(Unimodular) triangulations of lattice polytopes

Francisco Santos

Theorem 1: ("Dialations", Haase-Paffenholz-Pichnik, S., 2014) For every lattice $d$-polytope with volume $V$, the dialation $cP$ has regular unimodular triangulations where $c = d^{vd\,vd^2}$

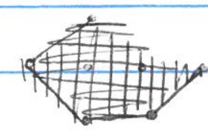
Theorem 2: ("Empty 4-simplices", Iglesias-Valino, Blanco-Haase-Hofmann, S.) The whole list of empty 4-simplices consist of

- 1 3-parameter family
- 2 2-parameter families
- 29+23 1-parameter families
- 2461 sporadic ones

Proof of Pick's Theorem:

$i$ = # interior points

$b$ = # boundary points



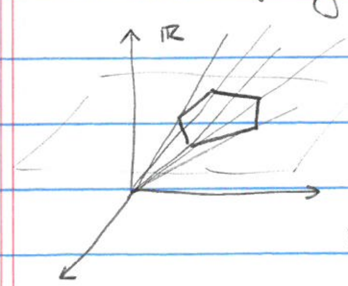
- $f_0 - f_1 + f_2 = 1$
- $3f_2 = 2f_1 - b$
- Area $= f_2/2$
- $f_0 = i + b$

b/c of unimodular triangulation

$\Rightarrow f_2/2 = i + b/2 - 1 =$ area

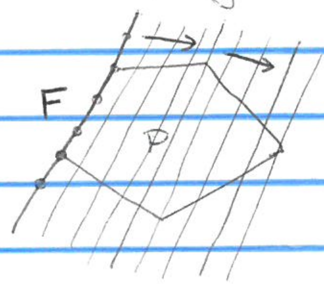
∎

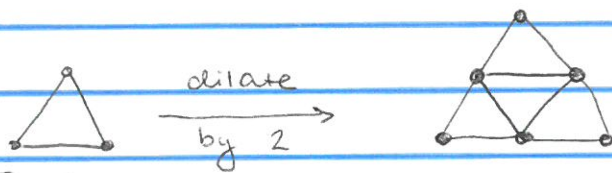
Ehrhart polynomial: $k \mapsto \#(kP \cap \mathbb{Z}^d)$




$\Big\}\ \mathbb{R}^{d+1}$

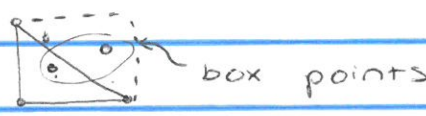
Compressed polytopes



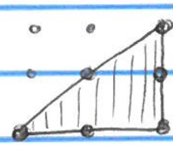




Empty
simplex

Reducing the volume of a single dilated simplex:




Hollow but not empty:



# (Unimodular) triangulations of lattice polytopes

Francisco Santos,U. Cantabria, Spain.
http://personales.unican.es/santosf

GTC intro workshop, MSRI, Sep. 5, 2017

# Definitions

Lattice polytope $P$ in $\mathbb{R}^d := \text{conv}(S)$, $\quad S \in \mathbb{Z}^d$, finite.

Unimodular simplex := vertices are an affine basis of $\mathbb{Z}^d$.
(Equivalently, normalized volume equal to 1)

(Lattice) subdivision of $P$: "face to face" decomposition into lattice subpolytopes.

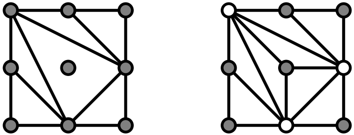
(Lattice triangulation) of $P$: same, into simplices.

Unimodular triangulation: triangulation into unimodular simplices.

# Dim 2 versus higher dim

**Proposition**

*Every lattice polygon has a unimodular triangulation.*



**Proof.**

Every lattice polytope can be triangulated into empty simplices (lattice simplices w.o. extra lattice points). In dim 2 all empty simplices are unimodular. □

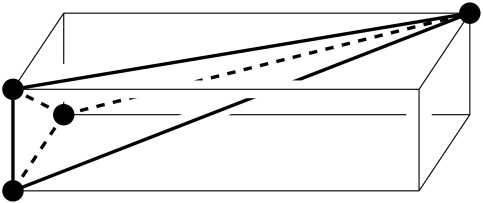
**Corollary (Pick's Theorem)**

$$Area(P) = |int(P) \cap \mathbb{Z}^2| + \frac{1}{2}|\partial P \cap \mathbb{Z}^2| - 1.$$

# Dim 2 versus higher dim

**Remark**

In dim $\geq 3$ there are empty non-unimodular simplices $\Rightarrow$ there are polytopes without unimodular triangulations.



# Integer programming

Let $A$ be an integer matrix and $b$ an integer vector. If the normal fan of $P = \{Ax \leq b\}$ has a unimodular triangulation (using only facet normals as vertices) then the system $Ax \leq b$ is totally dual integral.

In particular, all vertices of $P$ are integral ($P$ is a lattice polyhedron) and integer programming on $P$ is as easy" as linear programming.

# Counting lattice points

The *Ehrhart series* of a lattice polytope $P$ counts how many lattice points lie in $kP$, for $k \in \mathbb{N}$.

It is known that its generating function can be rewritten as

$$\sum_{k \geq 0} \#(kP \cap \mathbb{Z}^d)\ t^k \;=\; \frac{h^*_P(t)}{(1-t)^{d+1}},$$

for a certain polynomial $h^*_P$ of degree (at most) $\dim(P)$.

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**Theorem (Stanley 1996)**

*If $\mathcal{T}$ is a unimodular triangulation of $P$, then $h^*_P$ equals the h-polynomial $h_{\mathcal{T}}(x)$ of $\mathcal{T}$. (That is, the coeffs of $h^*_P$ coincide with the h-vector of $\mathcal{T}$).*

# Unimodular triangulation $\Rightarrow$ integrally closed

If $P$ has a unimodular triangulation then the cone $\sigma_P$ generated by $P \times \{1\} \subset \mathbb{R}^{d+1}$ is generated in degree one: every lattice point in $kP$, $k \in \mathbb{N}$, decomposes as the sum of $k$ points in $P$.

