f-vectors of simplicial complexes & polytopes: Tales & Tools, part I — 9/6/17

Eran Nevo — 10:30am

Combinatorics:

Simplicial complex $\Delta \subseteq 2^{[n]}$ s.t. $T \subseteq F \in \Delta \Rightarrow T \in \Delta$

i-dim faces: $f_{i-1}(\Delta) = \# \{ F \in \Delta \mid |F| = i \}$

f-vector: $f(\Delta) = (f_{-1}, f_0, f_1, \ldots)$

f_{-1}: 1 for $\emptyset$

$f($  $) = (1, 4, 5, 1)$

Which $f \in \mathbb{Z}^d_{\geq 0}$ are f-vectors of simplicial complexes?

Theorem: (Kruskal '63, Katona '68, Schutzenberger '59)

$\mathbb{Z}^d \ni (f_0, f_1, \ldots, \underset{\neq 0}{f_{d-1}}) = f(\Delta) \iff \forall 0 \leq i \leq d-1: \ 0 \leq f_{i+1} \leq f_i^{(i+1)}$

"upper shadow function"

Lemma: $\forall n, i \in \mathbb{Z}_{>0}$, $\exists !\ n = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}$

for $n_i > n_{i-1} > \ldots > n_j \geq 1$

Def: The upper shadow function of n, i is

$n^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \ldots + \binom{n_j}{j}$

Pf idea ($\Rightarrow$) combinatorial shifting, ($\Leftarrow$) compression

Example: $f_1 = 5 = \binom{3}{2} + \binom{2}{2}$

Multicomplexes: M = family of multisets, closed under inclusion

Example: $\{\emptyset, \{1\}, \{1,1\}, \{2\}, \{2,1\}\} = M$

$\leftrightarrow \{1, x, x^2, y, xy\}$

$f(M) = (1, 2, 2)$

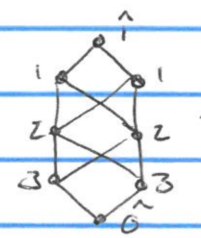
Balanced complexes: Δ s.c., $d = \max\{|F| \mid F \in \Delta\}$

$c: [n] \to [d]$ coloring s.t. $c|_F$ is injective $\forall F \in \Delta$

Example: graph balanced $\Leftrightarrow$ bipartite

Example: $\Delta(P)$

order complex



$= P$

Flag complexes: $\Delta = \{$cliques of $G\}$ G simple graph

Example: $G =$  $\longrightarrow$  $= \Delta(G)$

Complexes:

What f-vectors correspond to different groups?

Multicomplexes – Macaulay

Balanced – Frankl-Füredi-Kalai

Flag – Open!

Flag complex: $f_1 = 6$

$f_2 \in \{0, 1, 2, 4\}$

$\uparrow$ holes in the sequence

Frohmader: $\{f(\text{flag})\} \subseteq \{f(\text{balanced})\}$

Geometry

polytopes $P = \text{conv}(v_1, \dots, v_n) \quad v_i \in \mathbb{R}^d$

$=$ bounded intersection of finitely many half-spaces

Face-lattice of P

$L(P) = \{P\} \cup \{F = P \cap H \mid P \subseteq H^+\}$ graded by $\dim F = \dim(\text{aff } F)$

Euler-Poincaré: $\dim P = d \Rightarrow$

$$(-1)^{d-1} = \sum_{F \in L(P) - \{P\}} (-1)^{\dim F} = -f_{-1} + f_0 - f_1 + \dots$$

Simplicial polytopes: P s.t. all proper faces are simplices $\quad \Delta = \partial P = L(P) - \{P\}$ a s.c.

