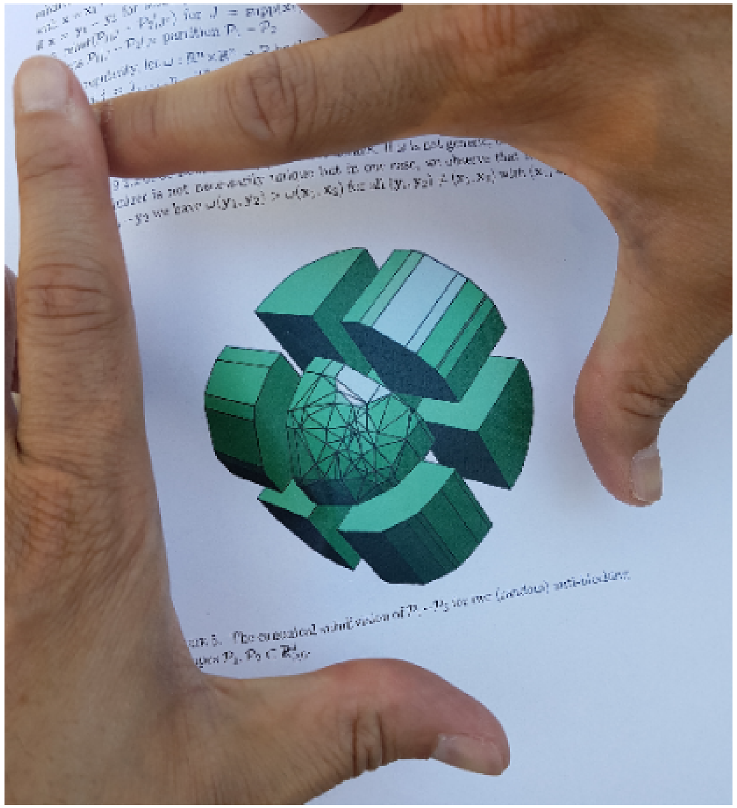
# ORDER, GEOMETRICALLY



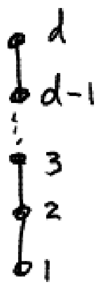

Raman Sanyal

Goethe University Frankfurt


... joint w/ T. Chappell, T. Friedl; C. Stump

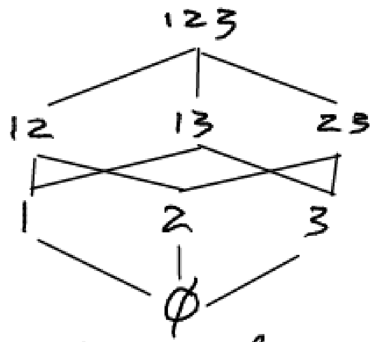
# Partially ordered sets (posets)

$(P, \preccurlyeq)$ P – finite set , $\preccurlyeq$ – partial order

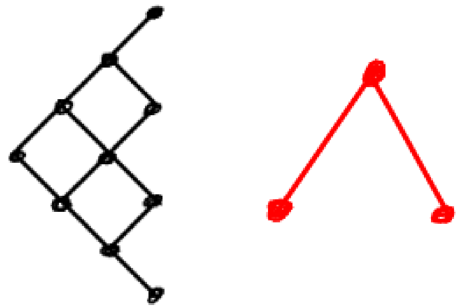



d-chain

d-antichain




Boolean lattice



Hasse diagrams

... face lattices, geometric lattices (matroids), ...

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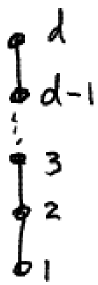


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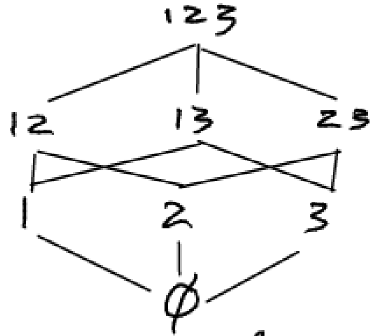


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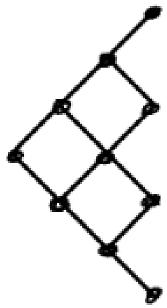

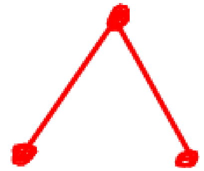

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... face lattices, geometric lattices (matroids), ...

Dim-2 poset: $\pi = (\pi_1, \dots, \pi_d)$ Permutation

$([d], \preccurlyeq_\pi)$ $i \preccurlyeq_\pi j :\iff i < j$ and $\pi_i < \pi_j$

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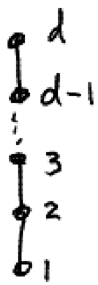


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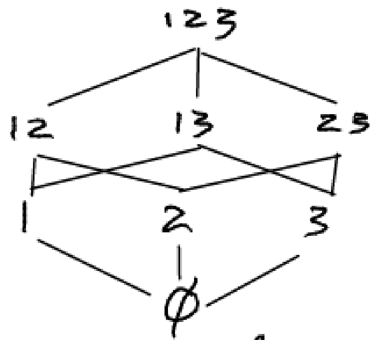


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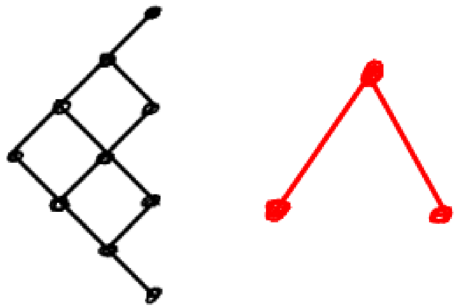

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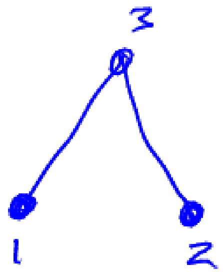
**Convention:** $P = \{a_1, a_2, \dots, a_d\}$ such that $a_i \preccurlyeq a_j \Longrightarrow i \leq j$

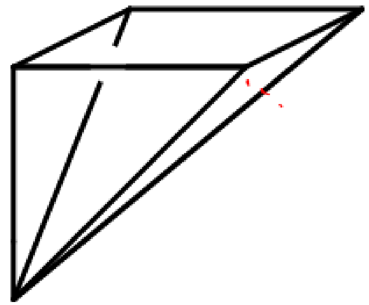
"naturally labelled"

# Stanley's "Two Poset Polytopes" (1984)

Order polytope $\mathcal{O}(P) = \left\{ f \in \mathbb{R}^P : \begin{array}{ll} 0 \leq f(a) \leq 1 & a \in P \\ f(a) \leq f(b) & a \prec b \end{array} \right\}.$

$P =$ 



$\mathcal{O}(P) =$ 

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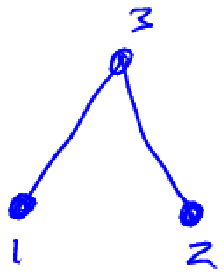
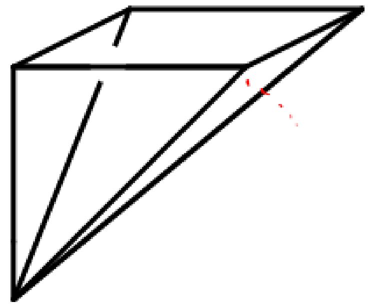
$P =$ 


$\mathcal{O}(P) =$ 

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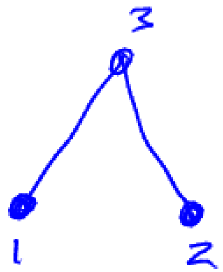
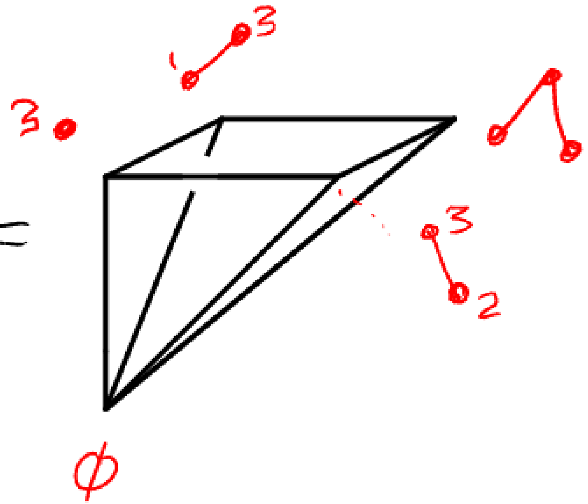
$P =$ 


$\mathcal{O}(P) =$ 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facets $\triangleq$ Minima $\uplus$ Maxima $\uplus$ Cover relations

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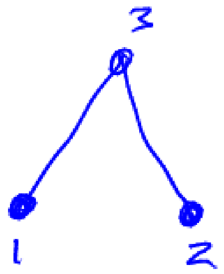
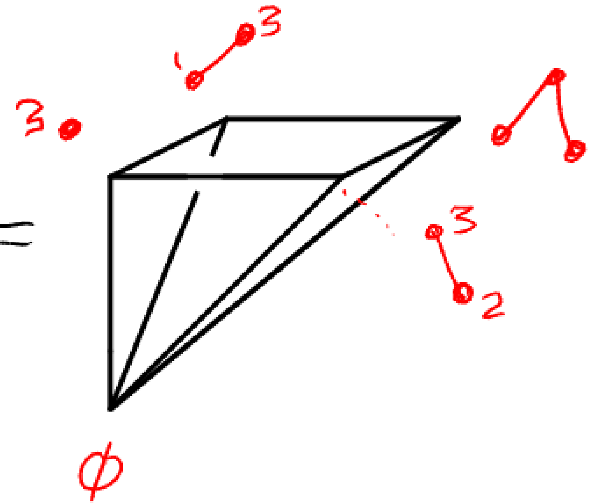
$P =$ 