**Definition**

We call $P$ *integrally closed* if this happens. (Other names exist: "integer decomposition property", "normal").

If $P$ is integrally closed:

- $P \cap \mathbb{Z}^d$ is the Hilbert basis for $\sigma_P$.
- The semigroup algebra $R_P = \mathbb{K}[\sigma_P \cap \mathbb{Z}^{d+1}]$ is an integral domain and generated in degree one.

# Smooth polytopes, "Oda's question"

Let $X_P = \operatorname{Proj} \mathbb{K}[\sigma_P \cap \mathbb{Z}^{d+1}] = \operatorname{Proj} R_P$ be the projective variety associated to $\sigma_P$ and consider its natural embedding $X_P \hookrightarrow \mathbb{P}^{n-1}$ (where $n = |P \cap \mathbb{Z}^d|$).

Then,

- $X_P$ is projectively normal $\Leftrightarrow$ $P$ is normal.
- $X_P$ is smooth $\Leftrightarrow$ $P$ is simple and every vertex cone is unimodular (we say then that "$P$ is smooth").

**Oda's conjecture**

Every smooth $X_P$ is projectively normal.

This would follow from

**Conjecture**

Every smooth lattice polytope $P$ has a unimodular triangulation.

# Algebraic geometry

*Regular* unimodular triangulations of $P$ correspond to certain (so-called crepant) resolutions of the singular point in the affine toric variety

$$U_P = \operatorname{Spec} \mathbb{K}[\sigma_P^\vee \cap \mathbb{Z}^{d+1}].$$

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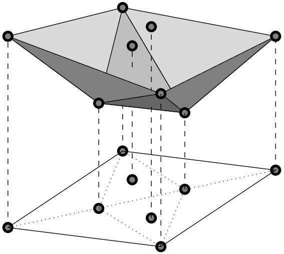
**Theorem (Knudsen-Mumford-Waterman, 1973)**

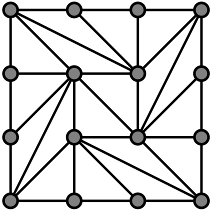
*For every lattice polytope $P$ there is a dilation factor $c \in \mathbb{N}$ such that $cP$ admits a regular unimodular triangulation.*

# Regular triangulations

A triangulation (unimodular or not) is called regular if its simplices are the domains of linearity of a piece-wise convex function $P \to \mathbb{R}$.



Regular



Non-regular

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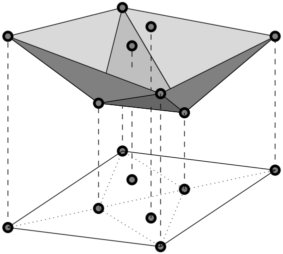

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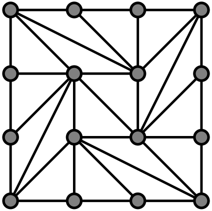

Non-regular

A quadratic triangulation is a regular, unimodular, and flag triangulation (flag:= every clique in the graph spans a simplex).

# Compressed polytopes

A compressed polytope $P$ is a polytope of width one with respect to every facet. That is, for every facet hyperplane $H$ of $P$, all vertices of $P$ not in $H$ lie in the next lattice translation of $H$.

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All compressed polytopes have regular unimodular triangulations.
In fact, all their pulling triangulations are unimodular.

We can use this to show that

**Theorem (S. 1996, Haase-Paffenholz-Piechnik-S 2014+ for flagness)**

*If a polytope $P$ has a (regular, flag) unimodular triangulation $T$ then every integer dilation $cP$ of it has one too.*

# Compresed polytopes

**Sketch of proof.**

Consider the dilation $cT$ of $T$, which subdivides $cP$ into dilated unimodular simplices.

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Any pulling refinement of this subdivision is unimodular. □

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**Definition**

Let $Q \subset \mathbb{R}^d$ and $P_i \subset \mathbb{R}^{d_i}$ for $i = 1, \ldots, n$ be lattice polytopes, and let $\phi : \mathbb{Z}^d \to \mathbb{Z}^n$ be an integer affine map with $\phi(Q) \subset \mathbb{R}_{\geq 0}$.

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$$Q \ltimes_\phi (P_1, \ldots, P_n) := \operatorname{conv}_{a \in Q} \left( \{a\} \times \prod \phi_i(a) P_i \right),$$

where $(\phi_1, \ldots, \phi_n)$ are the coordinates of $\phi$.

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- $\{\mathrm{pt}\} \ltimes_k (P)$ is the $k$-th dilation of $P$.
- The *chimney* (Haase-Paffenholz 2007)

$$\mathrm{chim}(Q, f, g) := \{(x, t) \in \mathbb{R}^{d+1} : f(x) \le t \le g(x)\}$$

associated to two integer functionals $f \le g$ on $Q$ is the semidirect product $Q \ltimes_{g-f} I$, where $I$ is a unimodular segment.

# Semidirect product

**Theorem (Aoki et al. 2008, HPPS 2014+)**

*If $Q, P_1, \ldots$,and $P_n$ admit unimodular triangulations, then every semidirect product $Q \ltimes_\phi (P_1, \ldots, P_n)$ admits one too.*

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**Remark**

Semidirect product is essentially equivalent to nested configurations [Aoki et al. 2008]. Aoki et al. prove the theorem above under the assumption that all factor triangulations are regular.

# Polytopes from root systems

(Crystallographic) root systems give examples of particularly nice lattices. It seems natural to look at lattice polytopes related to them. We can do this in two ways:

- Polytopes cut out by roots: facet normals belong to the root system (=:alcoved polytopes).

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We concentrate on the first type.

# Alcoved polytopes of type $A$

Payne (2009) has proved that all alcoved polytopes in the classical types $A$, $B$, $C$ and $D$ are integrally closed. This suggests they may all have unimodular triangulations.

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In type $A$ this is easy to show. Remember that.

