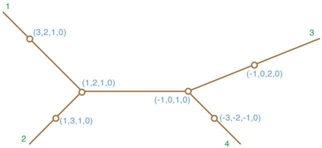
Idea: Obtain information about V from $\mathrm{Trop}(V)$.

o $\mathrm{Trop}(V)$ is simpler, but still contains information about V.

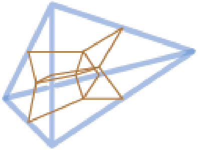
o $\mathrm{Trop}(V)$ is a polyhedral complex, we can do combinatorics.



conic in 2-space



line in 3-space



plane in 2-space

Tropicalization.

To tropicalize a projective variety, take **all** the equations it satisfies, and change:

$$\begin{array}{rcl} X+Y & \mapsto & \min(x,y) \\ X\cdot Y & \mapsto & x+y \\ \text{multiplicative scalar} & \mapsto & 0 \\ \text{additive scalar} & \mapsto & \text{ignore} \\ =0 & \mapsto & \text{is achieved} \geq \text{twice} \end{array}$$

For example, the 2-D surface in $\mathbb{C}^4$:

$$2X+Y=0, X^3+Z^2W+W^3=0$$

becomes the 2-D polyhedral complex in $\mathbb{R}^4$:

$\min(x,y)$, $\min(3x,2z+w,3w)$, $\min(3y,2z+w,3w)$,...
are achieved twice.

I am oversimplifying in some ways.

Tropical linear spaces = matroids

V = linear subspace of $\mathbb{C}^n$
Trop V = tropical linear space.

$w \in \operatorname{Trop} V$ $\leftrightarrow$ for each **circuit** $a_1 X_{i_1} + \cdots + a_k X_{i_k} = 0$ of V, $\min(w_{i_1}, \ldots, w_{i_k})$ is achieved $\geq$ twice.

Corollary: Trop V only depends on the matroid of V.

Example. $L = \{X \in \mathbb{R}^4 : X_1 - X_2 + X_3 = 0 \, , \, X_4 = 2X_3\}$

Circuits: $123, 34, 124$.

Trop L: $\min(w_1, w_2, w_3), \min(w_1, w_2, w_4), \min(w_3, w_4)$ att. $\geq$ twice.

Tropical linear spaces = matroids.

Example. (cont.) Circuits: $123, 34, 124$.

Trop L: $\min(w_1, w_2, w_3), \min(w_1, w_2, w_4), \min(w_3, w_4)$ att. $\geq$ twice.

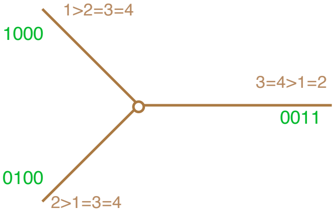
Whether $w \in \operatorname{Trop} L$ depends on relative order of $w_1, \ldots, w_4$.

- $w_3 = w_4$
- $w_1 > w_2 = w_3$ or $w_2 > w_1 = w_3$ or $w_3 > w_1 = w_2$.

$\downarrow$

Three rays: e_1, e_2, e_{34}

Q: What are these in general?



Tropical linear spaces = matroids

Definition. (Sturmfels 02) The **tropical linear space** of M is

$$\mathrm{Trop}(M) = \{w \in \mathbb{R}^E : \text{ for every circuit } C \text{ of } M, \ \min_{c \in C} w_c \text{ is achieved} \geq \text{twice}\}$$

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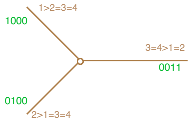
Let $e_{adf} = (1,0,0,1,0,1)$.

If $\mathcal{F} = \{\emptyset \subset F_1 \subset \cdots \subset F_r = E\}$ is a flag of flats

$$\sigma_{\mathcal{F}} := \mathrm{cone}(e_{F_1}, \ldots, e_{F_r}) \subseteq \mathbb{R}^E$$



Theorem. (A. - Klivans 03)
The tropical linear space Trop(M) has
- rays: e_F where F is a flat
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$$\mathrm{Trop}(M) = \bigcup_{\mathcal{F} \text{ flag}} \sigma_{\mathcal{F}}$$

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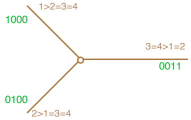

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So we can recover M from Trop(M). (for M simple.)

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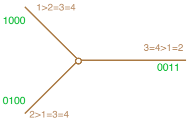

There is also an intrinsic tropical definition:

Theorem / Definition. (Fink '09)
A tropical linear space is an **abstract tropical variety** of **degree** 1.

So matroids arise very naturally – even non-linear ones!

Orthogonality for matroids

Theorem / Definition. If $\mathcal{B}$ is a matroid on E, then

$$\mathcal{B}^{\perp} = \{E - B : B \in \mathcal{B}\}$$

is also a matroid, the **orthogonal** or **dual** matroid $M^{\perp}$.