$\mathcal{O}(P) =$ 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$P =$ d-antichain $\mathcal{O}(P) = [0,1]^d$ $\qquad$ $P =$ d-chain $\mathcal{O}(P) =$ d-simplex

# Metric & Arithmetic properties

Order polytope $\mathcal{O}(P) = \left\{ f \in \mathbb{R}^P : \begin{array}{ll} 0 \leq f(a) \leq 1 & a \in P \\ f(a) \leq f(b) & a \prec b \end{array} \right\}.$

Linear extension of $(P = \{a_1, \dots, a_d\}, \preceq)$:

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**Ehrhart polynomial**

$$E(\mathcal{O}(P), n) := |n \cdot \mathcal{O}(P) \cap \mathbb{Z}^P| = \# \underset{P \to [n+1]}{\text{order preserving}} = \Omega_P(n+1)$$

**Order polynomial**

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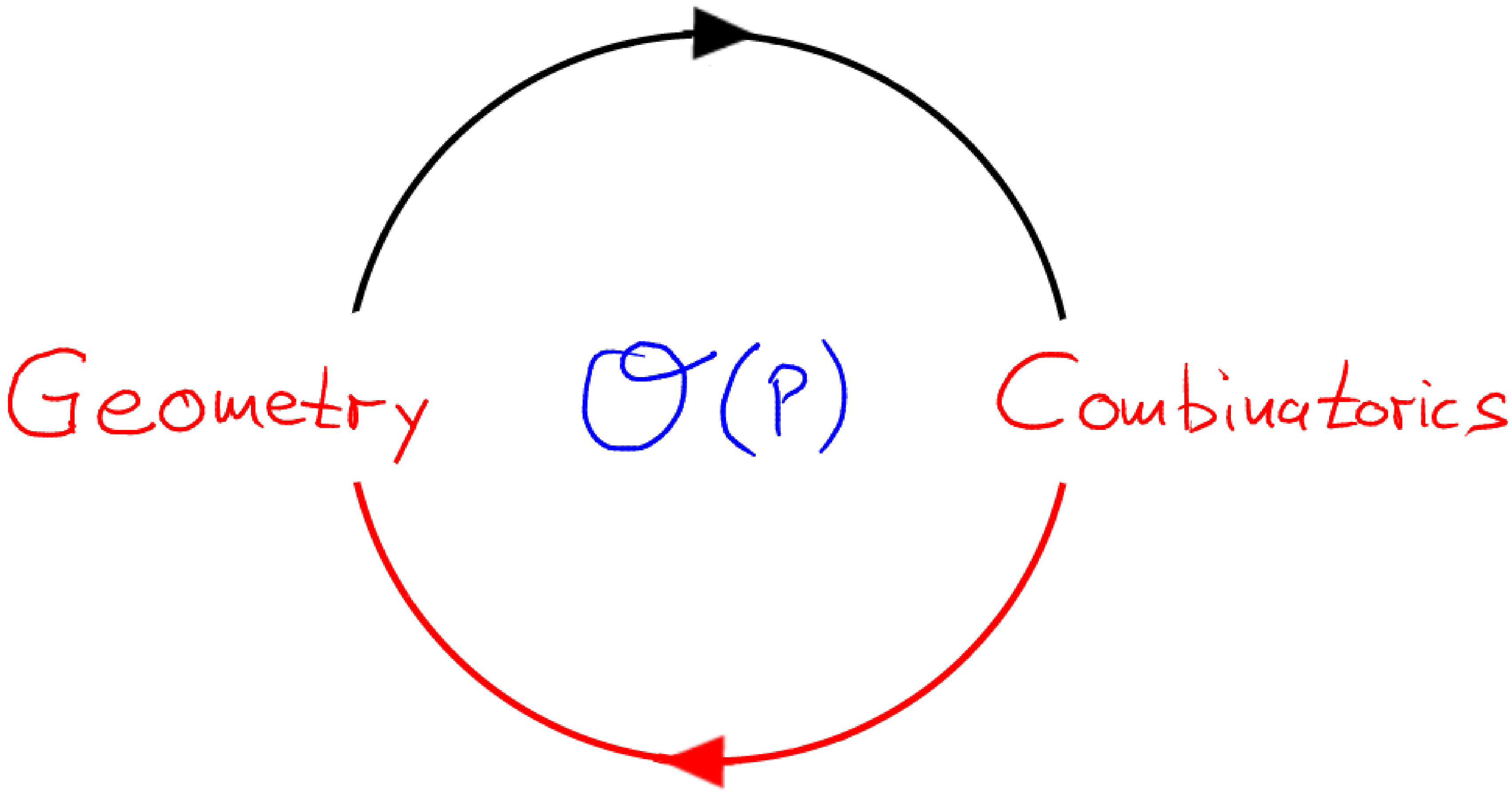
Order polynomial

$$\Omega_P(n) = \sum_{i=0}^{d} h_i^*(P) \binom{n+d-i}{d}$$

$h_i^*(P) = \#$ lin extensions with $i$ descents

# Geometric Combinatorics




→ Complete description

UPSHOT: Computing volumes is #P-hard.

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## Chain polytope

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Thm (Stanley). Piecewise linear & lattice preserving homeomorphism

$$\mathcal{O}(P) \longrightarrow \mathcal{C}(P)$$

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$e(P)$ & $\Omega_P$ depend only on comparability graph

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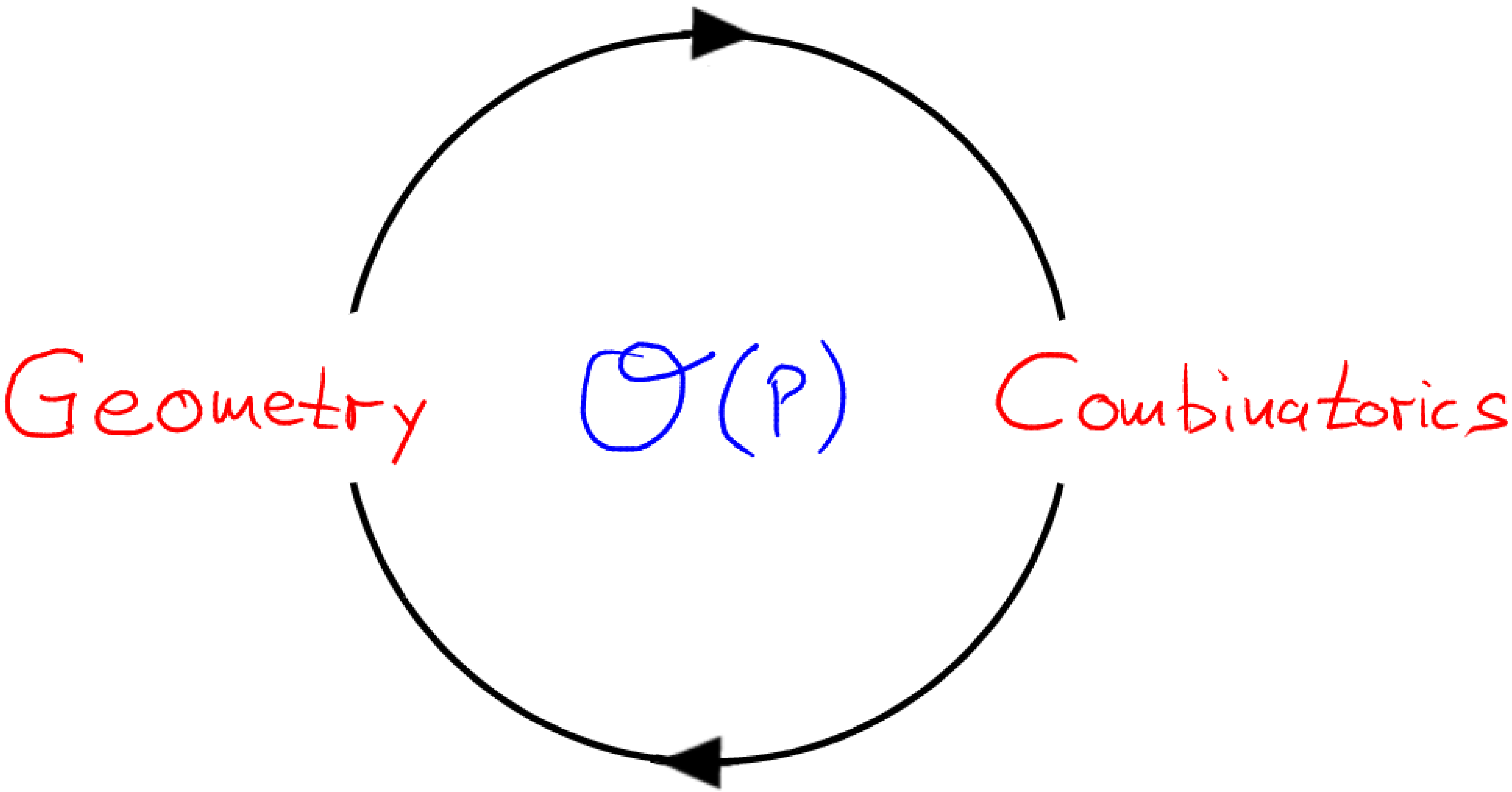

