"On Ehrhart positivity"
Fu Liu

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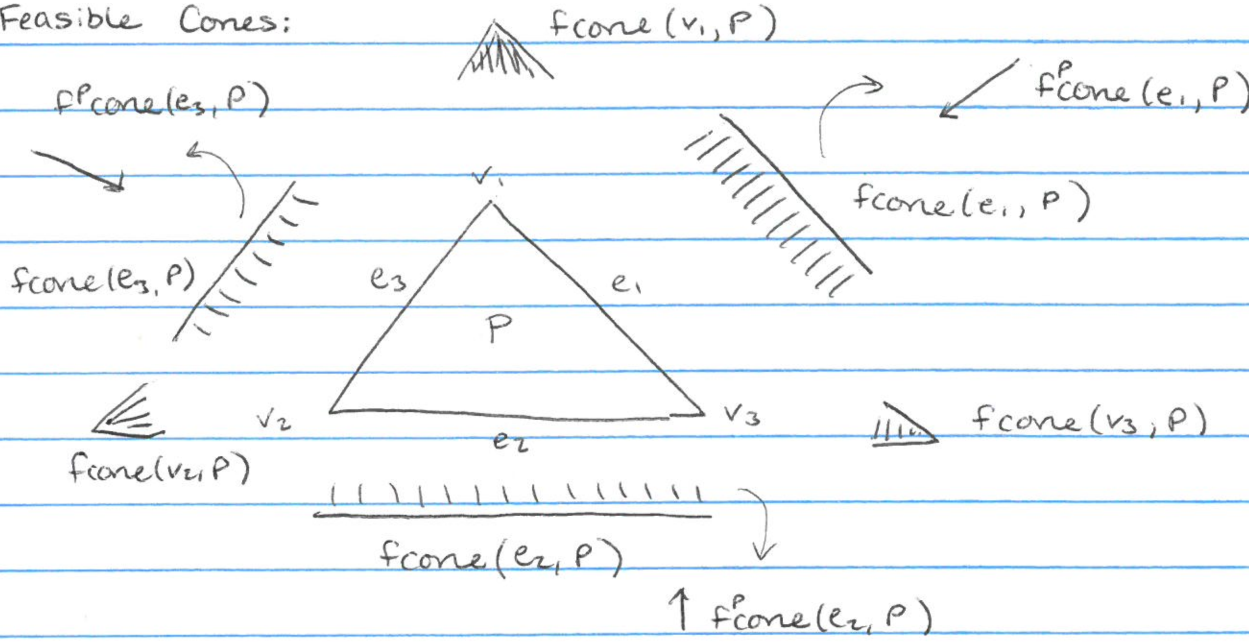

# On Ehrhart positivity

*Fu Liu*

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Introductory Workshop: Geometric and Topological Combinatorics

MSRI

September 6, 2017

# Outline

- Introduction
  - Polytopes and Ehrhart positivity
- McMullen's formula and consequences
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  - A positivity conjecture
- Positivity for generalized permutohedra (joint work with Castillo)
  - Generalized permutohedra
  - Reduction theorem
  - Partial results to the conjecture
- Other questions and results (joint work with Castillo-Nill-Paffenholz)

# PART I:

## Introduction

## Lattice points of a polytope

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**Definition.** For any polytope $P \subset \mathbb{R}^d$ and positive integer $t \in \mathbb{N}$, the $t$th dilation of $P$ is $tP = \{t\mathbf{x} \,:\, \mathbf{x} \in P\}$. We define

$$i(P, t) = |tP \cap \mathbb{Z}^d|$$

to be the number of lattice points in the $tP$.

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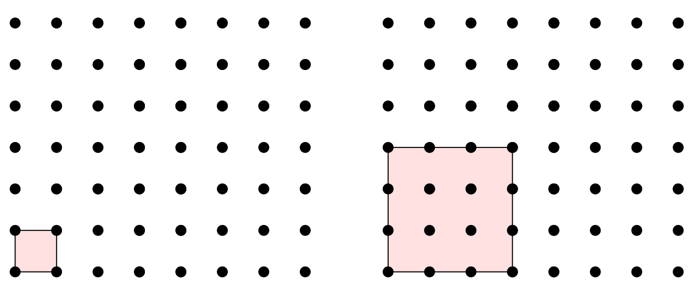
Example: For any $d$, let $\square_d = \{\mathbf{x} \in \mathbb{R}^d : 0 \leq x_i \leq 1, \forall i\}$ be the *unit cube* in $\mathbb{R}^d$. Then $t\square_d = \{\mathbf{x} \in \mathbb{R}^d : 0 \leq x_i \leq t, \forall i\}$ and $i(\square_d, t) = (t+1)^d$.



P 3P

## Theorem of Ehrhart (on integral polytopes)

**Theorem 1** (Ehrhart)**.** *Let $P$ be a $d$-dimensional integral polytope. Then $i(P,t)$ is a polynomial in $t$ of degree $d$.*

Therefore, we call $i(P,t)$ the *Ehrhart polynomial* of $P$.

# Coefficients of Ehrhart polynomials

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The leading, second, and last coefficient of $i(P,t)$ are **positive**.

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Question. What about the coefficient of $t^{d-2}, t^{d-3}, \dots, t^1$ in $i(P, t)$?

## Some negative results

- The *Reeve tetrahedron* $T_m$ is the polytope with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,m)$, where $m \in \mathbb{Z}_{>0}$. Its Ehrhart polynomial is

$$i(T_m, t) = \frac{m}{6}t^3 + t^2 + \frac{12-m}{6}t + 1.$$

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- In 2015, Hibi-Higashitani-Tsuchiya-Yoshida showed that each of coefficients of $t^{d-2}$, $t^{d-3}$, $\dots$, $t^1$ in $i(P,t)$ can be negative. Moreover, they can be simultaneously negative.

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- In 2016, Tsuchiya showed that any sign pattern is possible for the coefficients of $t^{d-2}, t^{d-3}, \dots, t^1$ in $i(P,t)$.

# Ehrhart positivity

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Characterize integral polytopes that are Ehrhart positive.

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In the literature, different techniques have been used to prove Ehrhart positivity.

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Polytope: Standard simplex.

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It can be computed that its Ehrhart polynomial is

$$\binom{t+d}{d}.$$

More explicitly, we have

$$\binom{t+d}{d} = \frac{(t+d)(t+d-1)\cdots(t+1)}{d!},$$

which expands positively in powers of $t$.

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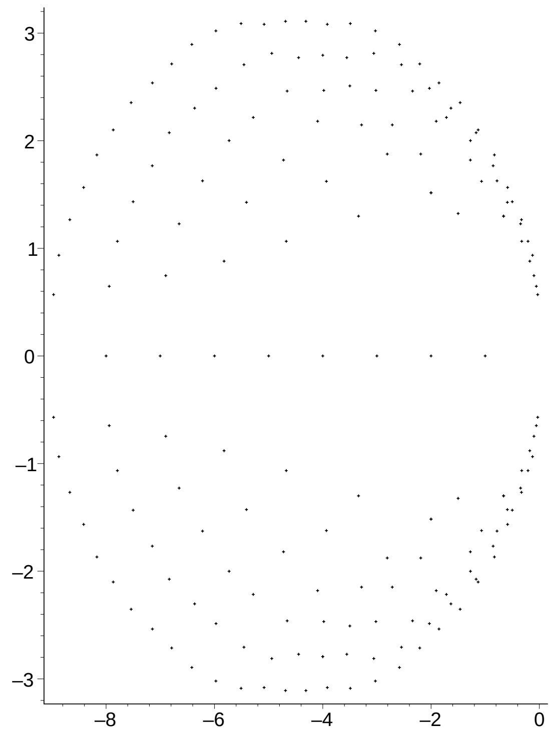
However, according to EC1, Exercise 4.61(b), every zero of $i(\Diamond_d, t)$ has real part $-1/2$. Thus it is a product of factors in the form of

$$(t + 1/2) \quad \text{or} \quad (t + 1/2 + ia)(t + 1/2 - ia) = t^2 + t + 1/4 + a^2,$$

where a is real, so Ehrhart positivity follows.

# More on roots

The following is the graph (Beck-DeLoera-Pfeifle-Stanley) of zeros for the Ehrhart polynomial of the Birkhoff polytope of doubly stochastic $n \times n$ matrices for $n = 2, \ldots, 9$.



## Example III

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**Polytope:** Zonotopes.

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**Definition.** A zonotope is the Minkowski sum of a set of vectors (in $\mathbb{R}^d$):

$$\mathcal{Z}(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k.$$

**Theorem 2** (Stanley)**.** *The coefficient of $t^i$ in $i(\mathcal{Z}(\mathbf{v}_1, \cdots, \mathbf{v}_k), t)$ is equal to*

$$\sum_{S=\{j_1,\dots,j_i\}\subseteq[k]} m(S),$$

*where $m(S)$ is the g.c.d. of all $i \times i$ minors of the $d \times i$ matrix*

$$M_S = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_{j_1} & \mathbf{v}_{j_2} & \cdots & \mathbf{v}_{j_i} \\ | & | & \cdots & | \end{bmatrix}.$$

The family of zonotopes includes:

- The unit cube $\Box_d = [0, 1]^d$ whose Ehrhart polynomial is

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- The *regular permutohedron*:

$$\begin{aligned}\Pi_{n-1} &= \mathrm{conv}\{(\sigma(1), \sigma(2), \cdots, \sigma(n)) \in \mathbb{R}^n : \sigma \in \mathfrak{S}_n\} \\ &\cong \sum_{1 \le i < j \le n} [e_i, e_j].\end{aligned}$$

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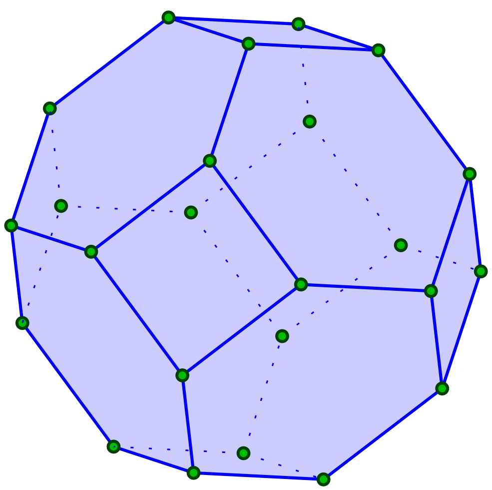

$$i(\Pi_3, t) = 1 + 6t + 15t^2 + 16t^3$$

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**Theorem 4** (L.). *For any rational polytope $P$, there exsits a polytope $P'$ with the same face lattice and Ehrhart positivity.*

Hence,

Ehrhart positivity is **not** a combinatorial property.