Linear relations (Dehn-Sommerville)

$h_i(P) = h_{d-i}(P) \quad \forall\, 0 \le i \le \lfloor \frac{d}{2} \rfloor$

where $\displaystyle\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x-1)^{d-i}$

$P = \Diamond^3$

f

1 take differences Stanley's trick

1 $\leftarrow$ 6

1 5 12

1 4 7 8

$h \rightarrow$ 1 3 3 1

Theorem (g-thm) (Billera-Lee, Stanley)

$\Leftarrow$ '80 $\Rightarrow$ '80

$h = h(P) \iff h_i(P) = h_{d-i}(P) \;\; \forall\, 0 \le i \le \lfloor d/2 \rfloor$ and

$g = (h_0, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$

$= f(M)$, M a multicomplex

Extremal cases:

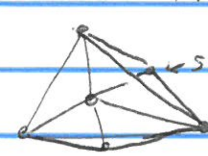
Lower Bound Thm (Barnette '71, '73):

$g_2 \ge 0 \iff f_1 \ge d f_0 - \binom{d+1}{2}$

(MPW-reduction) $\iff f_i(P) \ge f_i(\underbrace{S(d, f_0)}_{\text{stacked polytope}}) \quad \forall i$

and equality for some $i > 1$ iff P stacked.

Stacked polytope: $d = 3$



shallow pyramids over simplex

Kalai's proof: $P_{\le 1} = G$, graph

$\longrightarrow$ geometric matrix $R(G)$ $d f_0 \times f_1$

$\mathrm{rank}(R(G)) = d f_0 - \binom{d+1}{2}$

Generalized Lower Bound Theorem:

(Conj: McMullen - Walkup '71, pf: Murai - N.)

$g_i = 0 \iff P$ $(i-1)$-stacked

$P = \| \partial P (d-i) \|$

$\Delta(j) = \{ F \subseteq \Delta_0 \mid \binom{F}{\leq j} \subseteq \Delta \}$

Upper Bound Theorem: (McMullen '70)

$f_i(P) \leq f_i(C(d, f_0))$

cyclic polytope

$C(d,n) = \mathrm{conv}(x_1, \ldots, x_n)$ $\quad x_i = (t_i, t_i^2, \ldots, t_i^d)$

for $t_1 < \ldots \leq t_n$

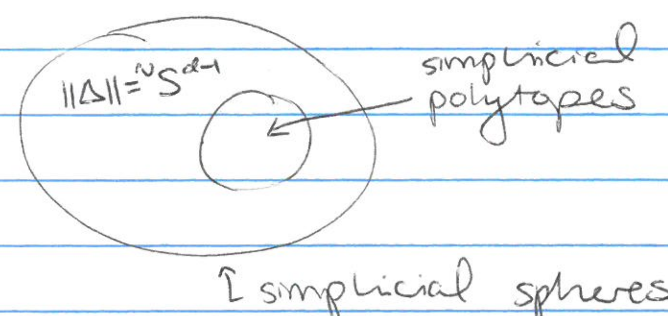
(moment curve)

Neighborly: $i \leq \lfloor d/2 \rfloor - 1 \Rightarrow f_i = \binom{[n]}{\leq i+1}$

g-conjecture (McMullen):

$$\left\{ f\left(\begin{matrix} \text{simplicial} \\ d\text{-polytopes} \end{matrix} \right) \right\} = \left\{ f\left(\begin{matrix} \text{simplicial} \\ (d-1)\text{-spheres} \end{matrix} \right) \right\}$$

(alg. version)



Classes of Simplicial Polytopes

Balanced - GLBT: (conj: Klee-Novik)

1. $\frac{h_0}{\binom{d}{0}} \leq \frac{h_1}{\binom{d}{1}} \leq \frac{h_2}{\binom{d}{2}} \leq \dots$

2. $\frac{h_i}{\binom{d}{i}} = \frac{h_{i-1}}{\binom{d}{i-1}} \Longrightarrow$ P is "balanced k-stacked"

Proof: 1. Juhnke-Kubitzke + Murai
2. Adiprasito

Balanced UBT (Stanley '79)

$h(P) = f(\Gamma)$ $\quad \Gamma$ = balanced simplicial complex