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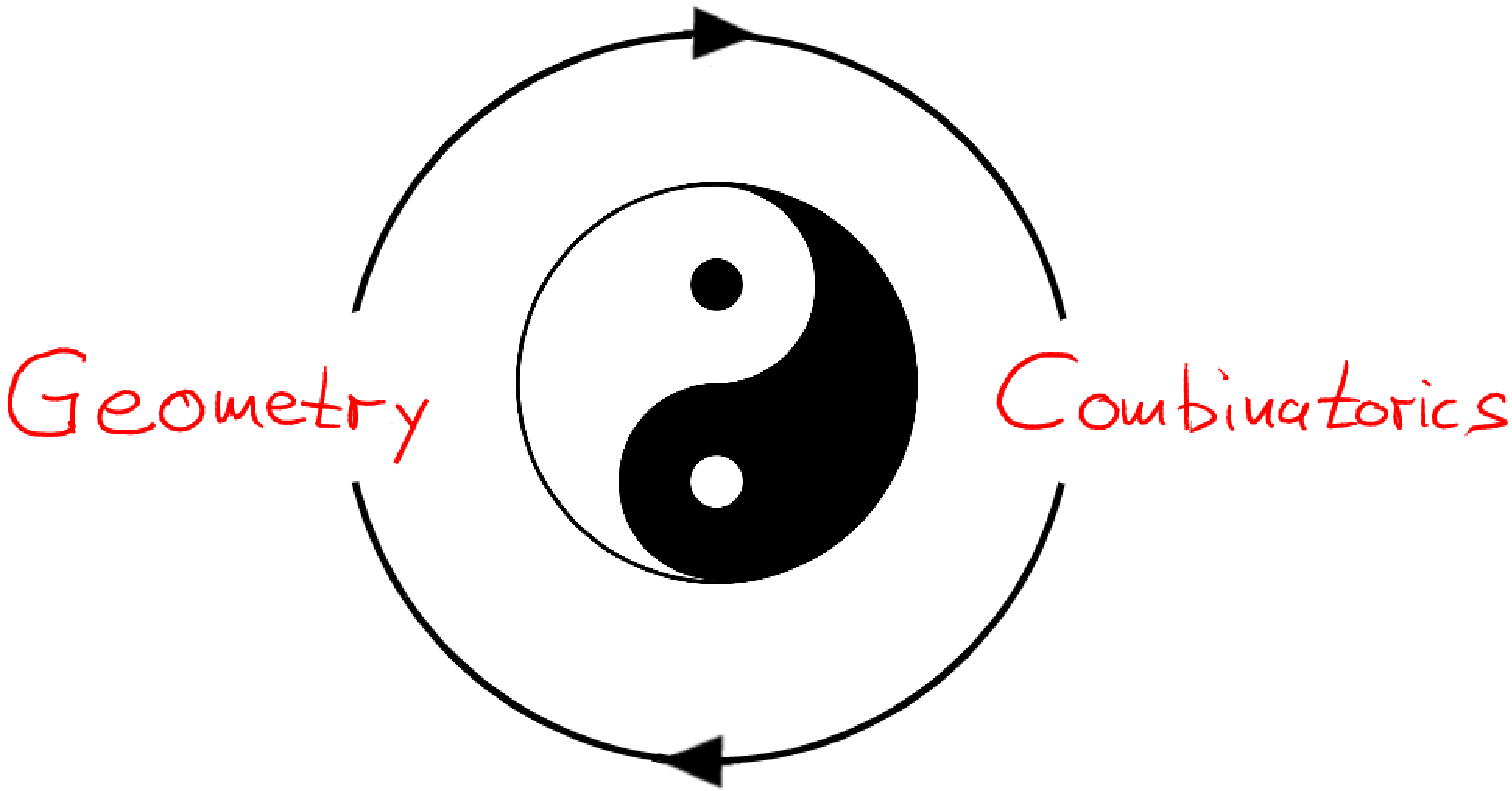

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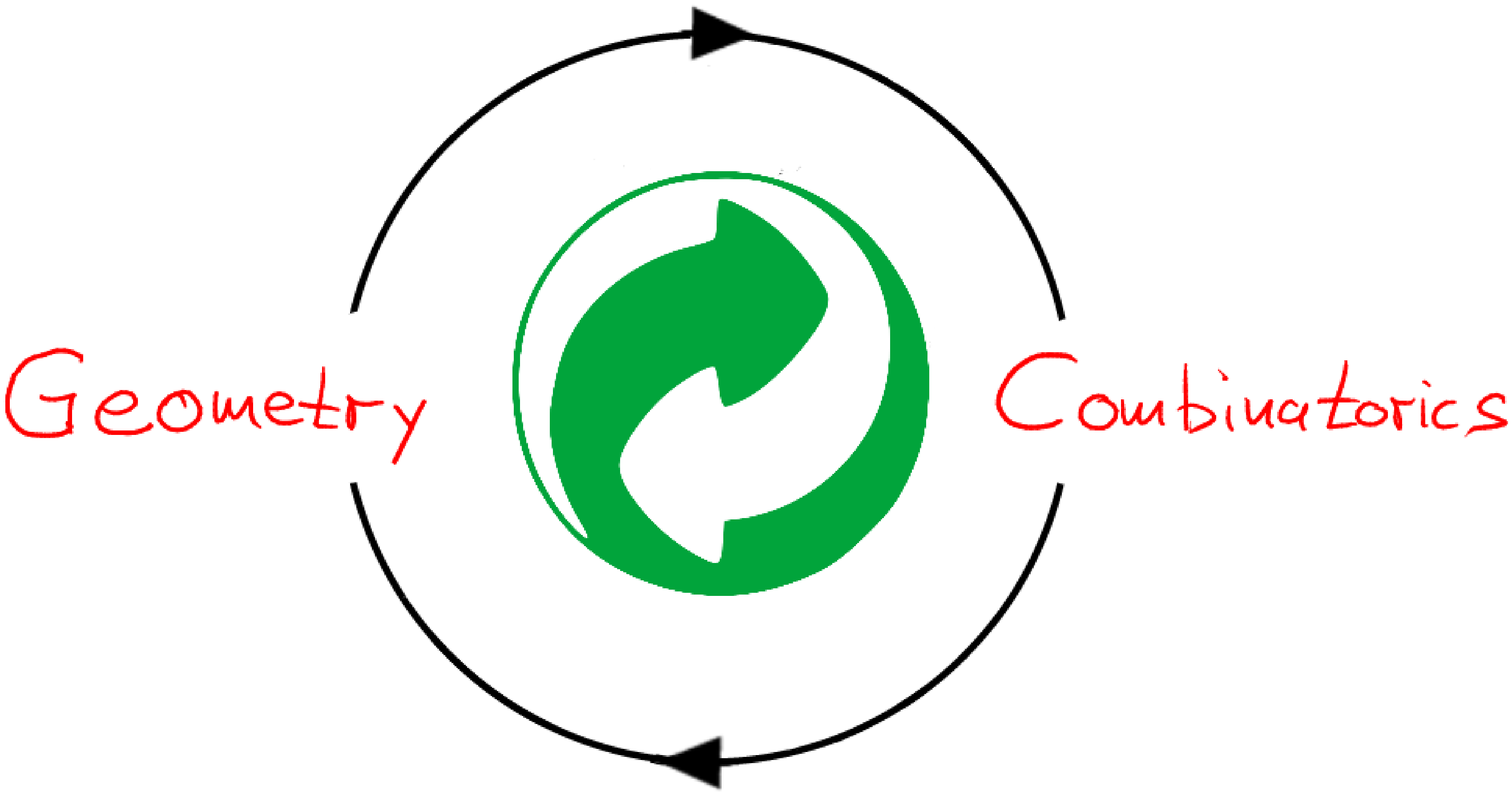
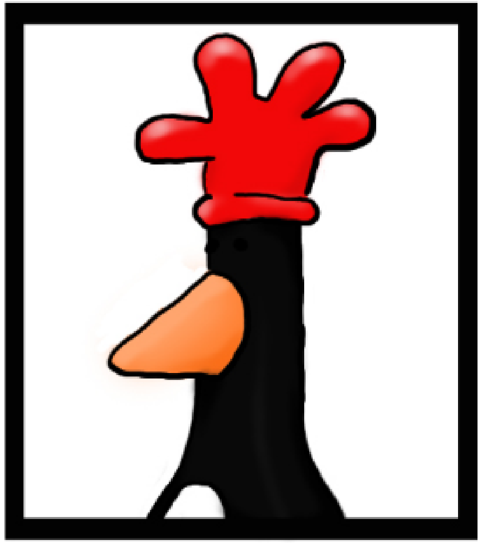


Attention!

**WANTED**



HAVE YOU SEEN THIS CHICKEN

$10,000 REWARD

Please report any familiar looking formulas!