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Hence:

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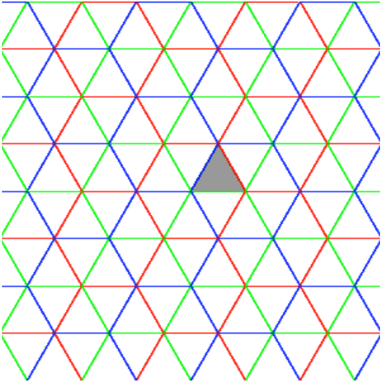

The canonical triangulation for the root system $A_2$

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## Theorem

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**Remark**

*If $\Delta = \text{conv}\{v_1, \dots, v_n\}$ is any lattice simplex with its vertices given in a specific order, we can consider the linear map sending its facet normals to the (normals of) the simple roots of type $A_n$, in that order. The preimage of the $A$-dicing gives a canonical triangulation of $c\Delta$, for every $c \in \mathbb{N}$, into simplices of the same volume as $\Delta$.*

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*This canonical triangulation will be important in our proof of the KMW theorem.*

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First slice $P$ by the hyperplanes corresponding to the "short roots" of type $B$. This gives a regular subdivision into compressed cells. Any pulling refinement of this is unimodular. □

The triangulation in the theorem may need to use simplices that are not alcoved.

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For other types:

- In $F_4$ and $E_8$ we have explicit examples of polytopes without r.u.t.'s
- In $C_n$, $D_n$, $E_6$ and $E_7$ we do not know.

## The KMW Theorem

x

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We recall the following classical theorem of Knudsen, Mumford, and Waterman (1973):

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*Given a polytope $P$, there is a factor $c = c(P) \in \mathbb{N}$ such that the dilation $c \cdot P$ admits a regular unimodular triangulation.*

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  - There are examples where $cP$ has a r.u.t. but $(c+1)P$ is not even integrally closed [Cox-Haase-Hibi-Higashitani 2012].
- What is a (good?) bound on $c(P)$ for a given $P$?

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- Neither the original KMW proof nor the reworking of it by Bruns and Gubeladze (2009) contains any explicit bound on the $c$ needed for a given $P$.
- Working out a bound from those proofs is not easy, and would certainly lead to a tower of exponentials of length related to the volume of $P$.

# The KMW Theorem

Concerning the last question:

- Neither the original KMW proof nor the reworking of it by Bruns and Gubeladze (2009) contains any explicit bound on the $c$ needed for a given $P$.
- Working out a bound from those proofs is not easy, and would certainly lead to a tower of exponentials of length related to the volume of $P$.
- The regularity part of the proof is not totally clear (it is omitted in [Bruns-Gubeladze 2009]).

# An effective KMW Theroem

**Theorem (Effective KMW Theorem, Haase-Paffenholz-Piechnik-S 2014+)**

*If a lattice polytope $P$ has a triangulation into lattice simplices of (lattice) volume bounded by $V$, then the dilation*

$$d!^{\mathrm{vol}(P)V!d^{d^2V}}P$$

*has a regular unimodular triangulation.*

Idea of proof: While $V > 1$, show that dilating $P$ sufficiently many times you can triangulate $cP$ into simplices of volume $< V$ (and get bounds on $c$).

# Canonical triangulation

Our proof is not substantially different from the previous ones, but uses a better "book-keeping" based on the canonical triangulation of dilations of an ordered simplex:

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The canonical triangulation of $c\Delta$ is the inverse image of the dicing triangulation of type $A$, under the natural affine map sending $\Delta$ to an alcoved simplex of type $A$.

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*If $\mathcal{T}$ is a triangulation of $P$, canonically refining each simplex of $c\mathcal{T}$ produces a triangulation of $cP$ in which:*

- *Volume of simplices is preserved. (Each simplex in the final triangulation has the volume of the simplex of of $\mathcal{T}$ that it refines).*

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- *Regularity is preserved.*

# Reducing the volume of a single dilated simplex

Let $\Delta$ be a non-unimodular simplex. let $\Lambda_\Delta$ be the lattice spanned by its vertices (rather, the linear lattice parallel to it...), so that $\mathrm{vol}(\Delta) = |\mathbb{Z}^d / \Lambda_\Delta|$. A box point is a non-zero element of this quotient.

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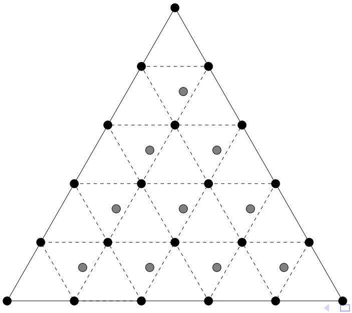

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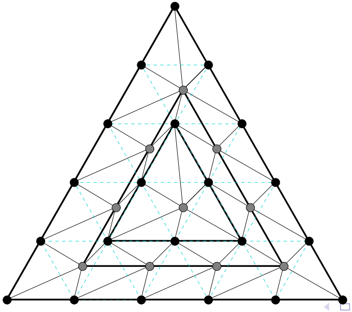

# Reducing the volume in a single simplex

**Lemma (Elementary volume reduction)**

*If $\mathcal{T}$ is a lattice triangulation on an ordered set of vertices and $F = \{v_0, \ldots, v_k\}$ is a non-unimodular face with a box point $m = (m_0, \ldots, m_k) \in \mathbb{Z}^d \setminus \Lambda_F$, then for every integer $c \in (k+1)\mathbb{N}$, $c \cdot \overline{\mathrm{Star}}(F; \mathcal{T})$ has a refinement $\mathcal{T}_m$ such that:*

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1. *The volume of every full-dimensional simplex $\Delta'$ in $\mathcal{T}_m$ is strictly less than the volume of simplex $\Delta$ for which $\Delta' \subset c\Delta$.*
2. *$\mathcal{T}_m$ induces the canonical triangulation on the boundary $c \cdot \partial\,\mathrm{Star}(F; \mathcal{T})$.*
3. *$\mathcal{T}_m$ is a regular refinement of $\mathcal{T}$, so if $\mathcal{T}$ is regular then $\mathcal{T}_m$ is regular.*

# Reducing the volume in several simplices at a time

Remarks:

- If we have box-points $m_1, \ldots, m_N$ for a family of simplices $F_1, \ldots, F_N$ with disjoint stars, the reduction lemma can be applied simultaneously to all of them, to reduce the volumes in all stars simultaneously.