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$$i(kP,t) = i(P,kt) = \left(c_d k^d\right) t^d + \left(c_{d-1} k^{d-1}\right) t^{d-1} + \cdots + (c_1 k) t + c_0.$$

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They have exactly the **same sign patterns**.

However, dilating each coordinate with different parameter works.

**Question**

Are there other geometric ways to prove Ehrhart positivity?

## Other polytopes observed to be Ehrhart positive

- CRY (Chan-Robbins-Yuen).
- Tesler matrices (Mezaros-Morales-Rhoades).
- Birkhoff polytopes. (Beck-DeLoera-Pfeifle-Stanley)
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## Littlewood Richardson

Ron King conjectured that the *stretch* Littlewood Richardson coefficients $c_{t\lambda,t\mu}^{t\nu}$ are polynomials in $\mathbb{N}(t)$.

# PART II:

## McMullen's formula and consequences

# McMullen's formula

**Definition.** Suppose $F$ is a face of $P$. The *feasible cone* of $P$ at $F$, denoted by $\mathrm{fcone}(F, P)$, is the cone of all feasible directions of $P$ at $F$.

The *pointed feasible cone* of $P$ at $F$ is $\mathrm{fcone}^p(F, P) = \mathrm{fcone}(F, P)/L$, where $L$ is the subspace spanned by $F$. In general, $\mathrm{fcone}^p(F, P)$ is $k$-dim'l pointed cone if $F$ is codimensional $k$.

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In 1975 Danilov asked if it is possible to assign values $\Psi(C)$ to all rational cones $C$ such that the following *McMullen's formula* holds

$$|P \cap \mathbb{Z}^d| = \sum_{F:\text{ a face of } P} \alpha(F, P) \,\mathrm{vol}(F).$$

where $\alpha(F, P) := \Psi(\mathrm{fcone}^p(F, P))$.

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where $\alpha(F, P) := \Psi(\mathrm{fcone}^p(F, P))$.

McMullen proved it was possible in a non-constructive and nonunique way.

# Different Constructions

There are at least three different constructions for $\Psi$.

- Pommersheim-Thomas: Need to choose a flag of subspaces.
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We will use Berline-Vergne's construction, which we will refer to as the *BV-construction*.

# A refinement of positivity

Applying McMullen's formula to the dilation $tP$ of $P$, we obtain

$$i(P,t) = |tP \cap \mathbb{Z}^d| = \sum_{F:\text{ a face of } P} \alpha(tF, tP)\,\mathrm{vol}(tF)$$

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Therefore,

$\alpha(F,P) > 0$ for all $k$-dim'l face $F$ $\Longrightarrow$ the coefficient of $t^k$ of $i(P,t) > 0$

Moreover,

all $\alpha$ positive $\Longrightarrow$ Ehrhart positive

## (BV-)$\alpha$-positivity

**Definition.** We say a polytope $P$ is *$\alpha$-positive* if all the $\alpha(F, P)$ are positive for a given $\alpha$ construction.

We will use *BV-$\alpha$-positive* for Berline-Vergne's construction.

# A refined conjecture

**Conjecture 5.** *The regular permutohedron* $\Pi_{n-1}$ *is BV-*$\alpha$*-positive.*

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Why do we care?

**Proposition 6.** *The above conjecture implies that all integral generalized permutohedra are Ehrhart positive.*

# A refined conjecture

**Conjecture 5.** *The regular permutohedron* $\Pi_{n-1}$ *is BV-*$\alpha$*-positive.*

## Why do we care?

**Proposition 6.** *The above conjecture implies that all integral generalized permutohedra are Ehrhart positive.*

## A few facts on generalized permutohedra

- A family of polytopes has nice combinatorial properties, first studied by Postnikov.
- Matroid polytopes belong to this family.
- Postnikov showed that a subfamily, called the $y$-family, has Ehrhart positivity. (Matroid polytopes do not belong to the $y$-family.)

# Ambition

**Example V**

Polytope: Generalized permutohedra.

Reason: $\alpha$-positivity.

# PART III:

## **Positivity for generalized permutohedra**

Based on joint work with Castillo.

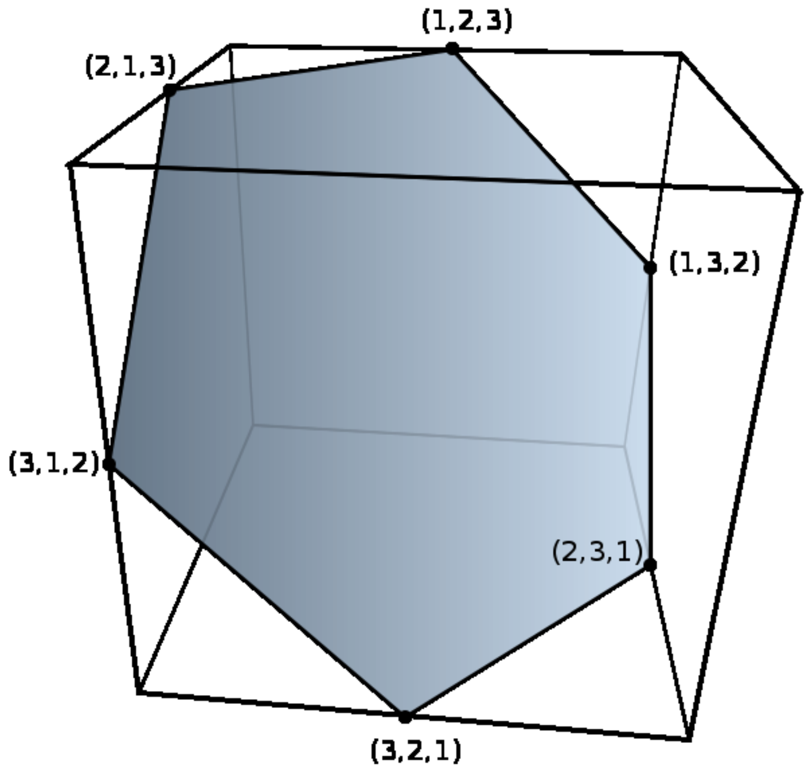
# Usual permutohedra

**Definition.** Suppose $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is a (nondecreasing) sequence. We define the *usual permutohedron*

$$\mathrm{Perm}(\mathbf{v}) := \mathrm{conv}\left\{ \left(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(n)}\right) \;:\; \sigma \in \mathfrak{S}_n \right\}.$$

- If $\mathbf{v} = (1, 2, \cdots, n)$, we get the *regular permutohedron* $\Pi_{n-1}$.

Example. $\Pi_2$:




Any usual permutohedron in $\mathbb{R}^n$ is $(n-1)$-dimensional.

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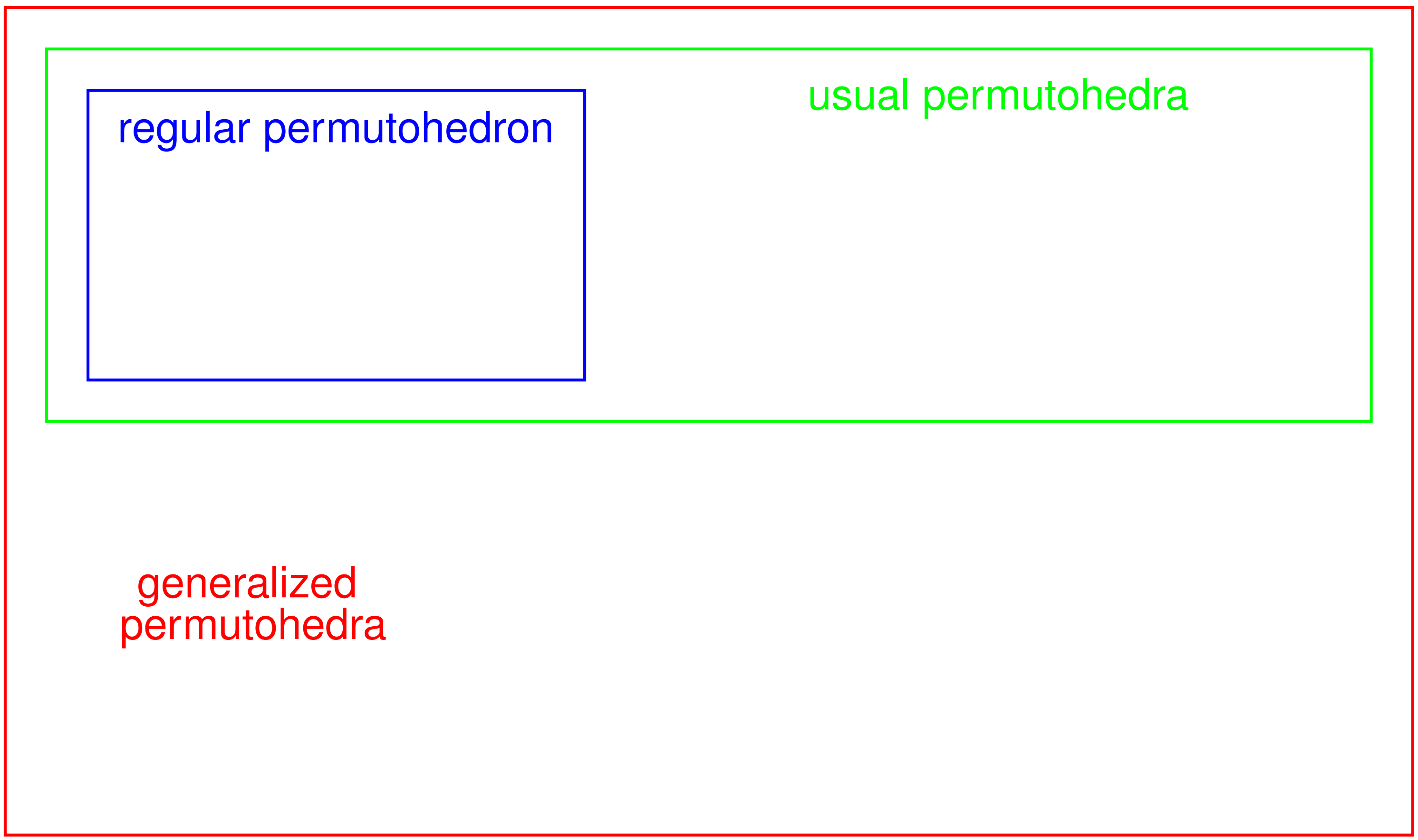

