Enumeration of Singular Subvarieties with Tangency Conditions

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Motivation

We always work over $\mathbb{C}$. δ-nodal curve = reduced+ has exactly δ nodes.
Let S be a smooth algebraic surface and L is a sufficiently ample line bundle.

Consider:

What is the number of δ-nodal curves in the linear system $|L|$ on S which pass through proper number of general points?

- On $\mathbb{P}^2$, they are computed by the recursive formula of Caparaso-Harris (1998).
- Similar formulas holds for rational ruled surfaces (Vakil, 2000).
- By Gottsche's conjecture, answers on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ completely determined the answer on all smooth surfaces.
- However, Gottsche's conjecture does not cover the case when curves are required to satisfy tangency conditions with another curve.

Göttsche's conjecture

Theorem (T-, Kool-Shende-Thomas)

For every integer $\delta \geq 0$, there exists a universal polynomial $T_\delta(x, y, z, t)$ of degree δ such that

$$T_\delta(L^2, LK_S, c_1(S)^2, c_2(S)) = \text{ number of } \delta\text{-nodal curves in } |L|$$

if L is δ-very ample.

For example, $T_1 = 3L^2 + 2LK_S + c_2(S)$.
L is called k-very ample if for all zero-dimensional closed scheme $\xi \subset S$ of length $k+1$, $H^0(S, L) \longrightarrow H^0(L|_\xi)$ is surjective.

$\Rightarrow$ This means all numbers of nodal curves can be computed by results on any four pairs of surfaces and line bundles, as long as their $(L^2, LK_S, c_1(S)^2, c_2(S))$ are linearly independent.

Generating function

The generating series of universal polynomials has the following form:

Theorem

$\sum_{\delta=0}^{\infty} T_\delta(L^2, LK, c_1(S)^2, c_2(S))x^\delta = A_1^{L^2} A_2^{LK_S} A_3^{c_1(S)^2} A_4^{c_2(S)}$ *for some power series* A_i *in* $\mathbb{Q}[[x]]^*$.

$\longrightarrow$ the generating function is multiplicative.

Theorem (Göttsche-Yau-Zaslow formula)

There exist two power series $B_1(q)$ *and* $B_2(q)$ *such that*

$$\sum_{\delta\geq 0} T_\delta(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^\delta = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

It's because $K3$ surfaces can provide two out of four pairs, and numbers curves on $K3$ surfaces are given by (quasi-)modular forms.

In this talk, we will try to answer the following questions:

Question 1:

Can we combine Caparaso-Harris' formula with Gottsche's conjecture to count singular curves with tangency conditions? If yes, what's the structure of the generating series of those numbers?

Known:

- There exist universal polynomials for curves with arbitrary given singularities [Li-T, Rennemo 2012].
- Block (2012) computed universal polynomials for curves with tangency conditions with a line on $\mathbb{P}^2$ using tropical geometry.

Application:

- New structures and relations for Caporaso-Harris invariants.
- New results such as number of degree d curves on $\mathbb{P}^2$ tangent to an elliptic curves.

Question 2:

Can we generalize Caporaso-Harris invariants and Gottsche's conjecture to higher dimension? What is the correct setting and which invariants do those numbers depend?

Known:

- The numbers of hypersurfaces with arbitrary given singularities in the linear system $|L|$ on a smooth variety X are given by universal polynomials of Chern numbers of L and X [Li-T, Rennemo 2012]. Moreover, the structure of generating series are similar.

Application:

- If one knows enough of these numbers, then others can be computed.

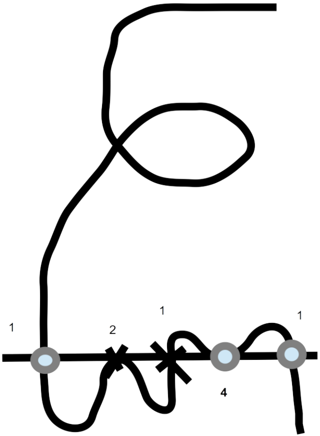
Question 3:

Compute the universal polynomials for nodal curves on surfaces with tangency conditions. In particular, does the generating series of degree d nodal curves on $\mathbb{P}^2$ tangent to an elliptic curve a modular form or other nice series?

Hope: Find B_1 and B_2 in Göttsche-Yau-Zaslow formula:

$$\sum_{\delta \geq 0} T_\delta(L^2, LK, c_1(S)^2, c_2(S))(DG_2)^\delta = \frac{(DG_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\Delta D^2 G_2/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Caporaso-Harris invariants



Fix a line H in $\mathbb{CP}^2$, consider degree d δ-nodal curves which intersect H at some fixed points of given multiplicities (recorded by α), and intersect H at some unspecified point of given multiplicities (recorded by β).