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- This happens, for example, for all simplices of prime volume:

**Corollary**

*Let $\mathcal{T}$ be a triangulation of a lattice polytope $P$ and assume that the maximal volume $V$ among all simplices in $\mathcal{T}$ is a prime. Then $(d+1)!\mathcal{T}$ can be refined to a triangulation with all simplices of volume $< V$.*

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... if every number was a prime, $(d+1)!^V\mathcal{T}$ would have a unimodular refinement ....

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We solve it by taking advantage of some properties of canonical triangulations.

# Canonical refinement, revisited

## Definition

An ordered $k$-simplex is a simplex with a specified order in its vertices. Two ordered simplices $\Delta = \text{conv}\{p_0, \dots, p_k\}$ and $\Delta' = \text{conv}\{p'_o, \dots, p'_k\}$ are called $A$-equivalent if

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## Lemma ($A$-equivalence)

1. *All the simplices in the canonical triangulation of $c\Delta$ are $A$-equivalent to $\Delta$.*
2. *If two simplices $\Delta$ and $\Delta'$ are $A$-equivalent then the $A$-dicing defined by $\Delta$ and by $\Delta'$ are the same, modulo a translation.*

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Part (2) of the previous lemma allows us to consider a box point for a simplex $\Delta$ as a box point for any other $A$-equivalent simplex $\Delta'$ (by the unique, modulo $L_\Delta$ translation sending one $A$-dicing to the other).

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The crucial property that we need is:

**Lemma**

*Let $\Delta$ and $\Delta'$ be two $A$-equivalent simplices in a triangulation $\mathcal{T}$, and let $m$ be a box point for both (in the above sense). Let $F$ and $F'$ be the faces of $\Delta$ and $\Delta'$ having $m$ in their relative interior. Then, either $F = F'$ or they have disjoint stars.*

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**Corollary**

*Let $\mathcal{T}$ be a triangulation of a lattice polytope $P$ and let $V$ be the maximal volume $V$. Let $N$ be the number of A-equivalence classes of maximal simplices of volume $V$ in $\mathcal{T}$.*
*Then, $(d+1)!^N \mathcal{T}$ can be refined to a triangulation with all simplices of volume $< V$.*

# An algorithm

To get a unimodular refinement of $cP$ for some constant $c$:

1. Construct any lattice triangulation $\mathcal{T}$ of $P$. Let $V$ be the maximal volume among its simplices and $N$ the number of $A$-equivalence classes of them.
2. While $N > 0$, apply the reduction lemma (that is, dilate by $d!$ and refine) to all the simplices in one of the $A$-equivalence classes of volume $V$. This reduces by (at least) one the number of them.
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Remark: in all steps regularity of the triangulation can be preserved.

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All these results heavily rely on the classification of empty 3-simplices (White 1964).

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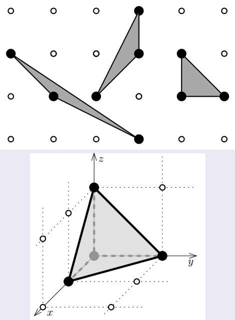

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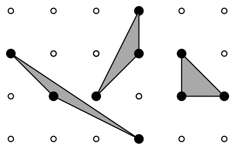

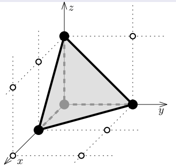

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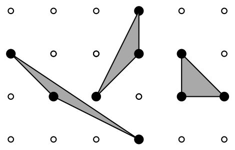

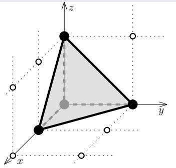

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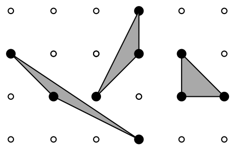

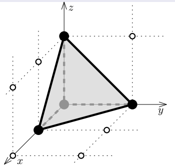


**Goal:** Classify empty simplices (in low dimensions). "Classify" means modulo lattice automorphisms ($\Leftrightarrow$ affine integer transformations).

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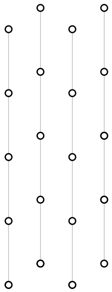

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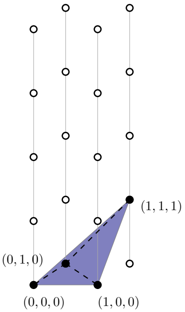

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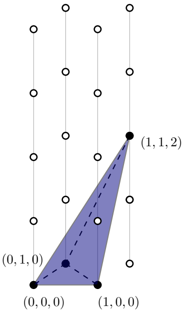

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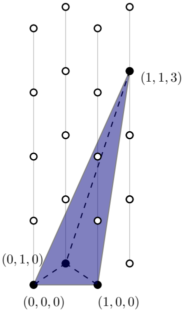

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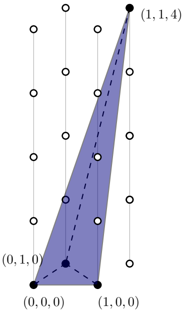

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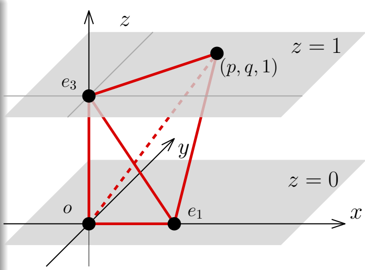
**Theorem (White 1964)**

*Every empty tetrahedron has* **width one** *with respect to a pair of opposite edges.*



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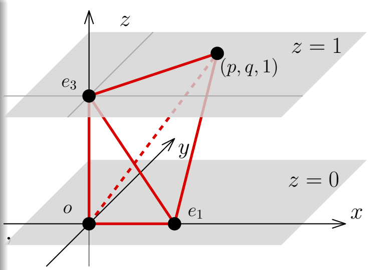
$\text{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$.
($q \in \mathbb{N}$, $p \in \mathbb{Z}$, $\gcd(p,q) = 1$).