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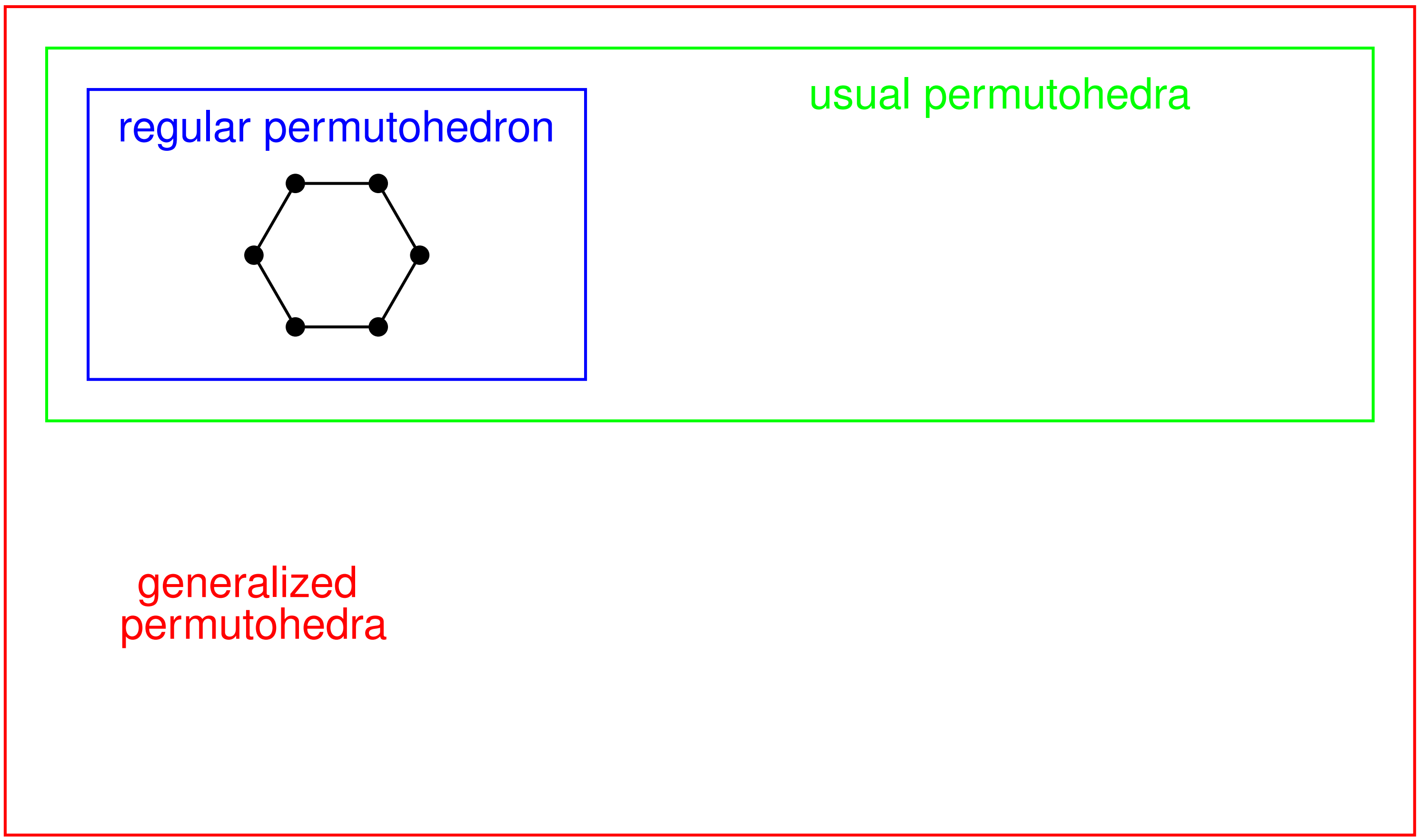
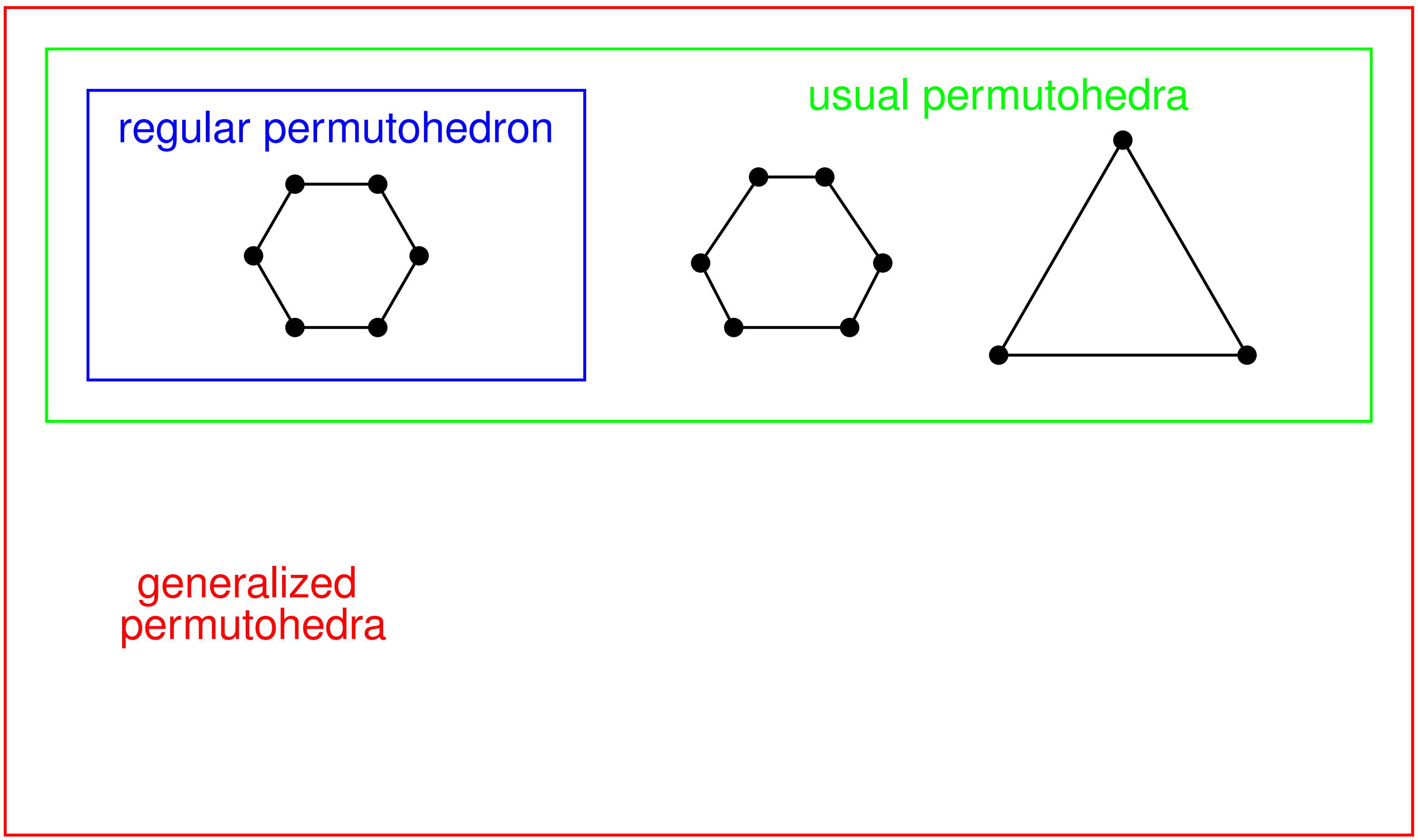


# Generalized permutohedra

**Definition** (Postnikov)**.** A *generalized permutohedron* is a polytope obtained from a usual permutohedron by moving the facets while keeping the normal directions.

For $n = 3$, we get $2$-dimensional polytopes.



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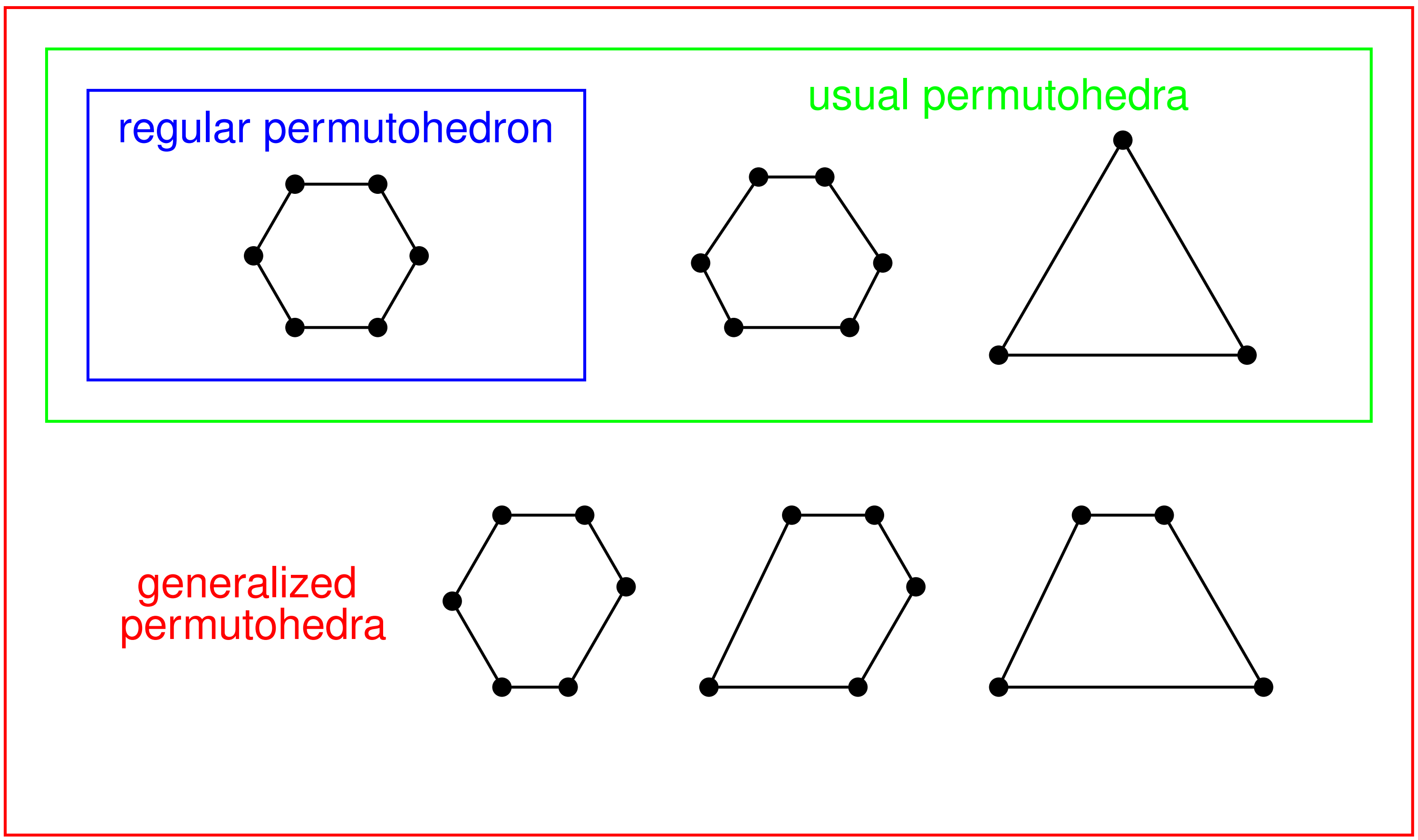