Source: Wallace&Gromit, The wrong trousers

# LIPSCHITZ POLYTOPES OF POSETS

joint work with Christian Stump

# Order cones & Lipschitz polytopes

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$f \in K(P)$ is K-Lipschitz if

$$f(b) - f(a) \leq k \cdot d_P(a,b) \qquad \forall a,b \in P$$

# Lipschitz Order Polytopes

$$\mathcal{L}(P) := \left\{ f \in \mathbb{R}^P : \begin{array}{ll} 0 \le f(a) \le 1 & a \in \min P \\ 0 \le f(b) - f(a) \le 1 & a \lessdot b \end{array} \right\}$$

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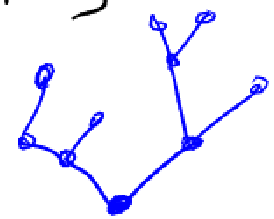
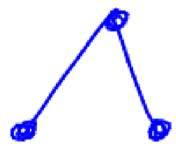
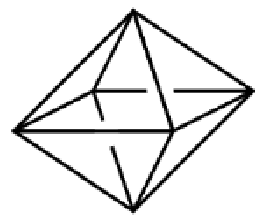


**Prop.** $P$ rooted tree $\implies$ $\mathcal{L}(P) \cong [0,1]^{|P|}$.

$\mathcal{L}($  $) \cong$ 

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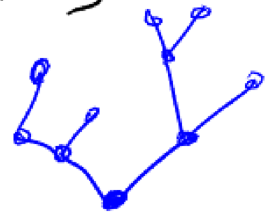

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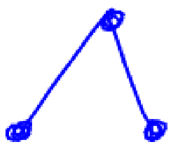
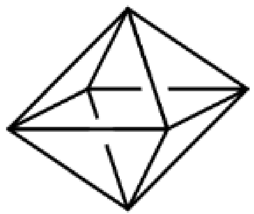
$\mathcal{L}($  $) \cong$ 

$\mathcal{L}(P)$ is lattice polytope, 2-level, bnded region of $G(P)$-Shi arr'ment ...

$P$ ranked $\Longrightarrow$ $\mathcal{L}(P)$ centrally-symm. & Gorenstein

$C = \{a_1 \lessdot a_2 \lessdot \cdots \lessdot a_k\}$ max chain , $\tau$ permutation

$\tau|_C = \tau(a_1), \tau(a_2), \ldots, \tau(a_k) \rightsquigarrow$ descents $\mathrm{des}_C(\tau)$

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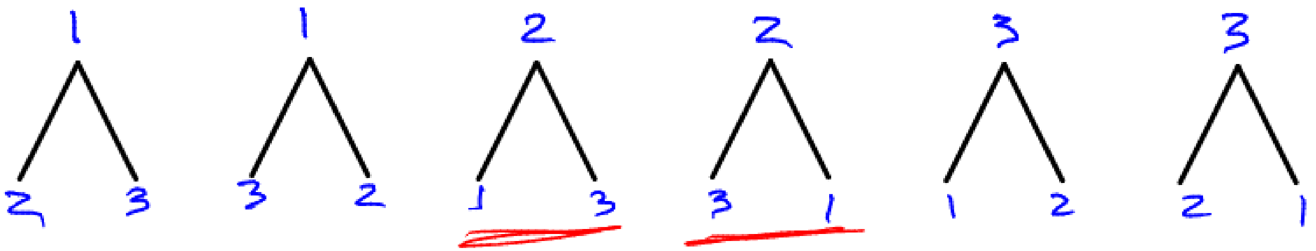

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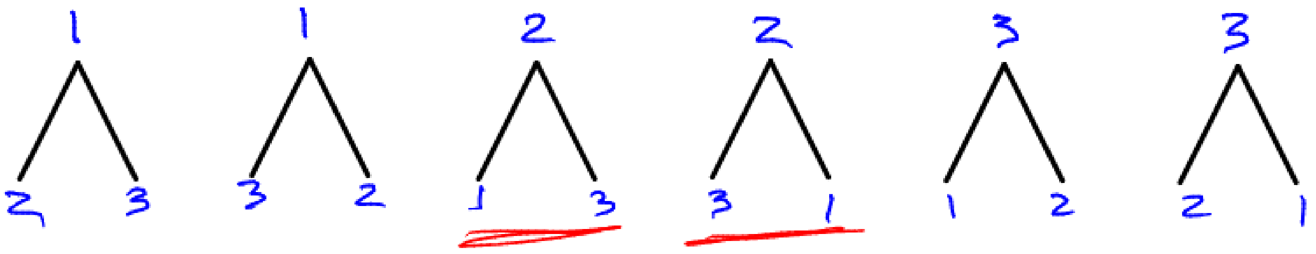


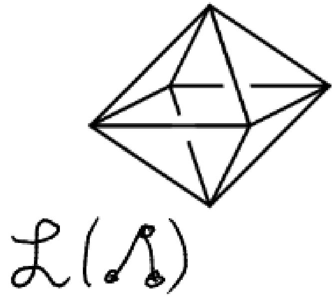

$DC(P) := \{\tau \text{ descent-compat.}\}$

Thm [SS'17]. $|P|! \cdot \mathrm{vol}(\mathcal{L}(P)) = |DC(P)|$.

$|DC(P)| = |P|! \iff P$ rooted tree

$C = \{a_1 \lessdot a_2 \lessdot \cdots \lessdot a_k\}$ max chain, $\tau$ permutation

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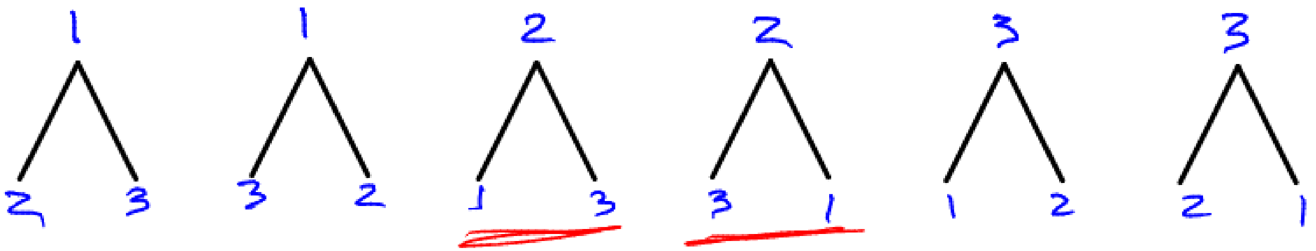


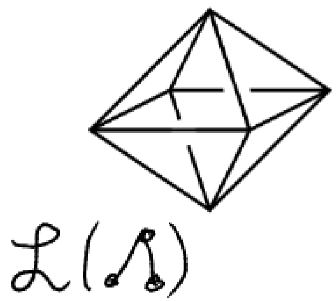

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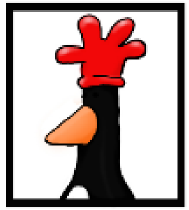

Thm [SS'17]. $\quad |P|! \cdot \mathrm{vol}(\mathcal{L}(P)) = |DC(P)|.$

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$\check{P} = \{ a_0 = \hat{0}, a_1, \dots, a_d \}$ , $\tau(a_0) := 0$ $\tau \in DC(P)$

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$\mathrm{Ldes}_P(\tau)$ = # pairs $(a,b) \in \check{P} \times \check{P}$ with $\tau(a) = \tau(b) - 1$

$\mathrm{des}_{P,\tau}(a) < \mathrm{des}_{P,\tau}(b)$ or $(\mathrm{des}_{P,\tau}(a) = \mathrm{des}_{P,\tau}(b)$ and $a > b)$

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<u>Thm</u> [SS'17].
$$\sum_{i=0}^{d} h_i^*(\mathcal{L}(P))\, z^i = \sum_{\tau\in DC(P)} z^{\mathrm{Ldes}_P(\tau)}$$

$P = d\text{-chain}$ : $\mathrm{Ldes}_P(\tau) = \text{big ascents}(\tau^{-1})$

$$\begin{array}{l}
0\ 1\ 2\ 3\\
0\ 1\bullet 3\ 2\\
0\bullet 3\ 1\ 2\\
0\bullet 3\ 2\ 1\\
0\bullet 2\ 3\ 1\\
0\bullet 2\ 1\bullet 3
\end{array}$$

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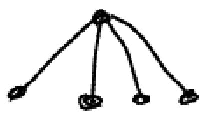
$P =$  : $Ldes_P(\tau) = des(\tau^{-1})$

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 0 | 1 $\bullet$ | 3 | 2 |
| 0 $\bullet$ | 3 | 1 | 2 |
| 0 $\bullet$ | 3 | 2 | 1 |
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| | | | |
|---|---|---|---|
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| 0 | 1 • | 3 | 2 |
| 0 • | 3 | 1 | 2 |
| 0 • | 3 | 2 | 1 |
| 0 • | 2 | 3 | 1 |
| 0 • | 2 | 1 • | 3 |

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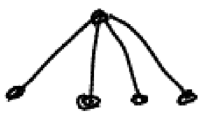
$P =$  : $\quad Ldes_P(\tau) = des(\tau^{-1})$

Cor. $P$ ranked. $\quad h_i^* = h_{d-1-i}^*$

Conjecture. $\quad h_i^* = \#\{\tau \in DC(P) : des(\tau^{-1}) = i\}$

$P$ ranked with maximum $\hat{1}$, $h = \mathrm{rank}(\hat{1}) + 1$.