The number of such curves is denoted by $N^{d,\delta}(\alpha, \beta)$. All can be computed from the recursive formula of Caporaso-Harris.

$\Leftarrow \alpha = (1, 1, 0, \dots)$ $\beta = (2, 0, 0, 1, \dots)$

Caporaso-Harris' formula:

$$N^{d,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} k\cdot N^{d,\delta}(\alpha+e_k,\beta-e_k) + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}N^{d-1,\delta'}(\alpha',\beta')$$

It is hard to see any polynomial structure from this formula.
Fomin and Mikhakin first used tropical geometry to show the number of degree d nodal curves are polynomials if $d \geq \delta$.
Block shows if $\sum \beta_i \geq \delta$, for any α, β there is a polynomial $N_\delta(\alpha_1,\alpha_2,\dots\beta_1,\beta_2,\dots)$ so that

$$N^{d,\delta}(\alpha,\beta) = 1^{\beta_1}2^{\beta_2}\cdots\frac{(|\sum\beta_i|-\delta)!}{\prod\beta_i!}N_\delta(\alpha_1,\alpha_2,\dots\beta_1,\beta_2,\dots).$$

When $\delta = 0$ it is the classical De Jonquières' formula: the number of sections of line bundle on curves vanishing at points of given multiplicities.
This is proved by tropical geometry and Block gave explicit formula for $\delta \leq 6$.

Higher dimension

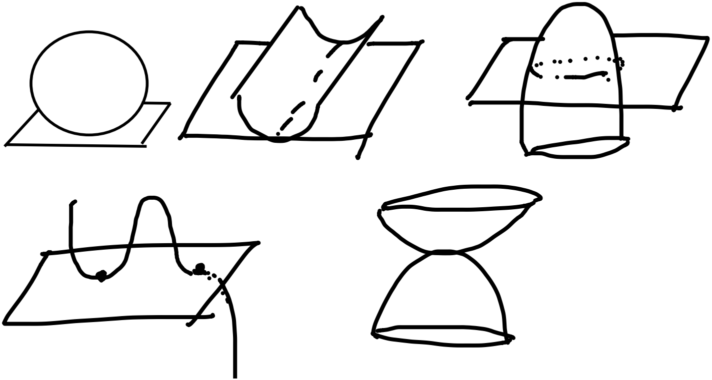
X: a smooth projective variety over $\mathbb{C}$ (any dimension)
E: a sufficiently ample vector bundle on X of rank r, D: a divisor in X

Question:

What is a proper generalization of Gottsche's conjecture? i.e. How to count singular subvarieties given by sections of E with tangency conditions with D?

- We can assume those subvarieties only have isolated complete intersection singularities (icis) and codimension r because E is sufficiently ample.
- We can also assume the intersection of subvariety with D has codimension $r+1$.
- But the tangency conditions with D can be complicated....

Which of these are intersecting transversally?



The tangency conditions of curves on surfaces with another curve is easy to describe because only the multiplicities matter and the singularity will be away from intersection by ampleness.

If a subvariety Y intersect a divisor D transversally, then $Y \cap D$ is smooth. So the "tangency condition" can be classified by the isolated singularity of $Y \cap D$ (we can't do non-isolated singularities now).

Certain singularity of $Y \cap D$ may force Y to be singular at the same point, which we call it induced singulaity. Singularities away from D are called non-induced singulaity

Right question:

How many reduced subvarieties cut out by sections of E have the prescribed non-induced icis and given tangency conditions with D?

Recall X: a smooth projective variety over $\mathbb{C}$ (any dimension)
E: a sufficiently ample vector bundle on X of rank r, D: a divisor in X

Theorem (T-)

For any collection of icis $\underline{\delta}$ and tangency conditions α (at assigned points) and β (at unassigned points), there exists a universal polynomial $T_{\alpha,\beta,\underline{\delta}}$ of Chern numbers of X, D, E which counts the number of curves in $|E|$ with non-induced singularity type δ and tangency condition (α, β) with a fixed smooth divisor D, if E is sufficiently ample.

This theorem answers our earlier question: general Caporaro-Harris invariants also have a polynomial structure and for much more general case!

Next we will see the structure of these polynomials.

Structure of generating series

For a fixed α, form generating series

$$S_\alpha(X,D,E) = \sum_{\underline{\delta},\beta} T_{\alpha,\beta,\underline{\delta}}(X,D,E) x_\alpha y_\beta z_{\underline{\delta}}.$$

Theorem (T-)

If $\alpha = (\alpha_1, \alpha_2, \ldots,)$, there exists constant power series A_i so that

$$S_\alpha(X,D,E) = A_1^{c_n(X)} A_2^{c_{n-1}(X)c_1(E)} \cdots A_*^{c_1^n(D)} \prod_i S_{\alpha_i},$$

where $n =$ dim X and each S_{α_i} is a constant power series which only depend on the icis α_i.