# Alternative definition

Let $V$ be the subspace of $\mathbb{R}^n$ defined by $x_1 + x_2 + \cdots + x_n = 0$. The *braid arrangement fan* denoted by $B_n$, is the complete fan in $V$ given by the hyperplanes

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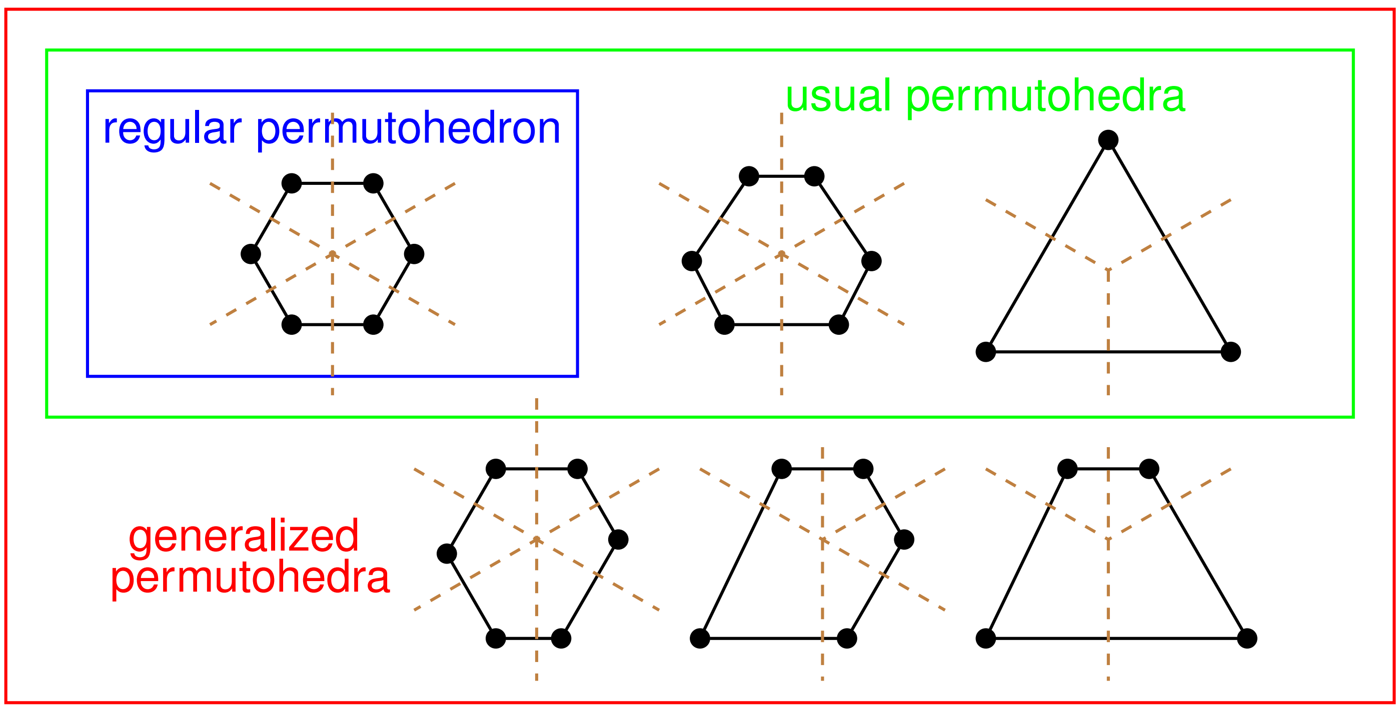

## Berline-Vergne's construction

For the rest of this part, we assume that $\alpha$ is the BV-construction.

**Important facts** about the BV-construction:

- Certain valuation property.
- Invariant under $O_n(\mathbb{Z})$ – orthogonal unimodular transformations, in particular invariant under rearranging coordinates with signs.

# Reduction Theorem

**Theorem 8.** *Suppose* $\alpha(F, \Pi_{n-1}) > 0$ *for any* $k$*-dimensional face* $F$ *of the regular permutohedron* $\Pi_{n-1}$*. Then* $\alpha(G, Q) > 0$ *for any* $k$*-dimensional face* $G$ *of any generalized permutohedron* $Q$ *in* $\mathbb{R}^n$*.*

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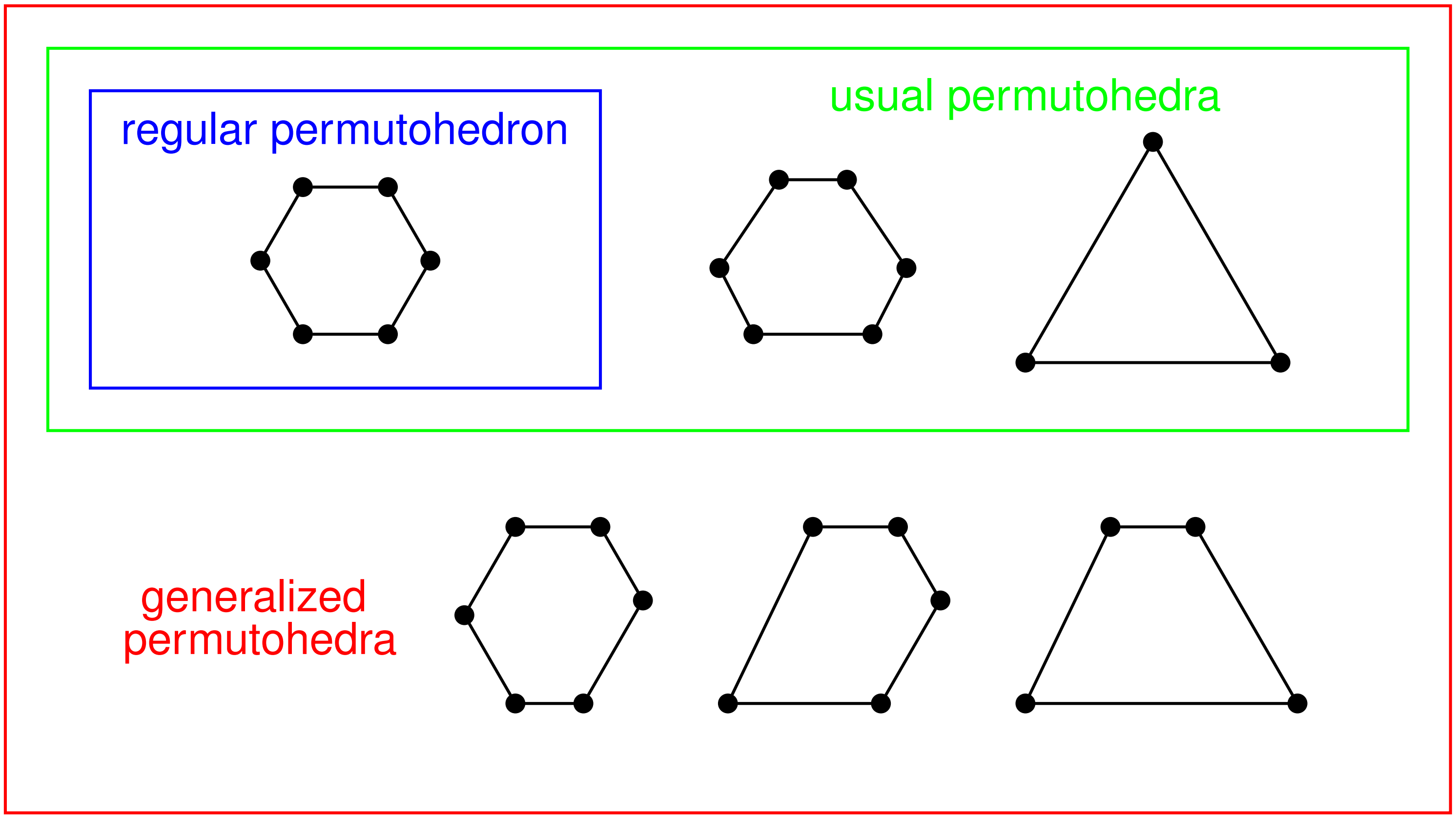