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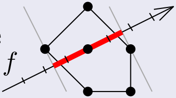

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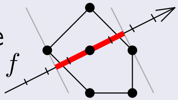


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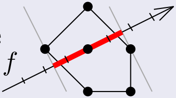


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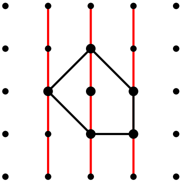

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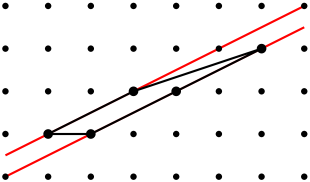

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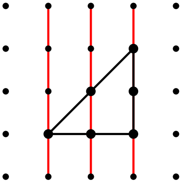

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- There are infinitely many empty 4-simplices of width two (e. g., $\Delta(2, 2, 3, D-6)$ when $\gcd(D, 6) = 1$).
- There are (at least) 178 of width three plus one of width 4).

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- There are infinitely many empty 4-simplices of width two (e. g., $\Delta(2, 2, 3, D-6)$ when $\gcd(D, 6) = 1$).
- There are (at least) 178 of width three plus one of width 4).

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Observe that $|\mathbb{Z}^d/L(\Delta)|$ equals the *(normalized) volume* (or the *determinant*) of $\Delta$.

In dimension $\geq 5$ there are non-cyclic empty simplices.

# Classification of terminal quotient singularities

Another classification comes from algebraic geometry, where *terminal quotient singularities* of a certain dimension are in bijection to empty simplices (together with a choice of a vertex to be the origin).

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1. There are 1+1+29 infinite families with three, two, and one parameter respectively.
2. Up to volume 419 there are some 4-simplices not in those families, but between 420 and 1600 there are none.

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1. There are 1+1+29 infinite families with three, two, and one parameter respectively.
2. Up to volume 419 there are some 4-simplices not in those families, but between 420 and 1600 there are none. They conjectured:

> CONJECTURE 1.4 (four-dimensional terminal lemma). *Fix* $p \geq 421$. *Up to the actions of* $(\mathbf{Z}/p\mathbf{Z})^*$ *and* $\mathbf{S}^4$, *each isolated four-dimensional terminal* $\mathbf{Z}/p\mathbf{Z}$-*quotient singularity of index* $p$ *is associated with one of the* $p$-*terminal quintuples given in Theorem* 1.3.

This conjecture was proved by Bover (2009) (and Sankaran (1990)).

# Hollow 3-polytopes of width three

**Theorem (Nill and Ziegler, 2011)**

*For each dimension $d$, all except finitely many hollow $d$-polytopes admit a lattice projection to a hollow $(d-1)$-polytope.*

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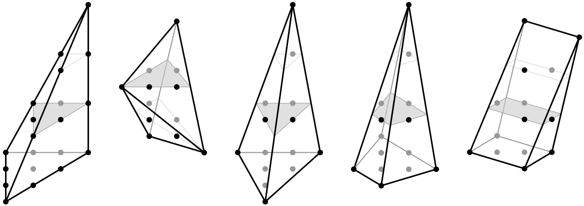
- Has width one.
- Projects to the only hollow lattice polygon of width larger than one (the second dilation of a unimodular triangle).
- Belongs to a final list with only twelve maximal ones (Averkov-Krümpelmann-Weltge, 2016): Seven of width two and five of width three.

# Hollow 3-polytopes of width three

**Theorem (Averkov-Wagner-Weismantel'11, A.-Krümpelmann-Weltge'15)**

There are 12 maximal hollow lattice 3-polytopes. Seven of width two plus the following five, of width three:



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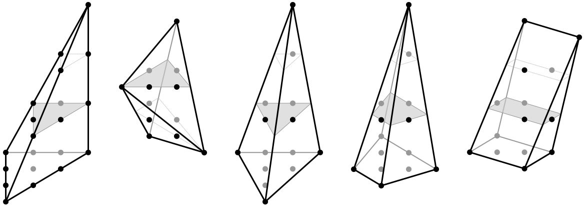

Remark: all proper subpolytopes of these have width $\leq 2$ (BHHS 16+). Hence, these five are **the only hollow** 3-**polytopes of width** $\geq 3$.

# Classification of empty 4-simplices, part 1. Volume bounds

**Theorem (Iglesias-S. 2017+)**

Let $P$ be an empty 4-simplex.

- If $\mathrm{width}(P) \geq 3$ then $\mathrm{Vol}(P) \leq 5058$.
- If $\mathrm{width}(P) = 2$ but $P$ does not project to a hollow 3-polytope then $\mathrm{Vol}(P) \leq 5184$.

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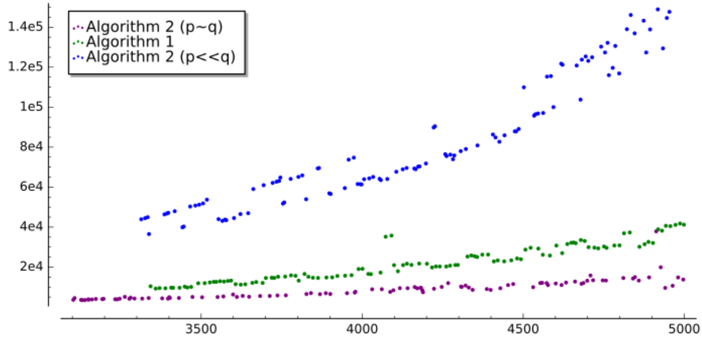
Ideas in proof: (1) reduce to dimension three and (2) use volume bounds for hollow 3-dimensional 3-polytopes.

(1) For width two, look at intermediate slice. For width $\geq 3$ show that either $P$ is “short n every direction” or it “projects to a wide hollow 3-polytope”.

(2) Uses several convex geometry tricks (covering minima, Minkowski Theorem, coefficient of asymmetry, etc).

# Classification of empty 4-simplices, part 2. Enumeration

We have enumerated all empty 4.simplices up to volume 7, 600. More than 10000 hours of computation have been used.




