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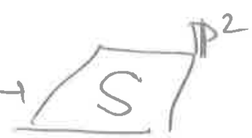
Talk Title: Reduced Gromov-Witten invariants from cuspidal curves

January 18, 2018 (11 am talk) w/ L. Battistella & F. Carocci.

- Reduced GW invariants: $GW_1^{red}(X)$
- inv. from cuspidal curves: $GW_1^{cusp.}(X)$
- Theorem: X = quintic 3-fold
  $GW_1^{red}(X) = GW_1^{cusp}(X)$

1. $\overline{M}_{g,n}(X,d) = \{ (C, p_1, \dots, p_n, f) \mid$ C nodal curve
   p_i marked points
   $f: C \to X$ $f_*[C] = d$
   every comp of $g=0$ has at least 3 sp. points
   $g=1$ has at least 1 sp. point $\} / \sim$

<u>Ex</u>: $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ → comps of deg 3 w/ gen point a map from a smooth cubic

$f: C \to$ 

→ comp w gen. pt

dim 10

→ comp. w/ g. pt

dim 9

<u>Remark</u>: We don't have quantum Lefschetz

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{P}^r \\ \downarrow \pi & & \\ \overline{M}_{g,n}(X,d) & & \end{array}$$

$X = \mathbb{P}^r$, $\mathcal{O}(a)$ on $\mathbb{P}^r$

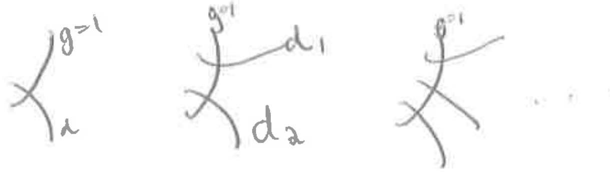
$\pi_* f^* \mathcal{O}(a)$ <u>not</u> a vector bundle

Reduced GW-invariants (Zinger)

- Blow up $\overline{M}_{1,m}(\mathbb{P}^r,d)$ until the main comp. becomes smooth.

Vakil-Zinger

$$\begin{array}{ccc} \overline{M}_{1,n}(\mathbb{P}^r,d) & & (C, p_i, f) \\ \downarrow \varepsilon & & \downarrow \\ \mathfrak{M}_{1,n}^{d} & & (C, p_i, d) \end{array}$$

Blow-up $\mathfrak{m}_{1,n}$ in the strata 

and end-up w/ $\widetilde{\mathfrak{m}}_{1,m}$.

$$\begin{array}{ccc} \widetilde{\overline{M}}_{1,n}(\mathbb{P}^r,d) & \longrightarrow & \overline{M}_{1,n}(\mathbb{P}^r,d) \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{m}}_{1,n}^{wt} & \longrightarrow & \mathfrak{m}_{1,n}^{wt} \end{array}$$

- it has a ~~smooth map~~ main comp. $\overline{M}_{1,m}(\mathbb{P}^r,d)^{main}$ which is smooth.

- $\tilde{\pi}_* \tilde{f}\mathcal{O}(a)$ is a vector bundle on $\widetilde{M}_{1,n}(\mathbb{P}^r,d)^{main}$
- $\widetilde{M}_{1,n}(\mathbb{P}^r,d)$ has a virtual class

$$p_*[\widetilde{M}_{1,n}(\mathbb{P}^r,d)]^{virt} = [\overline{M}_{1,n}(\mathbb{P}^r,d)]^{virt}.$$

$X \hookrightarrow \mathbb{P}^r$ is a hypersurface of degree a

$$i_*[M(X)^{red}]^{virt} = C_{top}(\tilde{\pi}_* \tilde{f}^*\mathcal{O}(a))[\widetilde{M}_{1,n}(\mathbb{P}^r,d)^{main}]$$

intersection numbers against $i_*[M(X)^{red}]^{virt}$ are called <u>reduced GW invariants</u> of X.

<u>Theorem</u>: (Li-Zinger, Chang-Li), X quintic three-fold.

$$GW_n(X) = GW_n^{red}(X) + \frac{1}{12} GW_0(X)$$

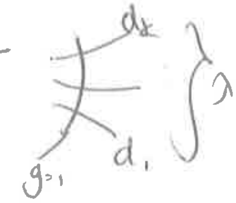
~~only~~ true for quintic three-fold
not true for four-fold.

Why you should believe this (proof hard):

If we had a nice splitting of the virtual class

$$[\tilde{M}_n(X,d)]^{virt} = [\tilde{M}_1(X,d)^{main}]^{virt} + [Z_\lambda]^{virt}.$$

S.t Z_λ are supported on a boundary map of type



Denote comp. of this type by D^λ.