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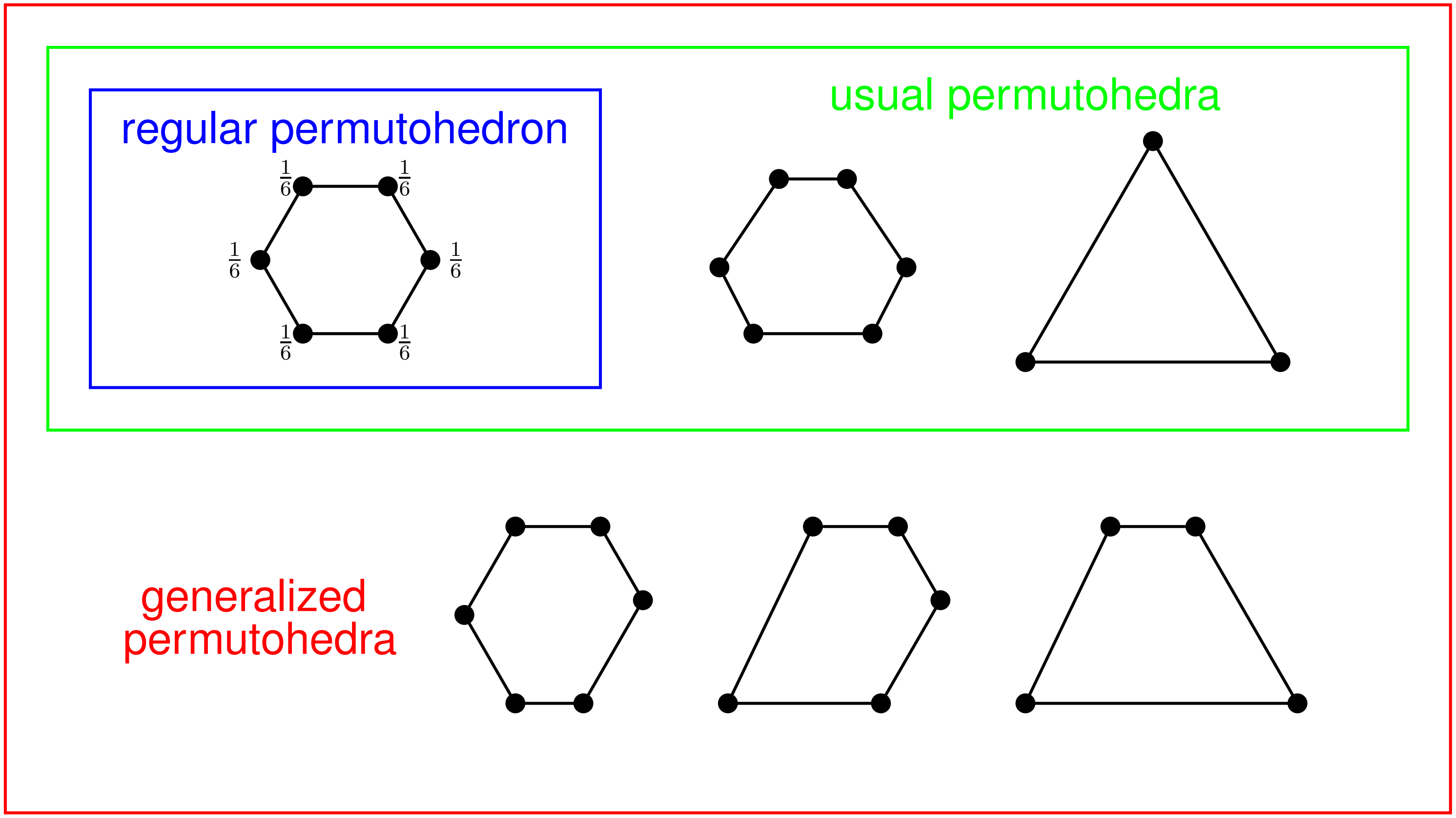

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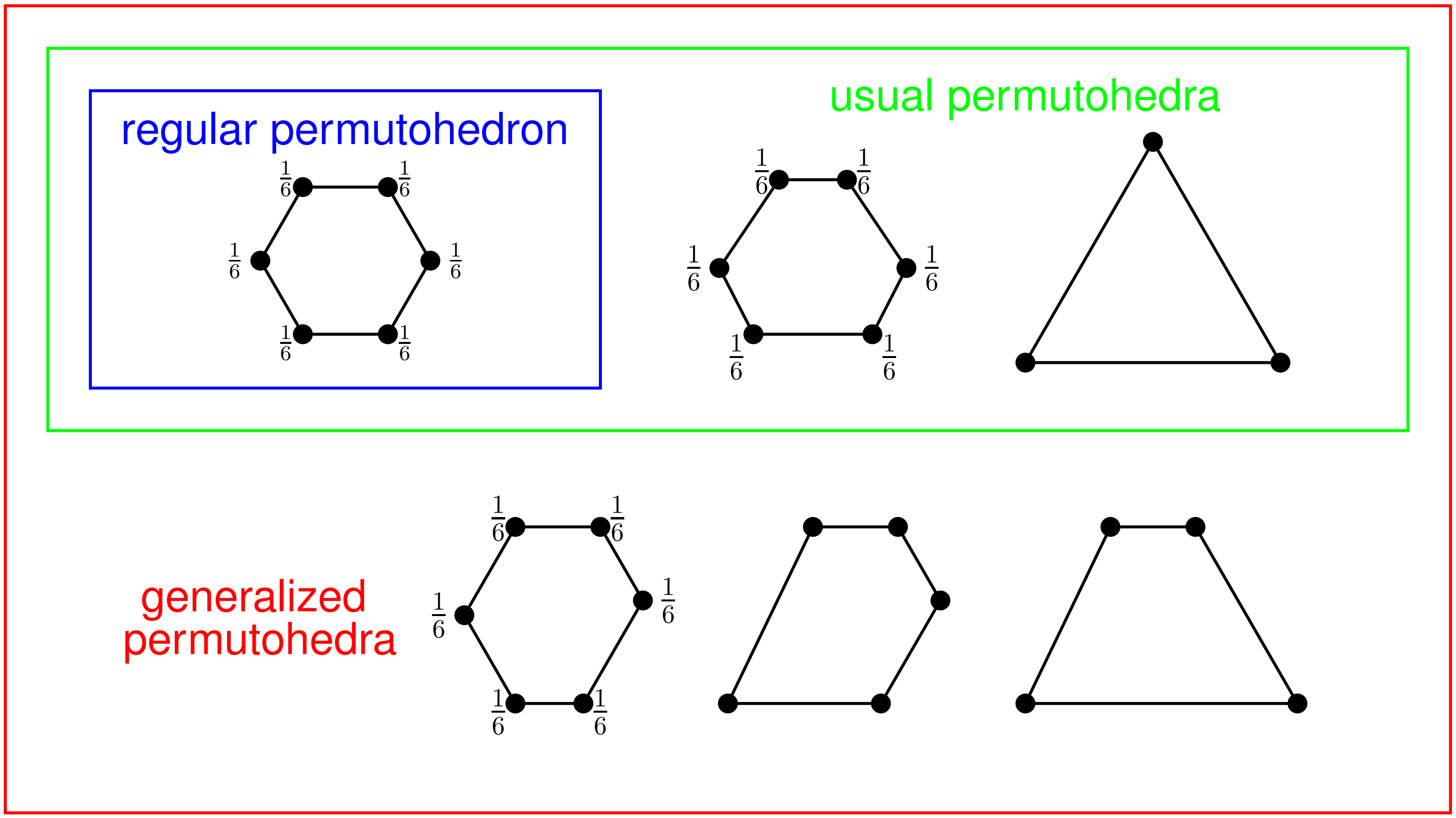

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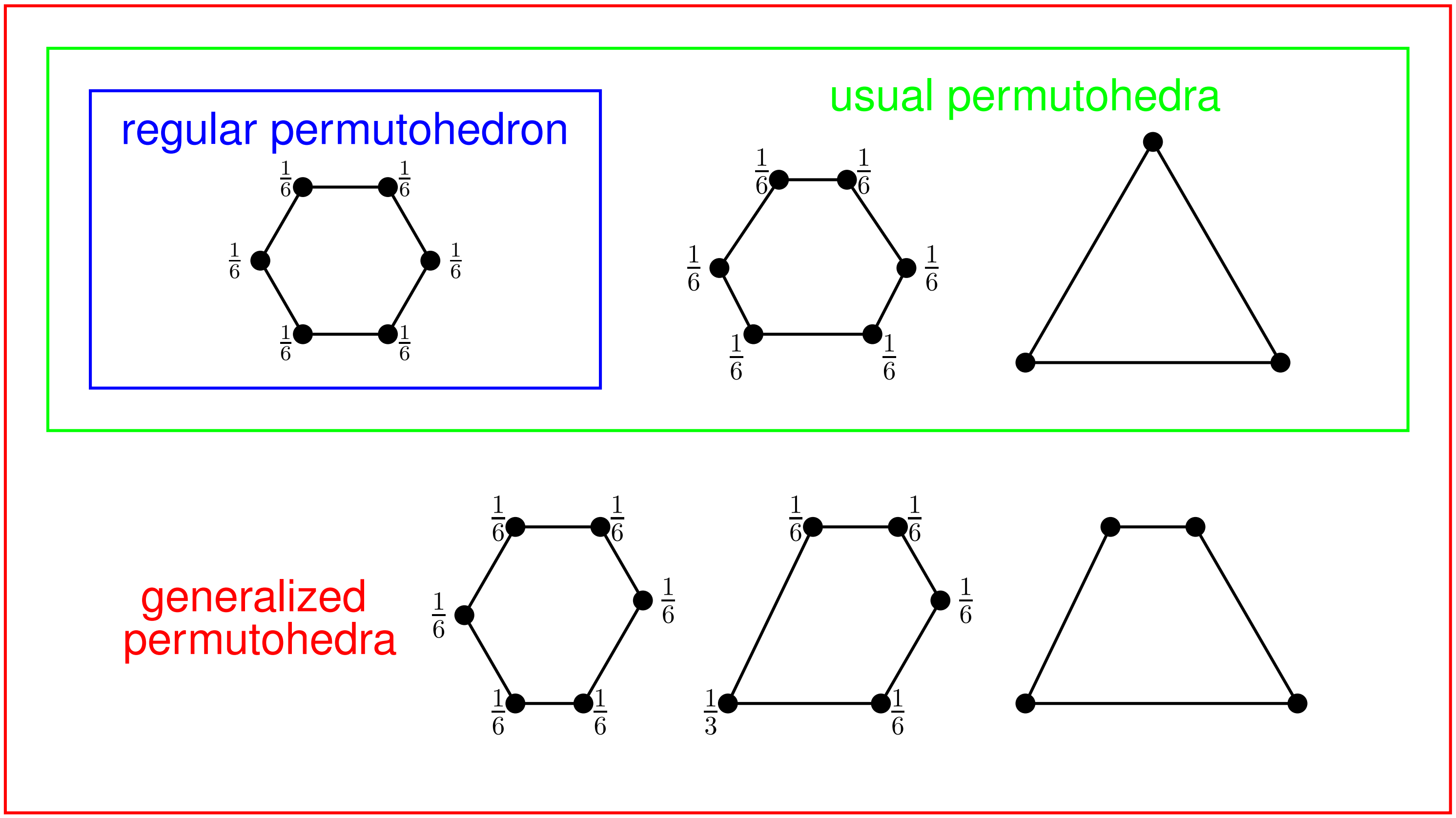