(P, K)-hypersimplex $\Delta(P,K) = \{ f \in \mathcal{L}(P) : K-1 \leq f(\hat{1}) \leq K \}$

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Properties of $\Delta(P,K)$ that we like about $\Delta(n,K)$ ?

P-descents $\quad \mathrm{des}_P(\tau) := \mathrm{des}_{P,\tau}(\uparrow) \qquad \tau \in DC(P).$

<u>Cor.</u> $\quad |P|! \cdot \mathrm{vol}\, \Delta(P,k) = |\{\tau \in DC(P) : \mathrm{des}_P(\tau) = k-1\}|.$

Generalizes Stanley's interpretation of $\mathrm{vol}\, \Delta(n,k)$

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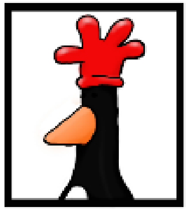
$$\Delta(P,k) \cong \Delta(P, n+1-k).$$

$$\Longrightarrow \quad \sum_{\tau \in DC(P)} t^{\mathrm{des}_P(\tau)} \quad \text{is palindromic}$$



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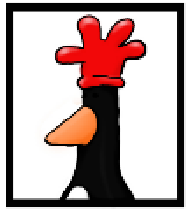

Half-open $(P,k)$-hypersimplex

$$\Delta^{◐}(P,k) = \{f \in \mathcal{L}(P) : k-1 < f(\hat{1}) \le k\} \qquad 1 < k \le n$$

**Thm** [SS'17]. $\displaystyle\sum h_i^*(\Delta^{◐}(P,k))\, z^i = \sum_{\mathrm{des}_P(\tau)=k} z^{\mathrm{Ldes}_P(\tau)}$

Generalizes Nan-Li's des-exc interpretation of $h^*(\Delta^{◐}(n,k))$.

# Two Double Poset Polytopes

joint work with Tom Chapell and Tobias Friedl

**Double posets** sets with 2 partial orders

[Malvenuto - Reutenauer 2011]

$$\mathbb{P} = (P, \leqslant_+, \leqslant_-) \longrightarrow P_+ = (P, \leqslant_+), \; P_- = (P, \leqslant_-)$$

## Double posets  sets with 2 partial orders

[Malvenuto – Reutenauer 2011]

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Stanley's *labelled* posets : $\leqslant_-$ total order ($\mathbb{P}$ "special")

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$(P, \leq)$ poset $\leadsto$ *induced* double poset $(P, \leq, \leq)$

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$(P, \leqslant)$ poset $\rightsquigarrow$ induced double poset $(P, \leqslant, \leqslant)$

$\pi$ permutation $\rightsquigarrow$ $\mathbb{P}_\pi = ([d], \leqslant_\pi, \leqslant_{\hat{\pi}})$ plane poset

$i \leqslant_\pi j \iff i < j \;\&\; \pi_i < \pi_j \qquad i \leqslant_{\hat{\pi}} j \iff i < j \;\; \pi_i > \pi_j$

MR'11 : Hopf algebras of double posets, Littlewood - Richardson rules...

# Double order polytopes

$$\mathcal{O}(\mathbb{P}) = \operatorname{conv}\left( 2\cdot\mathcal{O}(P_+)\times\{1\} \cup -2\cdot\mathcal{O}(P_-)\times\{-1\} \right)$$

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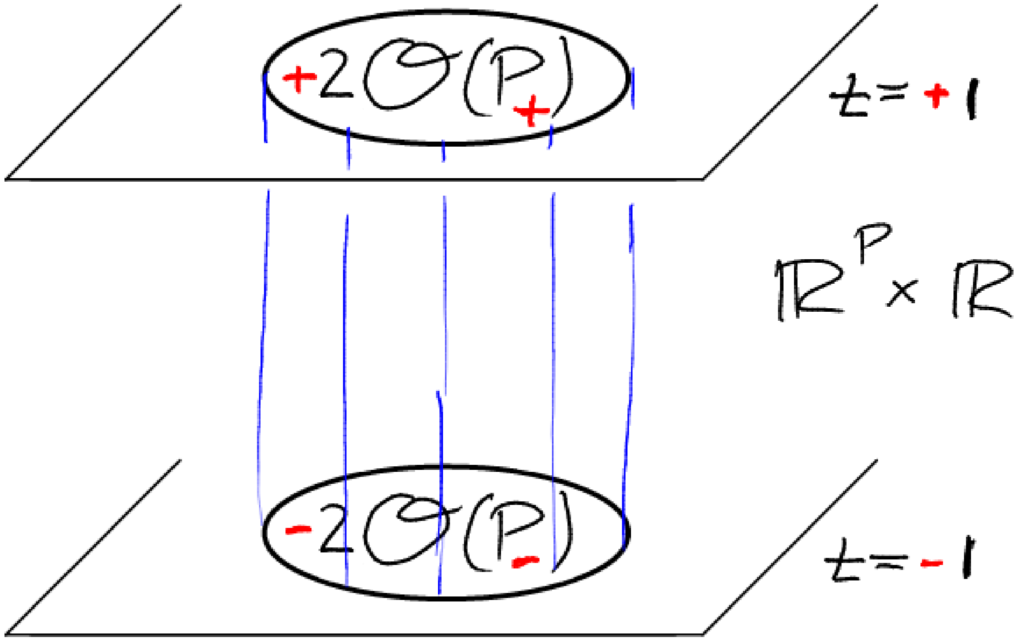

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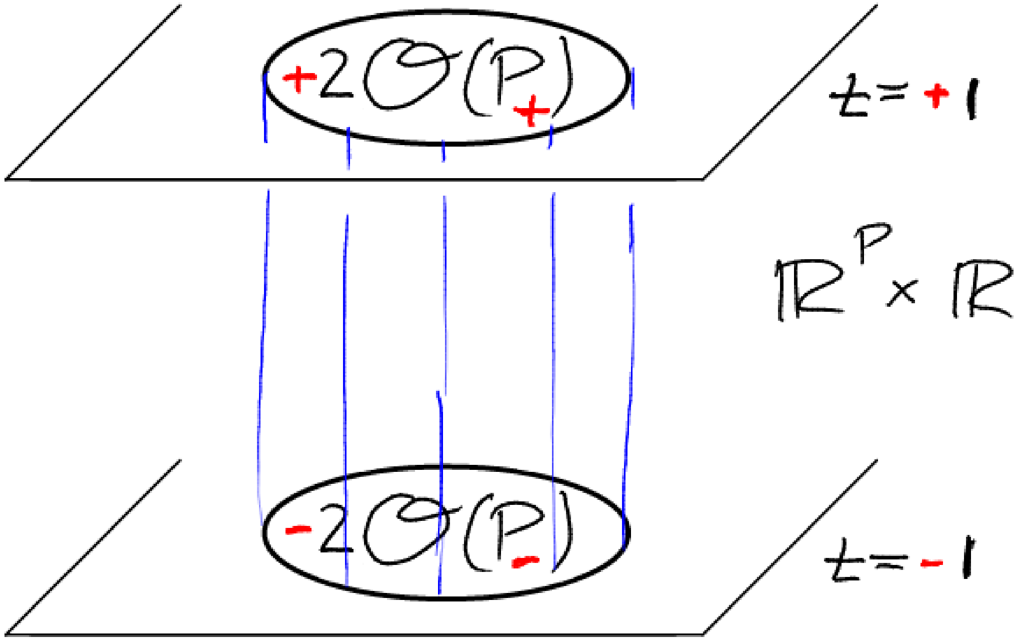


$$\dim \mathcal{O}(\mathbb{P}) = |P|+1$$

vertices $\hat{=}$

$$\mathrm{Filters}(P_+) \uplus \mathrm{Filters}(P_-)$$

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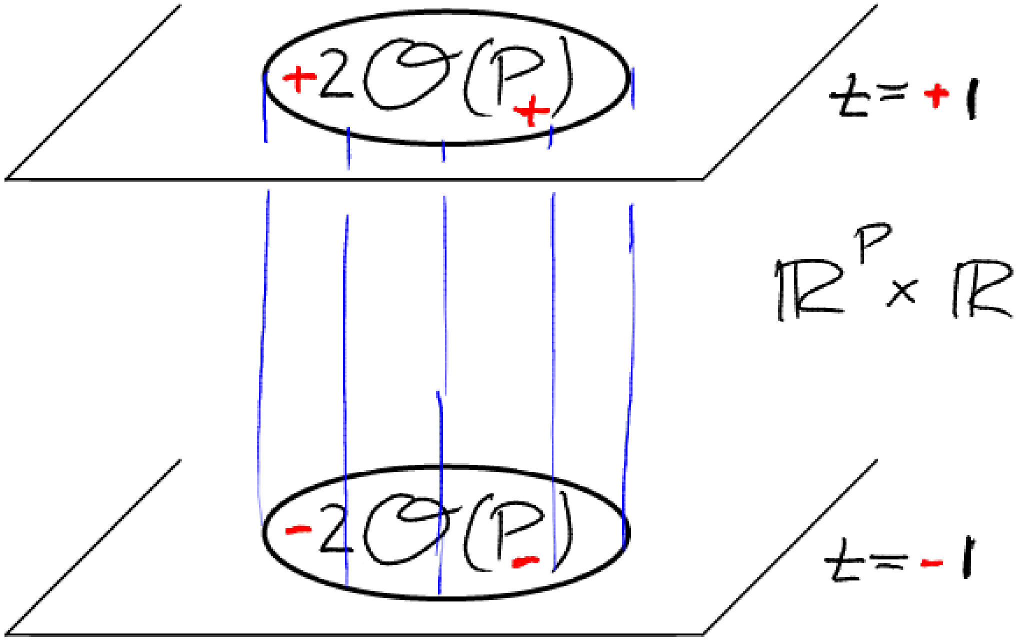