Computation time (sec.) for the list of all empty lattice 4-simplices of a given volume

# Classification of empty 4-simplices, part 3. Simplices projecting to lower dimension

Looking at hollow lifts of hollow polytopes (á la Blanco-Haase-Hofmann-S. 2017) we conclude that the hollow 4-simplices that project to lower dimensional hollow polytopes are:

1. Those of width one (i.e., projecting to dimension 1), which form a 3-parameter family.

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3. Those projecting to one of 29 (primitive) plus 23 (non-primitive) bipyramids of width two. Each forms a 1-parameter family. The first 29 correspond to the "stable quintuples" of Mori-Morrison and Morrison (1988). The other 23 form new "non-primitive quintuples".

# The 29 stable quintuples

| | |
|---|---|
| $\mathbb{Q}\{(9,1,-2,-3,-5)\}$ | $\mathbb{Q}\{(7,5,3,-1,-14)\}$ |
| $\mathbb{Q}\{(9,2,-1,-4,-6)\}$ | $\mathbb{Q}\{(9,7,1,-3,-14)\}$ |
| $\mathbb{Q}\{(12,3,-4,-5,-6)\}$ | $\mathbb{Q}\{(15,7,-3,-5,-14)\}$ |
| $\mathbb{Q}\{(12,2,-3,-4,-7)\}$ | $\mathbb{Q}\{(8,5,3,-1,-15)\}$ |
| $\mathbb{Q}\{(9,4,-2,-3,-8)\}$ | $\mathbb{Q}\{(10,6,1,-2,-15)\}$ |
| $\mathbb{Q}\{(12,1,-2,-3,-8)\}$ | $\mathbb{Q}\{(12,5,2,-4,-15)\}$ |
| $\mathbb{Q}\{(12,3,-1,-6,-8)\}$ | $\mathbb{Q}\{(9,6,4,-1,-18)\}$ |
| $\mathbb{Q}\{(15,4,-5,-6,-8)\}$ | $\mathbb{Q}\{(9,6,5,-2,-18)\}$ |
| $\mathbb{Q}\{(12,2,-1,-4,-9)\}$ | $\mathbb{Q}\{(12,9,1,-4,-18)\}$ |
| $\mathbb{Q}\{(10,6,-2,-5,-9)\}$ | $\mathbb{Q}\{(10,7,4,-1,-20)\}$ |
| $\mathbb{Q}\{(15,1,-2,-5,-9)\}$ | $\mathbb{Q}\{(10,8,3,-1,-20)\}$ |
| $\mathbb{Q}\{(12,5,-3,-4,-10)\}$ | $\mathbb{Q}\{(10,9,4,-3,-20)\}$ |
| $\mathbb{Q}\{(15,2,-3,-4,-10)\}$ | $\mathbb{Q}\{(12,10,1,-3,-20)\}$ |
| $\mathbb{Q}\{(6,4,3,-1,-12)\}$ | $\mathbb{Q}\{(12,8,5,-1,-24)\}$ |
| | $\mathbb{Q}\{(15,10,6,-1,-30)\}$ |

Table : The 29 stable quintuples of Mori-Morrison-Morrison. Each represents (the rational points in) a line through the origin, in $\mathcal{T}^4$.

# The 23 "non-primitive stable quintuples"

| | | | | | |
|---|---|---|---|---|---|
| $(0, 0, \frac{1}{2}, \frac{1}{2}, 0)$ | $+$ | $\mathbb{Q}\{(6, -2, -12, 4, 4)\}$ | $(0, 0, \frac{2}{3}, \frac{1}{3}, 0)$ | $+$ | $\mathbb{Q}\{(-9, 6, 3, 3, -3)\}$ |
| $(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(8, -6, 2, -8, 4)\}$ | $(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$ | $+$ | $\mathbb{Q}\{(9, -9, 3, -6, 3)\}$ |
| $(0, 0, \frac{1}{2}, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(8, -4, -12, 6, 2)\}$ | $(0, 0, \frac{1}{3}, \frac{2}{3}, 0)$ | $+$ | $\mathbb{Q}\{(-9, 3, 6, 6, -6)\}$ |
| $(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(4, 6, -2, -16, 8)\}$ | $(0, 0, \frac{1}{3}, \frac{2}{3}, 0)$ | $+$ | $\mathbb{Q}\{(12, -6, -12, 3, 3)\}$ |
| $(0, \frac{1}{2}, \frac{1}{2}, 0, 0)$ | $+$ | $\mathbb{Q}\{(2, -12, 4, 12, -6)\}$ | $(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$ | $+$ | $\mathbb{Q}\{(9, -18, 6, 6, -3)\}$ |
| $(\frac{1}{2}, 0, \frac{1}{2}, 0, 0)$ | $+$ | $\mathbb{Q}\{(12, -16, 8, -6, 2)\}$ | $(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$ | $+$ | $\mathbb{Q}\{(12, -18, 3, 6, -3)\}$ |
| $(0, \frac{1}{2}, 0, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(2, 12, -8, -12, 6)\}$ | $(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$ | $+$ | $\mathbb{Q}\{(12, -9, 3, -12, 6)\}$ |
| $(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(8, 6, -2, -24, 12)\}$ | $(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$ | $+$ | $\mathbb{Q}\{(6, -3, 6, -18, 9)\}$ |
| $(0, \frac{1}{2}, 0, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(6, -2, 8, -24, 12)\}$ | $(0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $+$ | $\mathbb{Q}\{(3, -18, 6, 18, -9)\}$ |
| $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, 0)$ | $+$ | $\mathbb{Q}\{(12, -12, 4, -8, 4)\}$ | $(\frac{1}{6}, 0, 0, \frac{2}{3}, \frac{1}{6})$ | $+$ | $\mathbb{Q}\{(6, -18, 6, 12, -6)\}$ |
| $(0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(4, 8, -4, -16, 8)\}$ | | | |
| $(0, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ | $+$ | $\mathbb{Q}\{(4, -16, 4, 16, -8)\}$ | | | |
| $(0\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$ | $+$ | $\mathbb{Q}\{(4, 12, -4, -24, 12)\}$ | | | |

Table : The 23 non-primitive quintuples. Each represents (the rational points in) a line in $\mathcal{T}^4$ not passing through the origin.