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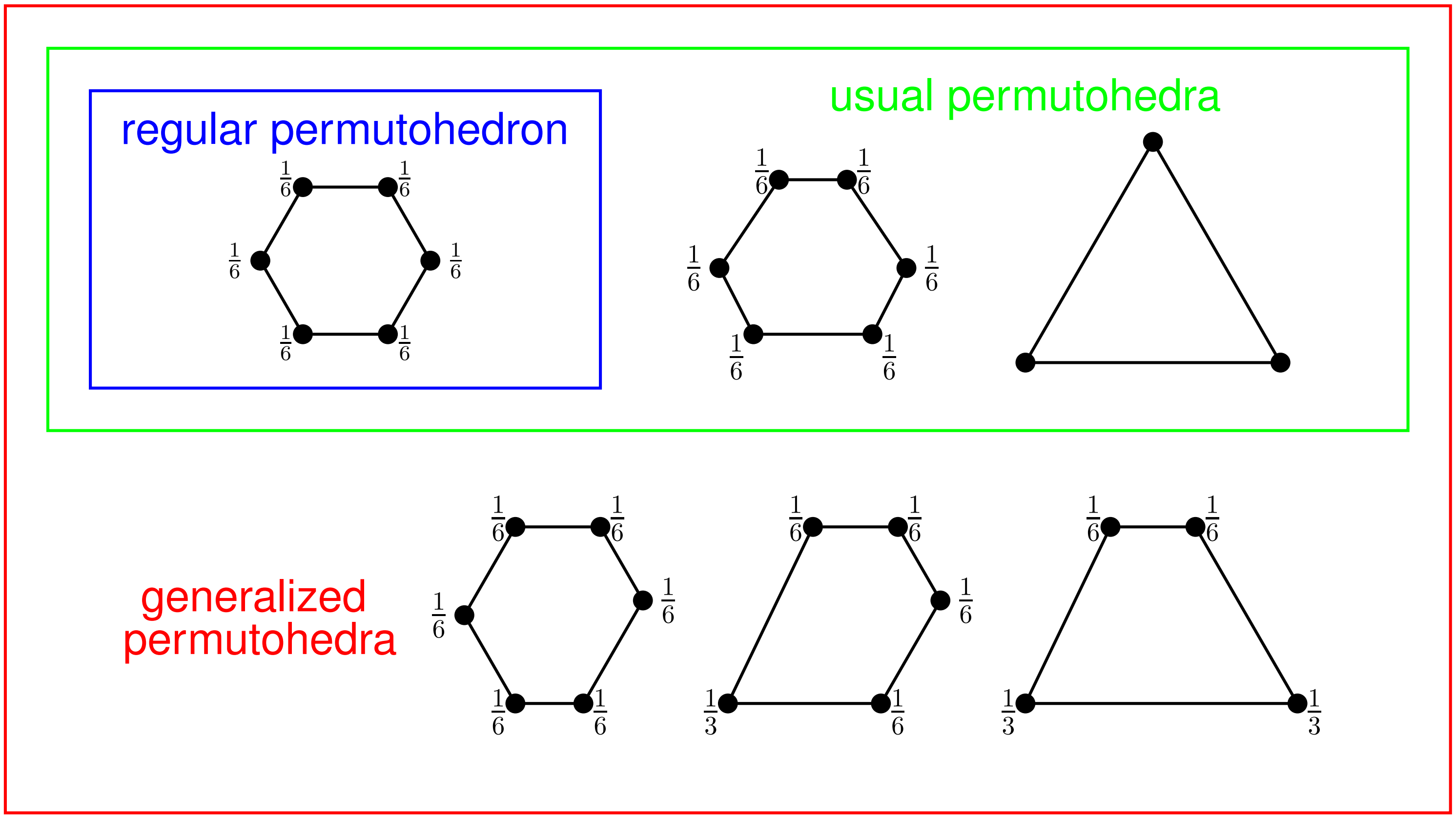

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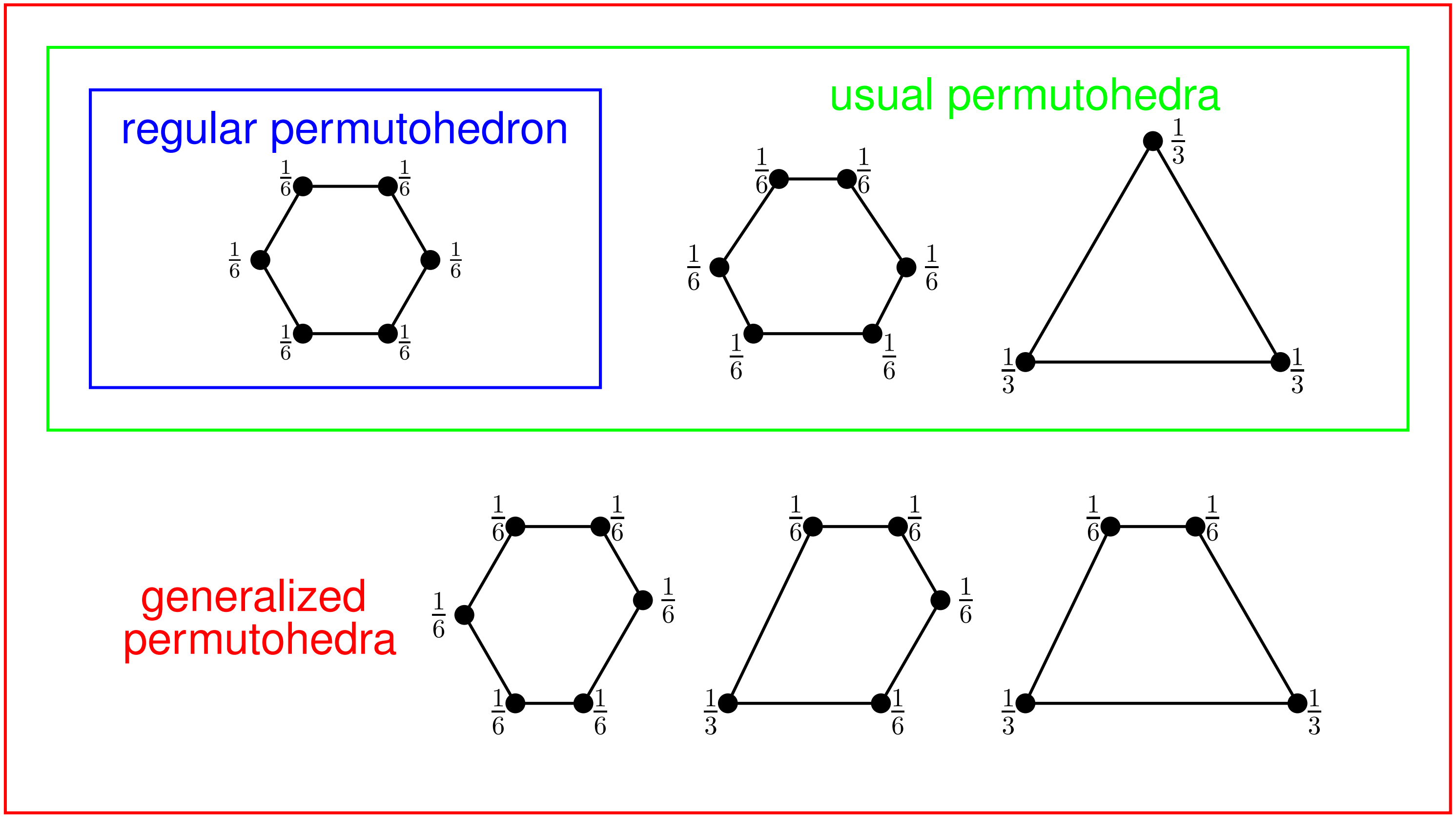

## A more general form of the reduction theorem

The reduction theorem is a consequence of the valuation property of the BV-construction for $\alpha$, thus does not only work for $\Pi_{n-1}$ and generalized permutohedra.

**Theorem 9.** *Suppose $Q$ is a deformation of $P$, or the normal fan of $P$ is a refinement of the normal fan of $Q$. If $\alpha(F, P) > 0$ for any $k$-dimensional face $F$ of $P$, then $\alpha(G, Q) > 0$ for any $k$-dimensional face $G$ of $Q$.*

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Applying the reduction theorem, we get:

**Corollary** (Castillo-L.).

*i. Any integral generalized permutohedron of dimension $\leq 6$ is Ehrhart positive.*

*ii. The third and fourth coefficients in the Ehrhart polynomial of any integral generalized permutohedron is positive.*

*iii. The linear coefficient in the Ehrhart polynomial of any integral generalized permutohedron of dimension $\leq 100$ is positive.*

# Proofs of the first two lemmas

Recall that

$$\alpha(F, P) := \Psi(\mathrm{fcone}^p(F, P)),$$

where $\Psi$ is a function that assigns values to all rational cones.

**Fact.** 1. Berline-Vergne's $\Psi$ is computed recursively. So lower dimensional cones are easier to compute.

2. If $F$ is a codimension $k$ face of $P$, then $\mathrm{fcone}^p(F, P)$ is $k$-dimensional.

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**Lemma** (Castillo-L.). $\alpha(F,\Pi_{n-1}) > 0$ *for any face $F$ of $\Pi_{n-1}$ of codimension $2$ or $3$.*

*Proof.* We have precise formulas for $\Psi$ of unimodular cones of dimension $\leq 3$. Applying these to regular permutohedra, we get $\alpha$-positivity for faces of codimension $\leq 3$. □

## The third lemma

**Lemma** (Castillo-L.). $\alpha(E, \Pi_{n-1}) > 0$ *for any edge* $E$ *of* $\Pi_{n-1}$ *of dimension* $\leq 100$.

The approaches used for the other two lemmas do not work. Since $\alpha(E, \Pi_{n-1})$ is $\Psi$ of an $(n-2)$-dimensional cone, which is very hard to compute directly.