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vertices $\hat{=}$

$\mathrm{Filters}(P_+) \,\dot\cup\, \mathrm{Filters}(P_-)$

Facets correspond to alternating chains / cycles of $\mathbb{P} \cup \{\hat{0},\hat{1}\}$

alternating chain in $\mathbb{P}$

$c_1 \prec_+ c_2 \prec_- c_3 \prec_+ \cdots \prec_\pm c_k$ or

$c_1 \prec_- c_2 \prec_+ c_3 \prec_- \cdots \prec_\mp c_k$ ... alternating cycle not every alternating cycle yields a facet !

$$P_- = P_+^{op} \implies \mathcal{O}(P) \cong \mathrm{prism}(\mathcal{O}(P_+))$$

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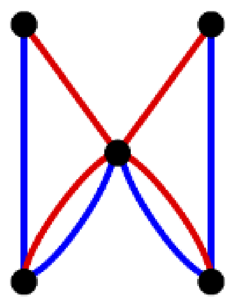
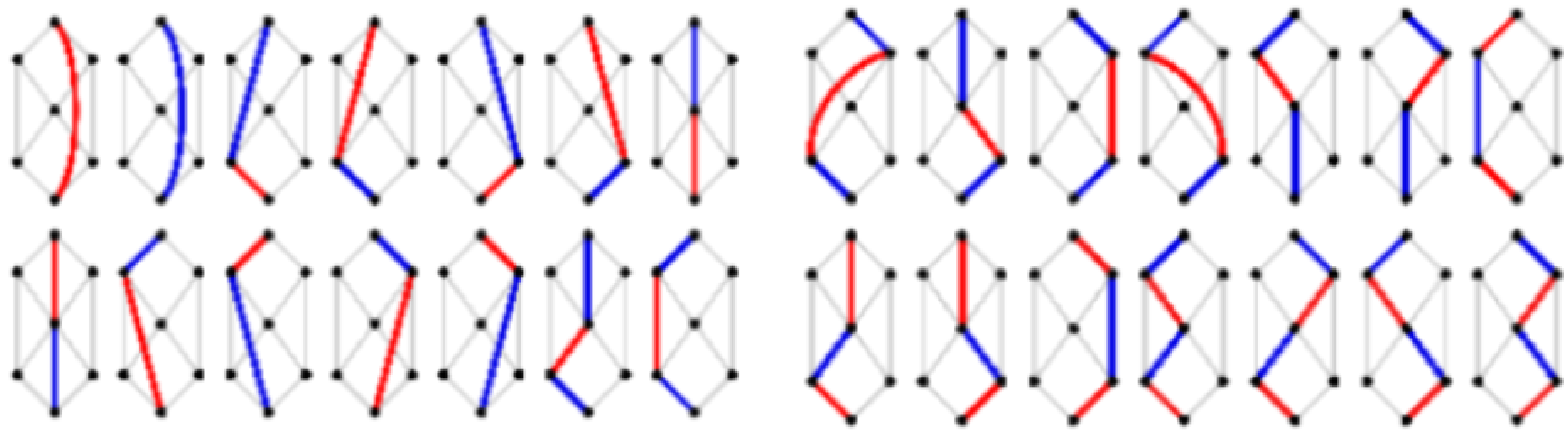
<u>Thm.</u> $\mathbb{P}$ compatible. Facets $\xleftrightarrow{1:1}$ alternating chains





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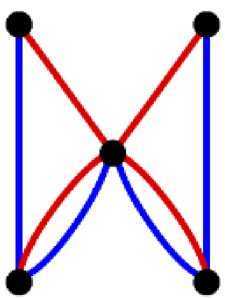
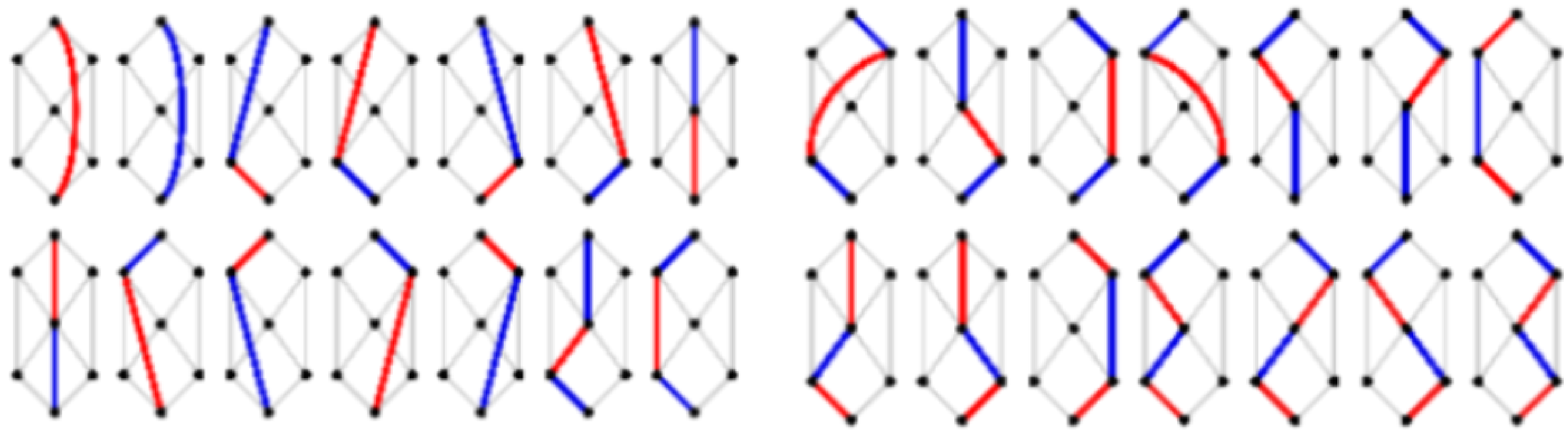
[Benjes'17+]: Facets of incompatible double posets





$\mathbb{P} = (P, \preceq_+, \preceq_-)$ $\quad P_+ = (P, \preceq_+)$ $\quad P_- = (P, \preceq_-)$

- $P_+ = P_- = d$-antichain $\quad \mathcal{O}(\mathbb{P}) = (d+1)$-cube
- $P_+ = P_- = d$-chain $\quad \mathcal{O}(\mathbb{P}) = (d+1)$-crosspolytope

$\mathbb{P} = (P, \preccurlyeq_+, \preccurlyeq_-)$  $P_+ = (P, \preccurlyeq_+)$  $P_- = (P, \preccurlyeq_-)$

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- $\mathbb{P} = (P, \preccurlyeq, \preccurlyeq)$ induced

  vertices $\xleftrightarrow{2:1}$ filters  facets $\xleftrightarrow{2:1}$ chains

- $\mathbb{P} = ([n], \preccurlyeq_\pi, \preccurlyeq_\pi)$ induced from dim-2 poset

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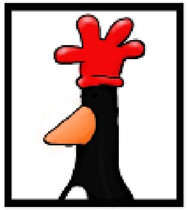

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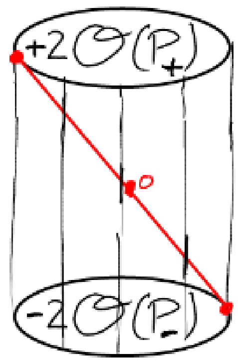
# Duals of double poset polytopes

$0 \in \mathcal{O}(\mathbb{P})$ by construction



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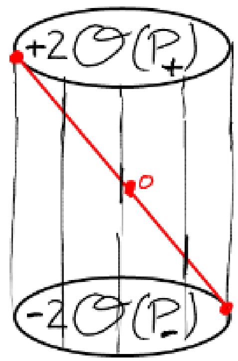

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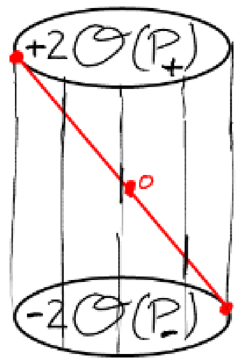


$P$ poset, Birkhoff lattice $\mathcal{J} = (\text{Filters}(P), \subseteq)$

Valuation $\nu : \mathcal{J} \longrightarrow \mathbb{R}$ $\quad \nu(\emptyset) = 0$

$$\nu(F \cup F') = \nu(F) + \nu(F') - \nu(F \cap F') \qquad F, F' \subseteq P \text{ filters}$$

Valuation polytope $\quad Val(P) := \{ \nu : \mathcal{J} \to [0,1] \text{ valuation} \}$
[Geissinger '81]