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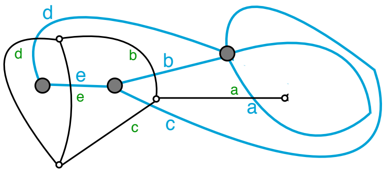
This generalizes:

- Dual graphs:
  abe spanning tree of G
  $\updownarrow$
  cd spanning tree of G^*



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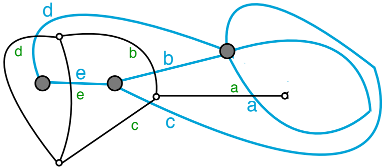

- Orthogonal complements:

  abe basis of W

  $\updownarrow$

  cd basis of $W^{\perp}$

$$W = \text{rowspace}\begin{bmatrix} 0 & 1 & 0 & .5 & 1 \\ 0 & 0 & 1 & .5 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W^{\perp} = \text{rowspace}\begin{bmatrix} 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix}$$

Tropical critical set variety

Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ = standard bases of $\mathbb{R}^E$ and $\mathbb{R}^E$.

Definition. (A. – Denham – Huh 17)
The **tropical critical set variety** $\mathrm{Crit}(M) \subset \mathbb{R}^E \times \mathbb{R}^E$ of M has
- rays: $e_F + f_G$ where F is a flat and G is a coflat with $F \cup G = E$.
- cones: $\tau_{\mathcal{F},\mathcal{G}} := \mathrm{cone}(e_{F_1} + f_{G_1}, \ldots, e_{F_{n-1}} + f_{G_{n-1}})$

for each pair $\mathcal{F}, \mathcal{G}$ of **compatible** flags of flats and coflats.

where

Definition. (A.–Denham–Huh 17) Say two flags

$\mathcal{F} = \{\emptyset \subseteq F_1 \subseteq \cdots \subseteq F_k = E\}$ of flats

$\mathcal{G} = \{E \supseteq G_1 \supseteq \cdots \subseteq G_k \subseteq \emptyset\}$ of coflats (flats of $M^\perp$)

are **compatible** if

$$\bigcap_{i=1}^{k} (F_i \cup G_i) = E, \qquad \bigcup_{i=1}^{k} (F_i \cap G_i) \neq E.$$

(Motivation: toric+trop geometry, hyperplane arrangements, Coxeter combin)

Trop M and Crit M as configuration spaces

...if there is time...

Tropical critical set variety

Why is this a useful construction?

Definition. (A.–Denham–Huh 17) The **Chow ring** of Crit(M) is

$$A_{M,M^\perp} = \mathbb{Z}[x_{F,G} : F \text{ flat}, G \text{ coflat}, F \cup G = E]/(I_M + J_M)$$

where

$$I_M = (x_{F_1,G_1} \cdots x_{F_k,G_k} : \{F_i\} \text{ and } \{G_i\} \text{ are not compatible})$$

$$J_M = \left(\sum_{i \in F \neq E} x_{F,G} - \sum_{j \in F \neq E} x_{F,G}, \sum_{i \in G \neq E} x_{F,G} - \sum_{j \in G \neq E} x_{F,G} : i,j \in E \right)$$

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It behaves like the Chow ring of a smooth proj. alg. variety; e.g.:

Poincaré duality	:	$A = A_0 \oplus \cdots \oplus A_{n-1}, \qquad A_i \cong A_{n-1-i}$
Hard Lefschetz theorem	:	$\dim A_0 \leq \cdots \leq \dim A_{(n-1)/2} \geq \cdots \geq \dim A_{n-1}$
Hodge-Riemann relations	:	imply log-concavity results

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the “hyperplane" and “cohyperplane" classes are

$$a = \sum_{i \in F \neq E} x_{F,G}, \qquad d = \sum_{i \in F,G} x_{F,G}$$

$A_{n-1} \cong A_0 = \mathbb{Z} \quad \Rightarrow \quad$ degree $n-1$ elements are just integers!

Theorem. (A.–Denham–Huh 17) In the **Chow ring** of Crit(M)

$$a^{r-1-i} d^{n-r-i} = h_i(\overline{BC}_<(M)) \qquad (1 \leq i \leq r-1)$$

This program should imply the log-concavity of $h_1, \ldots, h_{r-1}$.

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Trop(M)	Crit(M)
(A.–Klivans, 06) Trop(M) $=$ wedge of $\lvert\mu(M)\rvert$ $(r-1)$-spheres	?
(A.–Klivans–Williams, 06) (M oriented) • Trop$^+(M) =$ sphere. • $\lvert\mu(M)\rvert$ reorientations whose Trop$^+$ cover Trop(M).	?
(A.–Reiner–Williams, 06) ($M =$ root system) Trop$^+(M) =$ graph associahedron of Dynkin dgm	?

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3. Connection w Chern-Schwartz-MacPherson classes $c_k(M)$?
$\deg c_k = h_k(BC_<)$. (López de Medrano–Rincón–Shaw, 17)

muchas gracias.