## The symmetry property

**Lemma.** *The valuation* $\Psi$ *(from the BV-construction) is* symmetric about the coordinates*, i.e., for any cone* $C \in \mathbb{R}^n$ *and any signed permutation* $(\sigma, \mathbf{s}) \in \mathfrak{S}_n \times \{\pm 1\}^n$*, we have*

$$\Psi(C) = \Psi((\sigma, \mathbf{s})(C)),$$

*where* $(\sigma, \mathbf{s})(C) = \{(s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, \dots, s_n x_{\sigma(n)}) \ : \ (x_1, \dots, x_n) \in C\}$.

# Idea of the proof of the third lemma

Recall that the coefficient of $t^k$ in $i(P,t)$ is given by

$$\sum_{F\text{: a } k\text{-dimensional face of } P} \alpha(F,P)\,\mathrm{vol}(F).$$

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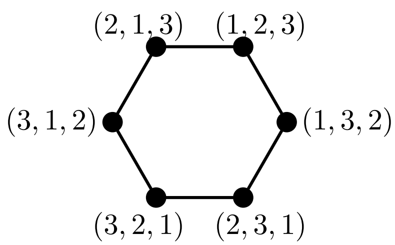
**General idea:** Suppose you have a family of polytopes such that

- they have same pointed feasible cones (for edges) up to signed permutations, and thus have the same $\alpha$-values;
- the Ehrhart polynomial of each polytope in the family is known (or at least the linear Ehrhart coefficient is known).

Then as long as you have enough "independent" polytopes in your family, you can figure out the $\alpha$-values.

# Idea of the proof of the third lemma (cont'd)

Example. When $n = 3$ : $\Pi_2 = \mathrm{Perm}((1,2,3)) = \mathrm{conv}\{\sigma \ : \ \sigma \in \mathfrak{S}_3\}$.




The pointed feasible cones of the six edges of $\Pi_2$ are

$\mathrm{Cone}((1,1,-2))$, $\mathrm{Cone}((2,-1,-1))$, $\mathrm{Cone}((1,-2,1))$,

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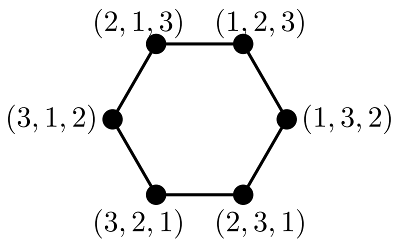


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By the symmetry property of $\Psi$, these cones all have the same value. Therefore, all $\alpha(E, \Pi_2)$ are a single value, say $\alpha$.

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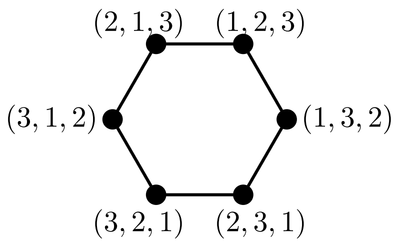


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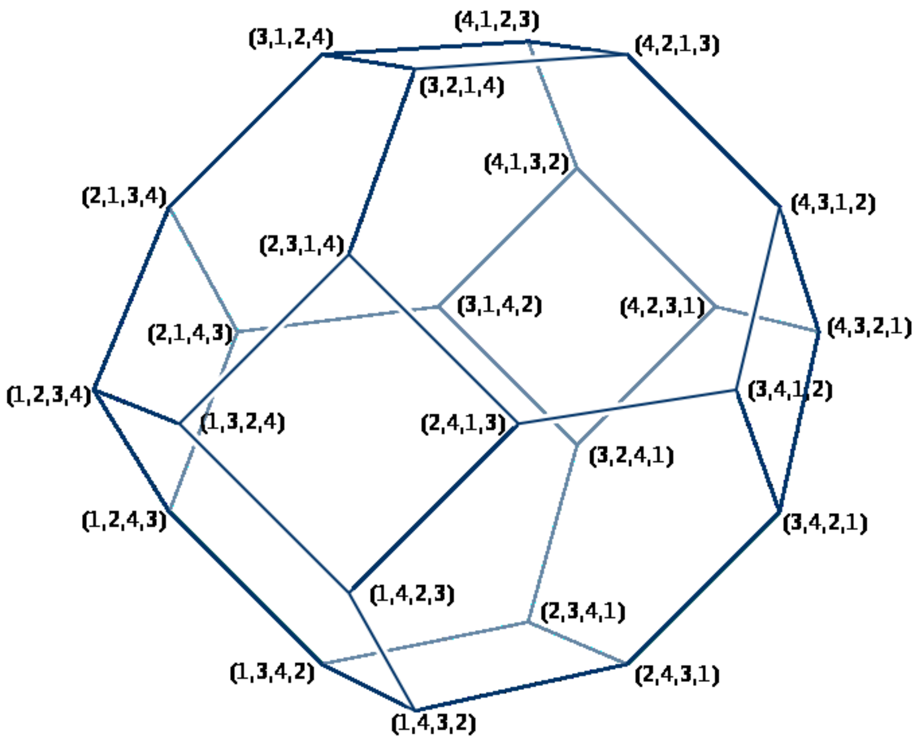
The Ehrhart polynomial of $\Pi_2$ is $3t^2 + 3t + 1$. Thus,

$$3 = \sum_E \alpha(E, \Pi_2) \cdot \mathrm{vol}(E) = 6\alpha \quad \Rightarrow \quad \alpha = 1/2 > 0.$$

# Idea of the proof of the third lemma (cont'd)

Example. When $n = 4$ : $\Pi_3 = \mathrm{Perm}((1,2,3,4)) = \{\sigma \ : \ \sigma \in \mathfrak{S}_4\}$.



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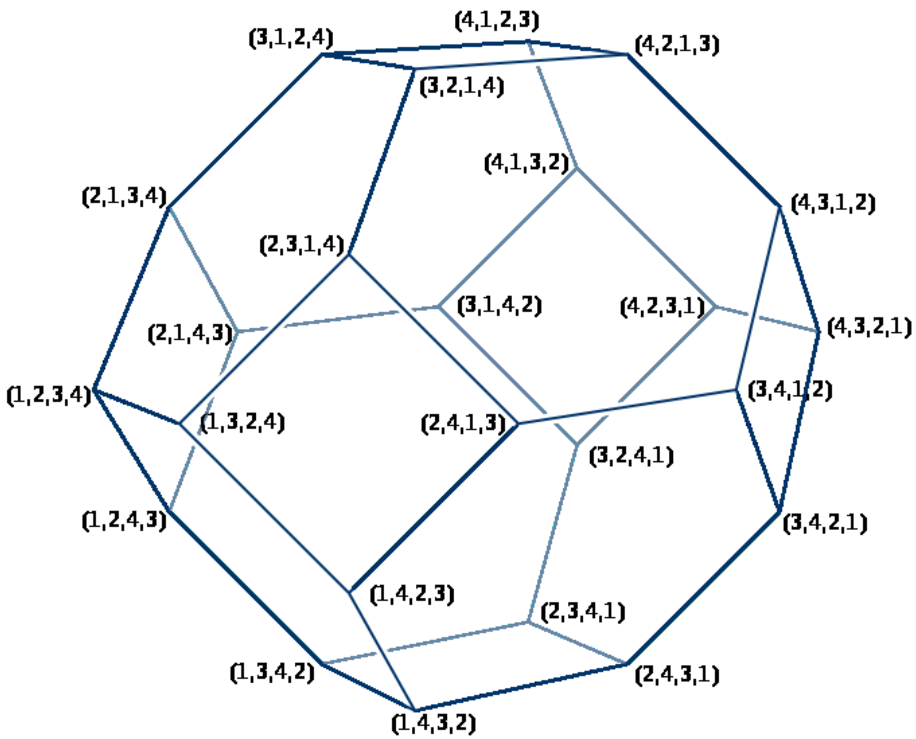


$\Pi_3$ have $36$ edges of two kinds. $24$ short edges have the same $\alpha$-values, say $\alpha_1$, and $12$ long edges have the same $\alpha$-values, say $\alpha_2$.

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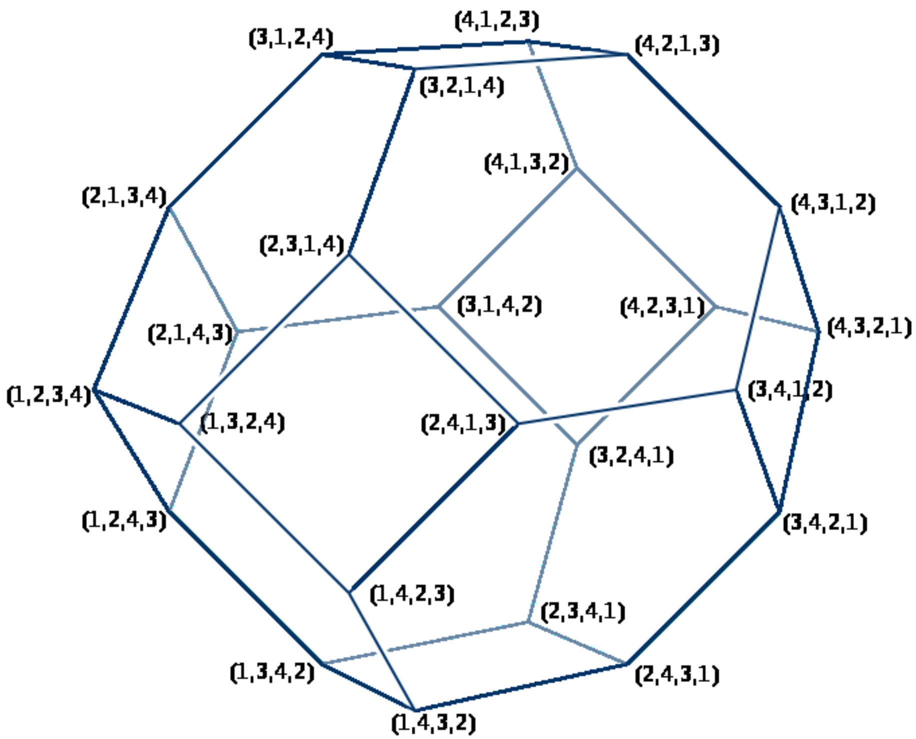