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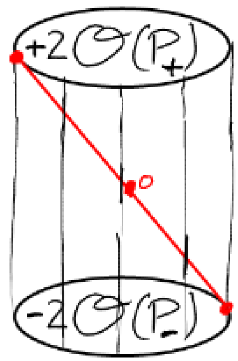


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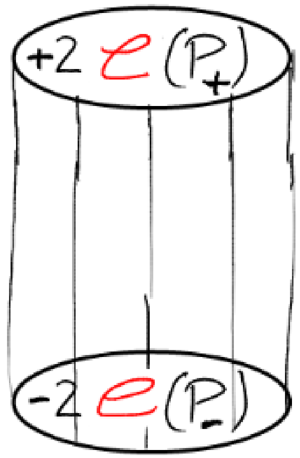
[Geissinger '81]

Thm [CFS'16]. $\mathbb{P} = (P, \preceq, \preceq)$ induced

$$\mathcal{O}(\mathbb{P})^{\Delta} \cong \text{conv}\left( Val(P) \times \{1\} \cup -Val(P) \times \{-1\} \right).$$

# Double chain polytopes

$$\mathcal{C}(\mathbb{P}) := \mathrm{conv}\left\{ \begin{array}{c} 2\cdot\mathcal{C}(P_+)\times\{1\} \\ \cup \\ -2\cdot\mathcal{C}(P_-)\times\{-1\} \end{array} \right\}$$



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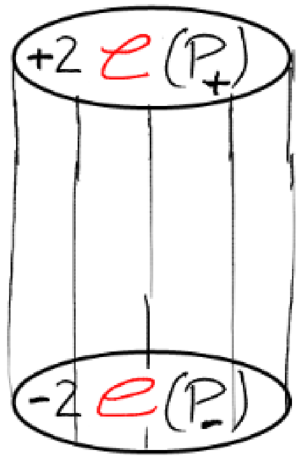


$\dim \mathcal{C}(\mathbb{P}) = |P| + 1.$

vertices $\triangleq$ Anti-chains$(P_+)$ $\dot\cup$ Anti-chains$(P_-)$

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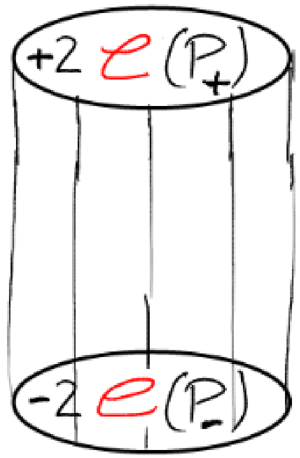


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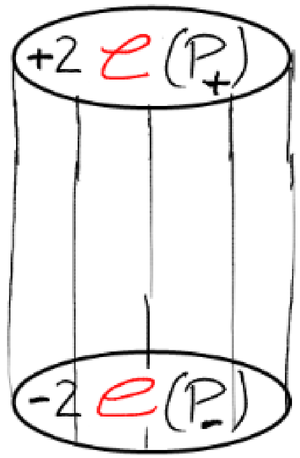


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# Ehrhart polynomials

**Reduced** double poset polytopes

$$\overline{\mathcal{O}}(\mathbb{P}) := \mathcal{O}(P_+) + (-\mathcal{O}(P_-))$$

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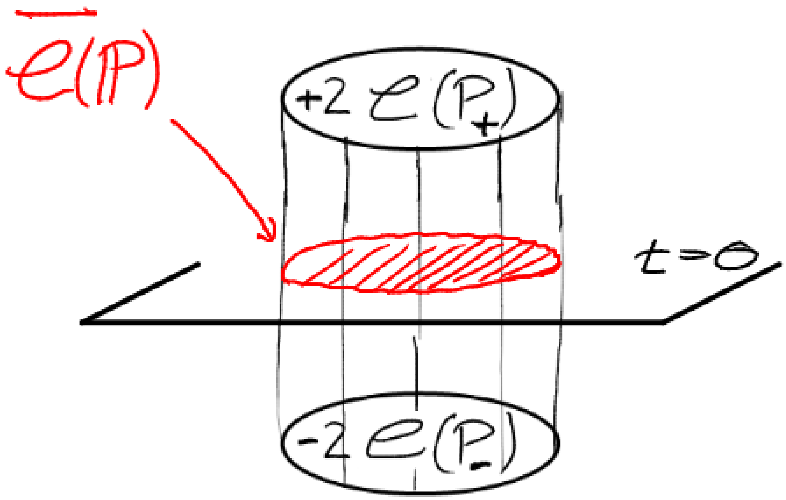

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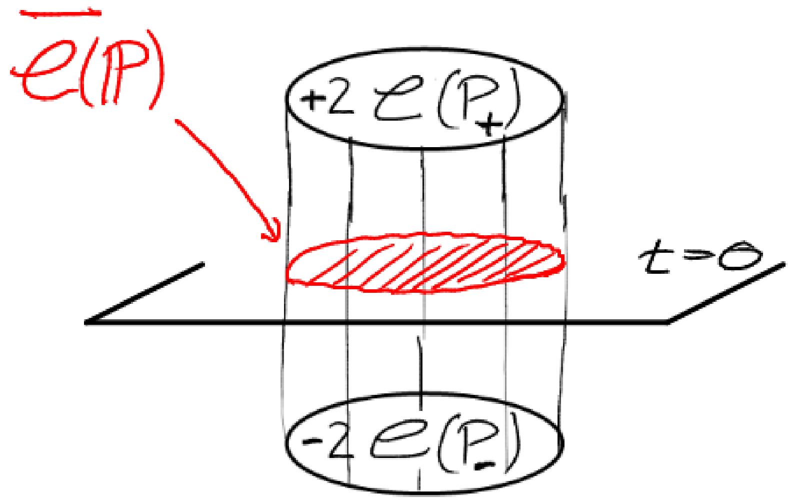


strictly order pres. $P \to [n-1]$

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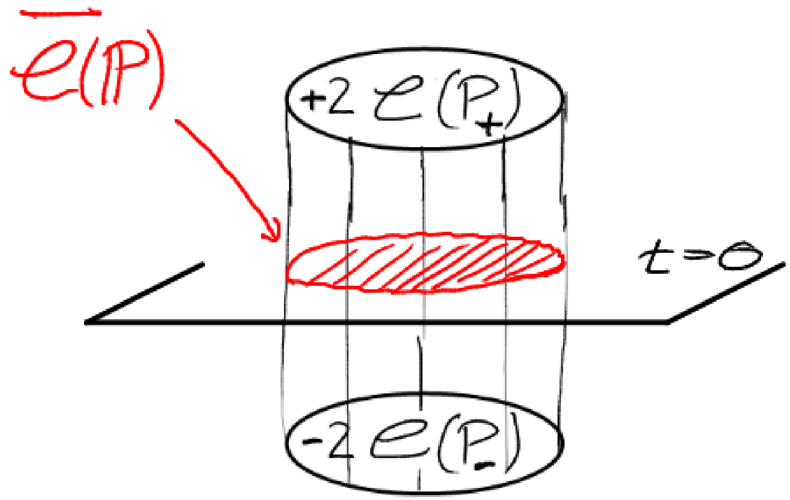


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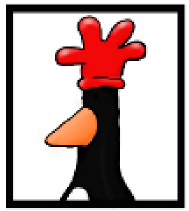

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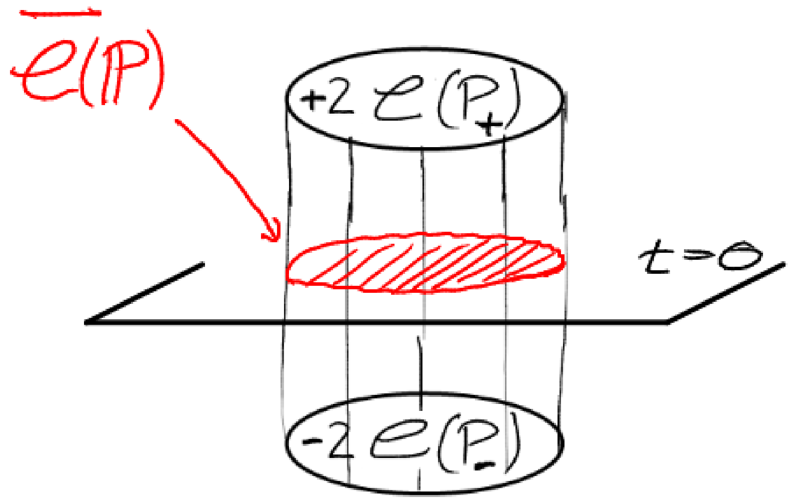


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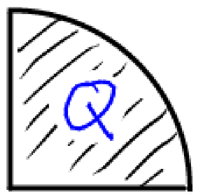
Q: What are the $h^*$-vectors of $\overline{\mathcal{O}}(\mathbb{P})$ resp. $\overline{\mathcal{C}}(\mathbb{P})$?