# Putting things together

By the bounds in "part 1" empty 4-simplices that do not project to hollow lower-dimensional polytopes have volume $< 6000$. We have the complete list of them, since we enumerated up to volume 7600 ("part 2"). Together with thw classification in "part 3" we have a complete classification of empty 4-simplices:

**Theorem (Iglesias-S. 2017+)**

Let $P$ be a 4-dimensional empty 4-simplex:

- If $P$ projects to a hollow 3-polytope then it is as in the previous slide (1+2+52 infinite families, depending on the projection).
- If $P$ does not project to a hollow 3-polytope then it has volume at most 419. There are 2461 classes of them, all of width two except for 178 classes of width three and **one** class of width four.

# Nbr. of sporadic 4-simplices (part 1 of 2)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $V=24:$ | 1 | $V=53:$ | 38 | $V=78:$ | 3 | $V=103:$ | 51 | $V=129:$ | 17 |
| $V=27:$ | 1 | $V=54:$ | 11 | $V=79:$ | 55 | $V=104:$ | 8 | $V=130:$ | 2 |
| $V=29:$ | 3 | $V=55:$ | 20 | $V=80:$ | 7 | $V=105:$ | 7 | $V=131:$ | 29 |
| $V=30:$ | 2 | $V=56:$ | 3 | $V=81:$ | 18 | $V=106:$ | 8 | $V=132:$ | 5 |
| $V=31:$ | 2 | $V=57:$ | 16 | $V=82:$ | 13 | $V=107:$ | 54 | $V=133:$ | 14 |
| $V=32:$ | 3 | $V=58:$ | 13 | $V=83:$ | 60 | $V=108:$ | 5 | $V=134:$ | 8 |
| $V=33:$ | 4 | $V=59:$ | 51 | $V=84:$ | 7 | $V=109:$ | 44 | $V=135:$ | 6 |
| $V=34:$ | 5 | $V=60:$ | 4 | $V=85:$ | 27 | $V=110:$ | 5 | $V=136:$ | 6 |
| $V=35:$ | 3 | $V=61:$ | 38 | $V=86:$ | 11 | $V=111:$ | 13 | $V=137:$ | 28 |
| $V=37:$ | 6 | $V=62:$ | 26 | $V=87:$ | 24 | $V=112:$ | 2 | $V=138:$ | 2 |
| $V=38:$ | 8 | $V=63:$ | 17 | $V=88:$ | 5 | $V=113:$ | 40 | $V=139:$ | 37 |
| $V=39:$ | 9 | $V=64:$ | 9 | $V=89:$ | 55 | $V=114:$ | 4 | $V=140:$ | 5 |
| $V=40:$ | 1 | $V=65:$ | 27 | $V=90:$ | 6 | $V=115:$ | 21 | $V=141:$ | 6 |
| $V=41:$ | 14 | $V=66:$ | 3 | $V=91:$ | 18 | $V=116:$ | 11 | $V=142:$ | 9 |
| $V=42:$ | 5 | $V=67:$ | 41 | $V=92:$ | 9 | $V=117:$ | 10 | $V=143:$ | 13 |
| $V=43:$ | 20 | $V=68:$ | 13 | $V=93:$ | 17 | $V=118:$ | 9 | $V=144:$ | 1 |
| $V=44:$ | 8 | $V=69:$ | 26 | $V=94:$ | 12 | $V=119:$ | 22 | $V=145:$ | 14 |
| $V=45:$ | 6 | $V=70:$ | 4 | $V=95:$ | 35 | $V=120:$ | 3 | $V=146:$ | 5 |
| $V=46:$ | 7 | $V=71:$ | 50 | $V=96:$ | 3 | $V=121:$ | 18 | $V=147:$ | 10 |
| $V=47:$ | 30 | $V=72:$ | 3 | $V=97:$ | 46 | $V=122:$ | 9 | $V=148:$ | 7 |
| $V=48:$ | 5 | $V=73:$ | 44 | $V=98:$ | 9 | $V=123:$ | 17 | $V=149:$ | 26 |
| $V=49:$ | 17 | $V=74:$ | 18 | $V=99:$ | 13 | $V=124:$ | 8 | $V=150:$ | 2 |
| $V=50:$ | 8 | $V=75:$ | 22 | $V=100:$ | 8 | $V=125:$ | 25 | $V=151:$ | 19 |
| $V=51:$ | 16 | $V=76:$ | 14 | $V=101:$ | 41 | $V=127:$ | 24 | $V=152:$ | 6 |
| $V=52:$ | 6 | $V=77:$ | 19 | $V=102:$ | 3 | $V=128:$ | 9 | $V=153:$ | 9 |

# Nbr. of sporadic 4-simplices (part 2 of 2)

$V = 154$ : 3
$V = 155$ : 12
$V = 156$ : 2
$V = 157$ : 11
$V = 158$ : 10
$V = 159$ : 9
$V = 160$ : 3
$V = 161$ : 13
$V = 163$ : 17
$V = 164$ : 6
$V = 165$ : 1
$V = 166$ : 7
$V = 167$ : 18
$V = 168$ : 3
$V = 169$ : 13
$V = 170$ : 2
$V = 171$ : 6
$V = 172$ : 3
$V = 173$ : 15
$V = 174$ : 3
$V = 175$ : 8
$V = 176$ : 4
$V = 177$ : 5
$V = 178$ : 2
$V = 179$ : 21
$V = 180$ : 1

$V = 181$ : 13
$V = 182$ : 5
$V = 183$ : 5
$V = 184$ : 5
$V = 185$ : 7
$V = 186$ : 2
$V = 187$ : 7
$V = 188$ : 5
$V = 189$ : 2
$V = 190$ : 2
$V = 191$ : 8
$V = 192$ : 1
$V = 193$ : 12
$V = 194$ : 3
$V = 196$ : 4
$V = 197$ : 13
$V = 199$ : 11
$V = 200$ : 4
$V = 201$ : 3
$V = 202$ : 2
$V = 203$ : 7
$V = 204$ : 1
$V = 205$ : 4
$V = 206$ : 4
$V = 207$ : 2
$V = 208$ : 1
$V = 209$ : 10