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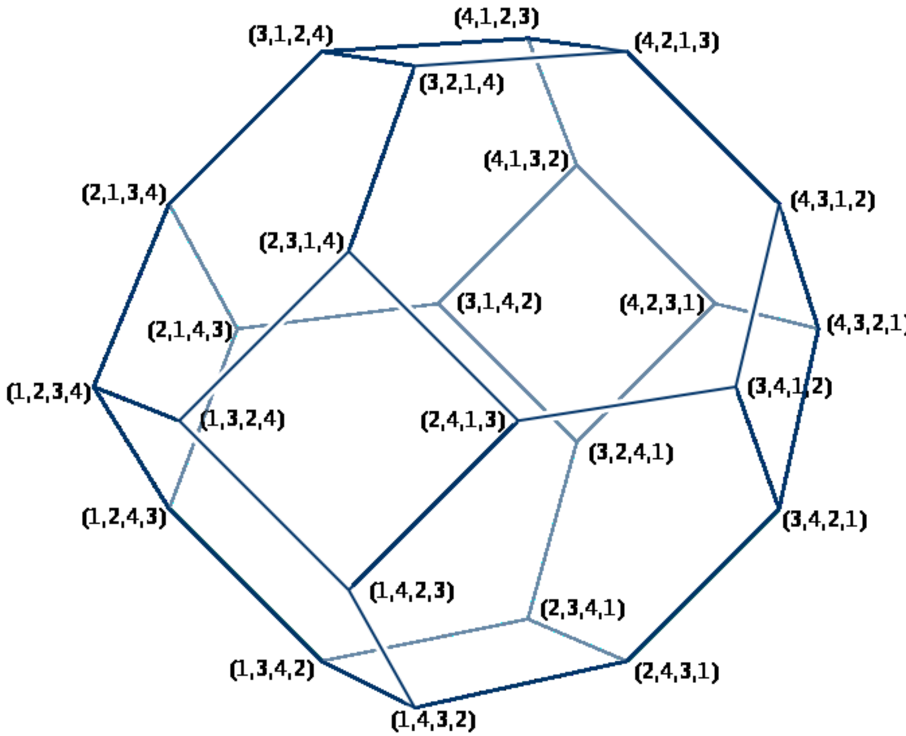


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*Not enough equations!*

## Idea of the proof of the third lemma (cont'd)

Consider the hypersimplex $\Delta_{2,4} = \mathrm{Perm}((0,0,1,1))$. It has $12$ edges whose corresponding pointed feasible cones are the same as that of the $12$ long edges of $\Pi_3$. So they all have $\alpha$-values $\alpha_2$.

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Therefore, we solve the $2 \times 2$ linear system, and get

$$\alpha_1 = \frac{11}{72} > 0, \qquad \alpha_2 = \frac{7}{36} > 0.$$

# Idea of the proof of the third lemma (cont'd)

For arbitrary $n$: The linear Ehrhart coeffcient of some polytopes in the $y$-family can be easily described. Using these, we were able to set up an explicit triangular linear system for $\{\alpha(E, \Pi_{n-1}) \ : \ E$ is an edge of $\Pi_{n-1}\}$ for any $n$.

**Remark.** The number "$100$" in the lemma can be pushed further.

# PART IV:

## Other questions and results

Based on joint work with Castillo, Nill and Paffenholz.

## Questions and Answers

- Recall that a $d$-dimensional integral polytope $P$ is called *smooth* if each vertex is contained in precisely $d$ edges, and the primitive edge directions form a lattice basis. Bruns asked the following question:

  Question 1. Is smooth integral polytope always Ehrhart positive?

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  Question 2. Are there polytopes $P$ that are Ehrhart positive but not BV-$\alpha$-positive? **Yes.**

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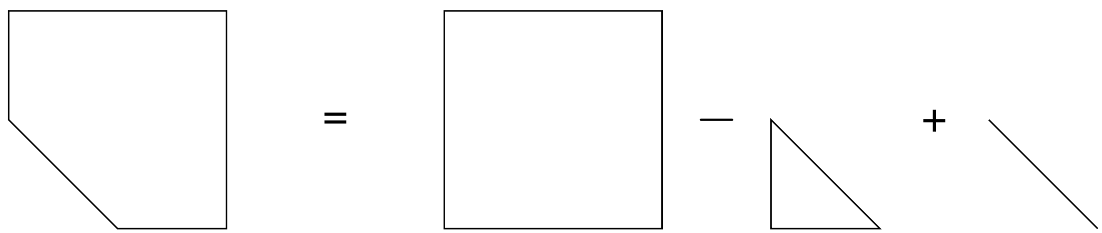
  Question 2. Are there polytopes $P$ that are Ehrhart positive but not BV-$\alpha$-positive? **Yes.**

  Question 3. If some cone in $\Sigma$ is BV-$\alpha$-negative, can we always construct a polytope with normal fan $\Sigma$ that is *not* Ehrhart positive? **No.**

# Idea of the constructions

i. Chiseling cubes:

$P_d(a, b)$: cutting one vertex off $a\square_d$ at distance $b$.



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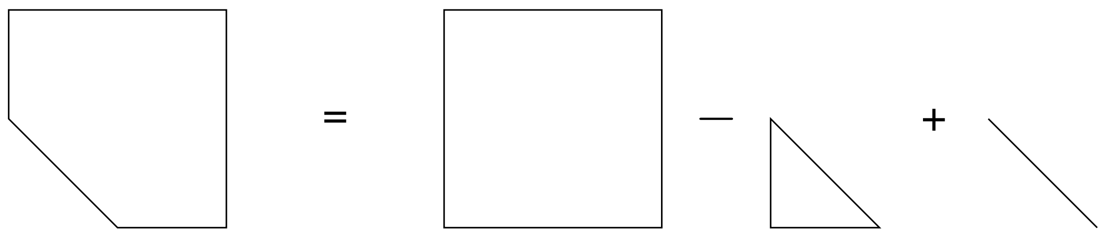


Figure 1: Inclusion-Exclusion for $P_2(2, 1)$

ii. Use inclusion-exclusion to compute BV-$\alpha$-values for $P_d(a, b)$ and search for negative values. Negatives appear at $d = 7$.

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i. Chiseling cubes:

$P_d(a,b)$: cutting one vertex off $a\square_d$ at distance $b$.

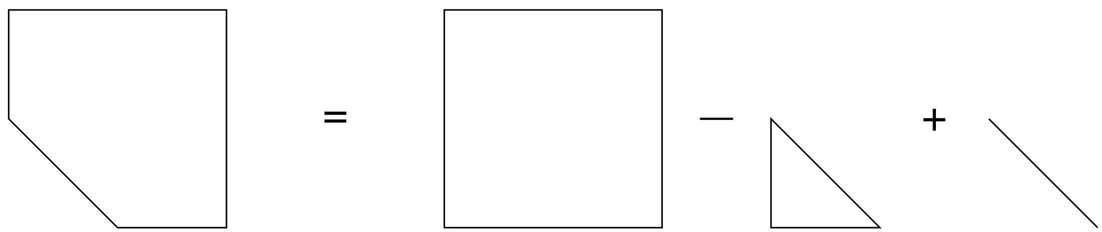

Figure 1: Inclusion-Exclusion for $P_2(2,1)$

ii. Use inclusion-exclusion to compute BV-$\alpha$-values for $P_d(a,b)$ and search for negative values. Negatives appear at $d = 7$.

**Results** $P_d(a,b)$ has negative BV-$\alpha$-values for any $d \geq 7$, but any polytope that has the same normal fan as $P_d(a,b)$ is Ehrhart positive.

# Idea of the constructions

iii Chiseling cubes more:

$Q_d(a, b)$: cutting all vertices off the $a\square_d$ at distance $b$.

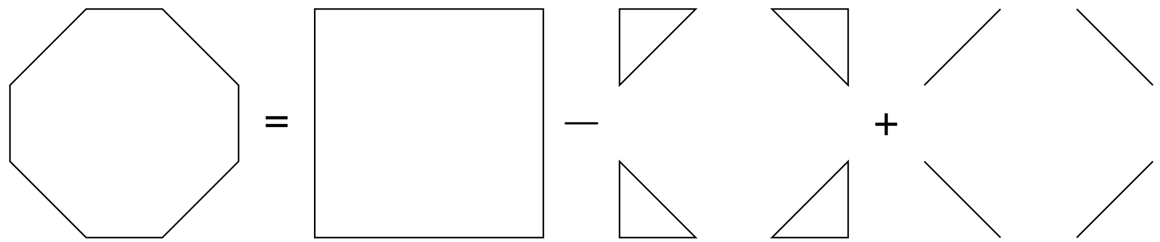

Figure 2: Inclusion-Exclusion for $Q_2(3, 1)$

# Idea of the constructions

iii Chiseling cubes more:

$Q_d(a, b)$: cutting all vertices off the $a\Box_d$ at distance $b$.

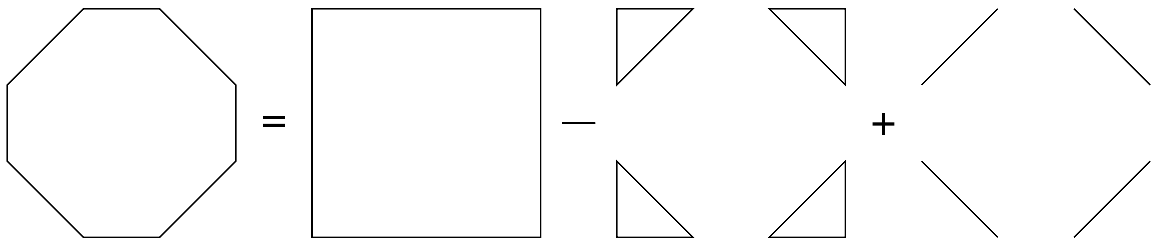

Figure 2: Inclusion-Exclusion for $Q_2(3, 1)$

**Results** For any $d \geq 7$, the smooth polytope $Q_d(a, b)$ is *not* Ehrhart positive.