# Minkowski sums of antiblocking polytopes

[Fulkerson '71.] Polytope $Q \subset \mathbb{R}^d_{\geq 0}$ is antiblocking (AB)

$$y \in Q, \quad x \in \mathbb{R}^d \quad \underset{\text{for } i=1,\ldots,d}{0 \leq x_i \leq y_i} \implies x \in Q$$

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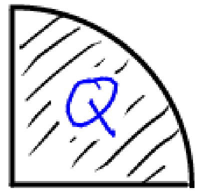


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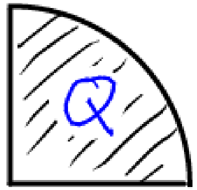


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Thm [CFS'17]. $P, Q \subseteq \mathbb{R}^d$ full-dim antiblocking polytopes

If $P$ & $Q$ have a regular, unimodular, & flag triangulation, then so does $P \boxminus Q$.

$J \subseteq [d]$ $\quad Q|_J := \{ x \in Q : x_i = 0 \ \forall i \notin J \} \subseteq \mathbb{R}^J$

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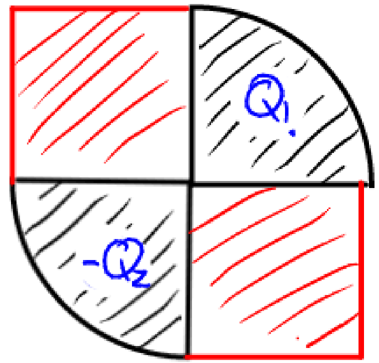
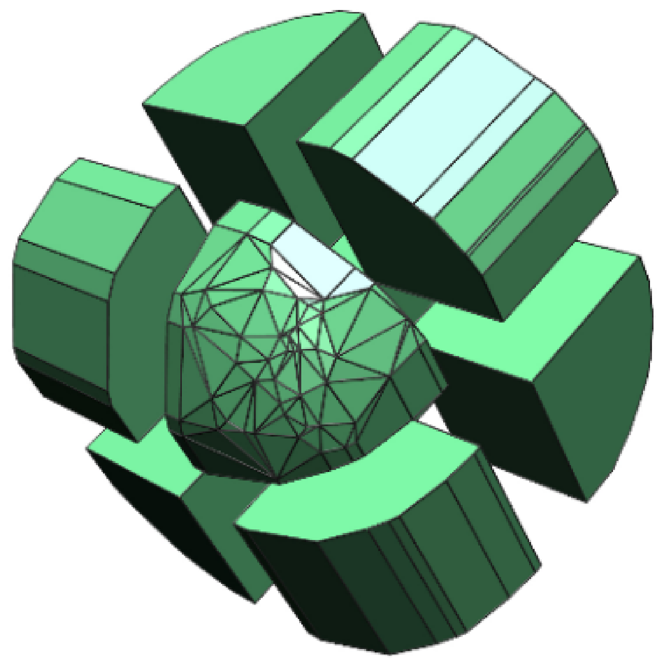
<u>Lemma</u> [CFS'16]. $Q_1, Q_2 \subseteq \mathbb{R}^d_{\geq 0}$ AB

$$Q_1 + (-Q_2) = \bigcup_{J \subseteq [d]} \underbrace{Q_1|_J + (-Q_2|_{J^c})}_{\cong \, Q_1|_J \times Q_2|_{J^c}}$$



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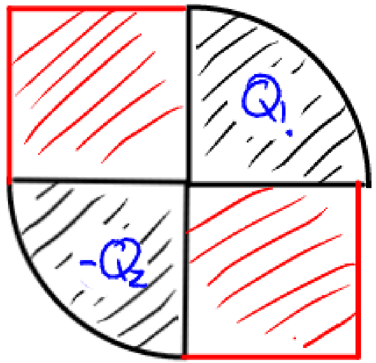
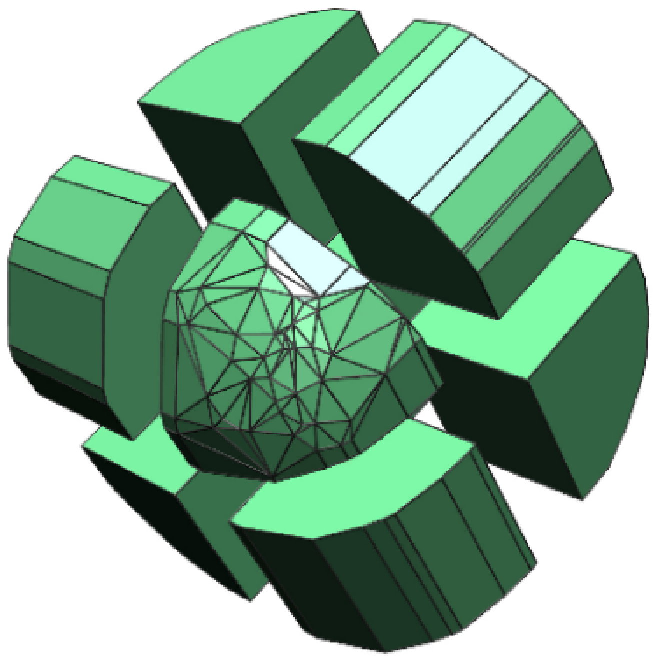



<u>Cor.</u> $Q_1, Q_2 \subset \mathbb{R}^d_{\geq 0}$ AB

$$\operatorname{vol}(Q_1 + (-Q_2)) = \sum_J \operatorname{vol} Q_1|_J \cdot \operatorname{vol} Q_2|_{J^c}$$

Thm via Cayley trick.

... formulas for Ehrhart quasi-polynomials



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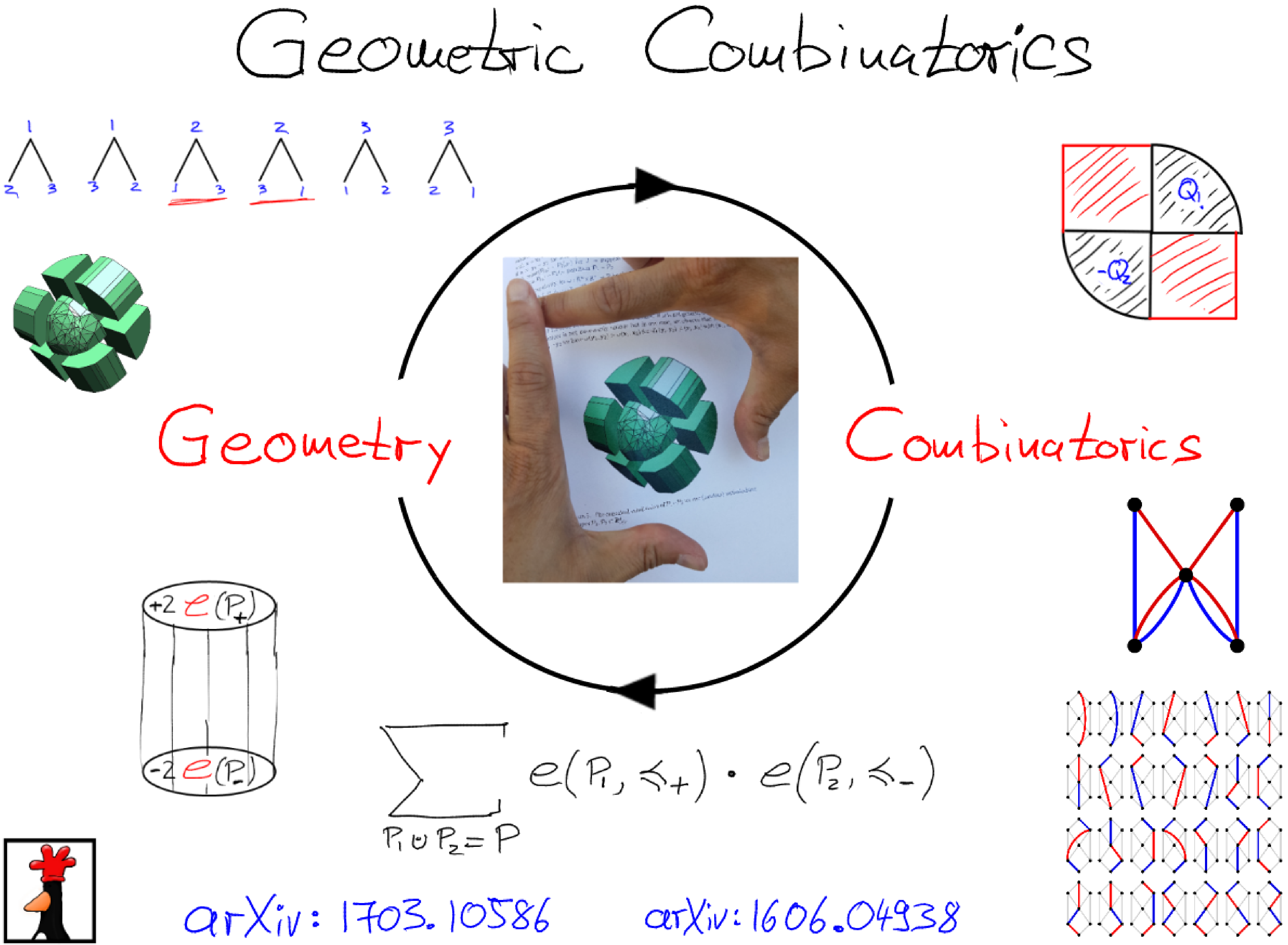
- squarefree quad Gröbner bases of affine semigroup rings of $\mathcal{O}(\mathbb{P})$ & $\mathcal{C}(\mathbb{P})$ "Double Hibi rings"


Geometric Combinatorics
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arXiv: 1703.10586
arXiv: 1606.04938