D^1: 

$= \overline{M}_{1,1} \times \overline{M}_{0,n+1}(X,d)$

$$\downarrow$$

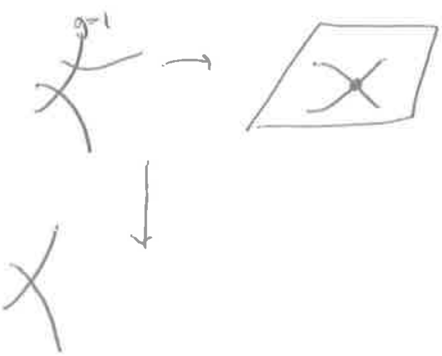
$$\overline{M}_{0,n}(X,d)$$

If the class $[Z_\lambda]^{virt}$ pushes forward to a multiple of $[\tilde{M}_{0,m}(X,d)]^{virt}$ then we get a contribution to $GW_1(X)$ from $GW_0(X)$.

$$D^2 = \overline{M}_{1,2} \times \overline{M}_{0,1}(X,d_1) \times \overline{M}_{0,1}(X,d_2)$$

$$\downarrow$$

$$\overline{M}_{0,1}(X,d_1) \times \overline{M}_{0,1}(X,d_2)$$



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2. Invariants from cuspidal curves.

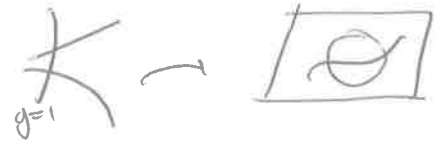
Viscardi

$\overline{M}_{1,n}(X,d)^{(1)} = \{(C, p_1, \dots, p_m, f) \mid$ C nodal or cuspidal curve, $f_*[C] = d$, on any contracted component
- $g=0$ has at least 3 special points
- $g=1$ has at least 2 special points $\}$

Theorem (Viscardi)

$\overline{M}_{1,n}(X,d)^{(1)}$ is a proper DM-stack.

Ex: $M_{1,0}(\mathbb{P}^2,3)^{(1)} \longrightarrow$ main comp



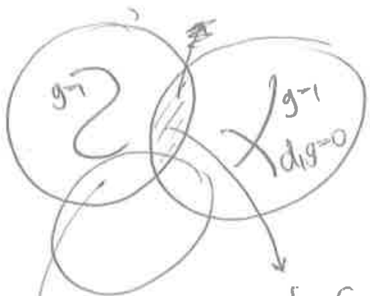
- $\overline{M}_{1,n}(X,d)^{(n)}$ have a virtual class

  we denote inv. by $GW_1(X)^{cusp}$

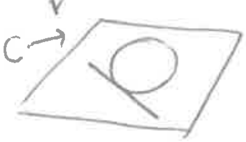
3. Theorem (Battistella, Carocci, –) X quintic 3-fold.

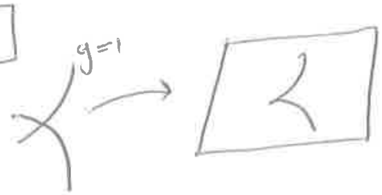
$$GW(X)^{red} = GW_1(X)^{cusp}$$

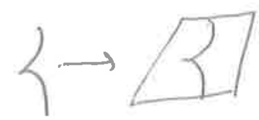
Thm (Vakil)



$f: C \to$ 

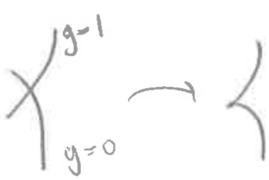
$f: C \to$ 

replace $f:$ 

$f:$ 

Idea of the proof:

Prop: $\varphi: \mathcal{M}_{1,0}^{aul,st} \longrightarrow \mathcal{M}_{1,0}(1)$
$= \{(C,p_1,\dots,p_r) / C$ is nodal or cuspidal, p_i-points$\}$



$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{1,n}(X,d)^{(1)} \arrow[d] \\
\mathfrak{M}_{1,n}^{wt,st} \arrow[r, "\varphi"] & \mathfrak{M}_{1,n}^{wt,st}(1)
\end{tikzcd}$$

$Z \overset{i}{\hookrightarrow} M_{1,n}(X,d)$ $\quad Z = \overline{M}_{1,n}(X,d)^{main} \cup_{\lambda \neq (1,0)} D_\lambda$

<u>Idea</u>: Use the Chang-Li machinery for this Z.