$V = 211$ : 4
$V = 212$ : 2
$V = 213$ : 3
$V = 214$ : 2
$V = 215$ : 5
$V = 216$ : 1
$V = 218$ : 5
$V = 219$ : 4
$V = 220$ : 1
$V = 221$ : 3
$V = 222$ : 1
$V = 223$ : 7
$V = 225$ : 2
$V = 226$ : 4
$V = 227$ : 9
$V = 229$ : 6
$V = 230$ : 3
$V = 232$ : 1
$V = 233$ : 9
$V = 234$ : 1
$V = 235$ : 3
$V = 237$ : 1
$V = 238$ : 2
$V = 239$ : 3
$V = 241$ : 6
$V = 244$ : 2

$V = 245$ : 3
$V = 247$ : 3
$V = 248$ : 3
$V = 249$ : 2
$V = 250$ : 1
$V = 251$ : 5
$V = 254$ : 1
$V = 256$ : 2
$V = 257$ : 3
$V = 259$ : 2
$V = 261$ : 1
$V = 263$ : 7
$V = 265$ : 1
$V = 267$ : 1
$V = 268$ : 1
$V = 269$ : 2
$V = 271$ : 4
$V = 272$ : 1
$V = 274$ : 1
$V = 275$ : 1
$V = 278$ : 2
$V = 283$ : 2
$V = 287$ : 1
$V = 289$ : 4
$V = 290$ : 1
$V = 291$ : 1
$V = 292$ : 1

$V = 293$ : 5
$V = 299$ : 2
$V = 304$ : 1
$V = 308$ : 1
$V = 310$ : 1
$V = 311$ : 1
$V = 313$ : 1
$V = 314$ : 1
$V = 317$ : 1
$V = 319$ : 2
$V = 321$ : 1
$V = 323$ : 1
$V = 331$ : 1
$V = 332$ : 1
$V = 334$ : 2
$V = 335$ : 1
$V = 347$ : 1
$V = 349$ : 2
$V = 353$ : 1
$V = 355$ : 1
$V = 356$ : 1
$V = 376$ : 1
$V = 377$ : 2
$V = 397$ : 1
$V = 398$ : 1
$V = 419$ : 1

# Nbr. of sporadic t.q.s. of prime volume (MMM vs. us)

TABLE 1.14

| $p$ | $S_p$ | $p$ | $S_p$ | $p$ | $S_p$ | $p$ | $S_p$ |
|---|---|---|---|---|---|---|---|
| 2 | 0 | 73 | 220 | 179 | 105 | 283 | 10 |
| 3 | 0 | 79 | 275 | 181 | 65 | 293 | 25 |
| 5 | 0 | 83 | 300 | 191 | 40 | 307 | 0 |
| 7 | 0 | 89 | 275 | 193 | 60 | 311 | 5 |
| 11 | 0 | 97 | 230 | 197 | 65 | 313 | 5 |
| 13 | 0 | 101 | 201 | 199 | 55 | 317 | 5 |
| 17 | 9 | 103 | 255 | 211 | 20 | 331 | 5 |
| 19 | 13 | 107 | 270 | 223 | 35 | 337 | 0 |
| 23 | 28 | 109 | 220 | 227 | 45 | 347 | 5 |
| 29 | 39 | 113 | 200 | 229 | 30 | 349 | 10 |
| 31 | 30 | 127 | 120 | 233 | 45 | 353 | 5 |
| 37 | 50 | 131 | 145 | 239 | 15 | 359 | 0 |
| 41 | 76 | 137 | 140 | 241 | 30 | 367 | 0 |
| 43 | 110 | 139 | 185 | 251 | 25 | 373 | 0 |
| 47 | 100 | 149 | 130 | 257 | 15 | 379 | 0 |
| 53 | 195 | 151 | 95 | 263 | 35 | 383 | 0 |
| 59 | 260 | 157 | 55 | 269 | 10 | 389 | 0 |
| 61 | 186 | 163 | 85 | 271 | 20 | 397 | 5 |
| 67 | 205 | 167 | 90 | 277 | 0 | 409 | 0 |
| 71 | 250 | 173 | 75 | 281 | 0 | 419 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| $V = 29$ : | 15 | $V = 113$ : | 200 | $V = 229$ : | 30 |
| $V = 31$ : | 10 | $V = 127$ : | 120 | $V = 233$ : | 45 |
| $V = 37$ : | 30 | $V = 131$ : | 145 | $V = 239$ : | 15 |
| $V = 41$ : | 66 | $V = 137$ : | 140 | $V = 241$ : | 30 |
| $V = 43$ : | 100 | $V = 139$ : | 185 | $V = 251$ : | 25 |
| $V = 47$ : | 150 | $V = 149$ : | 130 | $V = 257$ : | 15 |
| $V = 53$ : | 190 | $V = 151$ : | 95 | $V = 263$ : | 35 |
| $V = 59$ : | 255 | $V = 157$ : | 55 | $V = 269$ : | 10 |
| $V = 61$ : | 186 | $V = 163$ : | 85 | $V = 271$ : | 20 |
| $V = 67$ : | 205 | $V = 167$ : | 90 | $V = 283$ : | 10 |
| $V = 71$ : | 250 | $V = 173$ : | 75 | $V = 293$ : | 25 |
| $V = 73$ : | 220 | $V = 179$ : | 105 | $V = 311$ : | 5 |
| $V = 79$ : | 275 | $V = 181$ : | 65 | $V = 313$ : | 5 |
| $V = 83$ : | 300 | $V = 191$ : | 40 | $V = 317$ : | 5 |
| $V = 89$ : | 275 | $V = 193$ : | 60 | $V = 331$ : | 5 |
| $V = 97$ : | 230 | $V = 197$ : | 65 | $V = 347$ : | 5 |
| $V = 101$ : | 201 | $V = 199$ : | 55 | $V = 349$ : | 10 |
| $V = 103$ : | 255 | $V = 211$ : | 20 | $V = 353$ : | 5 |
| $V = 107$ : | 270 | $V = 223$ : | 35 | $V = 397$ : | 5 |
| $V = 109$ : | 220 | $V = 227$ : | 45 | $V = 419$ : | 5 |

# The end

# The end

THANK YOU