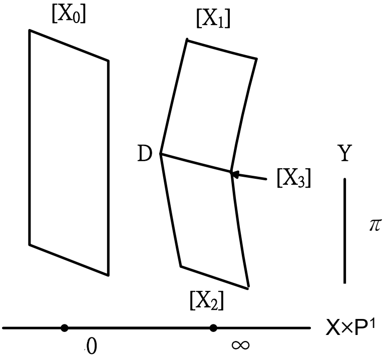
My undergraduate student did a project to use Block's formula to compute S_{α_i} for surfaces and verified the theorem for the first dozens of terms.

Idea of the proof: degeneration

Definition: Let X_i be smooth projective schemes. We call

$$[X_0] = [X_1] + [X_2] - [X_3]$$

a double point relation if there exist a family Y such that



1. X_0 is the smooth fiber over 0.
2. fiber over ∞ has two components X_1 and X_2
3. X_1 and X_2 intersect transversally along a smooth divisor D
4. $X_3 \cong \mathbb{P}_D(N_{X_1/D} \oplus \mathcal{O}_D)$

Chern numbers are invariants!

Definition: Define $\omega_* = \bigoplus_{X \text{ smooth projective}} \mathbb{Z}[X]/$ double point relations

Theorem (Levine and Pandharipande, 2009)

$\omega_* \cong$ *the complex cobordism ring*

$$\omega_* \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_r)} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times \ldots \times \mathbb{P}^{\lambda_r}]$$

Corollary:

For every smooth projective curve C, $[C] = (1 - g(C))[\mathbb{P}^1]$.

For every smooth projective surface S,

$$[S] = *[\mathbb{P}^2] + *[\mathbb{P}^1 \times \mathbb{P}^1].$$

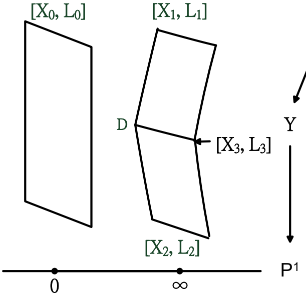
Every smooth projective scheme can degenerated into sums of products of projective scheme.

Double point relation for pairs

Definition: Let X_i be smooth projective surfaces and L_i be line bundles on X_i. We call

$$[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3]$$

a double point relation if there exist a family Y and line bundle L on X such that



1. $[X_0] = [X_1] + [X_2] - [X_3]$ is a dp relation
2. $L|_{X_i} = L_i$ for $i = 0, 1, 2$
3. $L_3 =$ the pullback of $L|_D$ to X_3

Definition

Define the algebraic cobordism group $\omega_{2,1}$ to be the free abelian group spanned by all pairs (of smooth projective surfaces and line bundles) modulo double point relations.

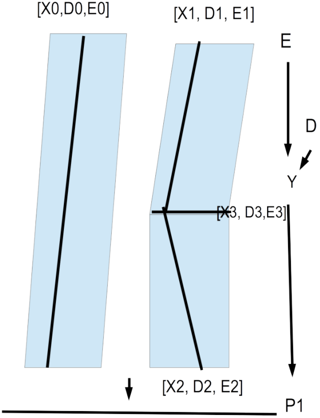
Theorem (T-, Lee-Pandharipande)

$$\omega_{2,1} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{(L^2, LK, c_1(S)^2, c_2(S))} \mathbb{Q}^4$$

is an isomorphism.

This is why the Gottsche's conjecture holds, because the number of nodal curves is an invariant of the algebraic cobordism group!

Algebraic Cobordism of bundles and divisors on varieties



Definition: Let E_i be vectors bundles and D_i be smooth divisors on smooth varieties X_i. Call

$$[X_0, D_0, E_0] \\ =[X_1, D_1, E_1] + [X_2, D_2, E_2] - [X_3, D_3, E_3]$$

a double point relation if

1. $[X_0, E_0] = [X_1, E_1] + [X_2, E_2] - [X_3, E_2]$ is a dp relation
2. Everything intersect transversally.

Let $n = \dim X$, $r=$ rank(E).

Definition. Define the algebraic cobordism group $\nu_{n,r,1}$ to be the free abelian group spanned by $[X, D, E]$ modulo double point relations.

Theorem (T-(2016))

$$\nu_{n,r,1} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \textit{the vector space of all Chern numbers of } X, D, E$$

.

This is why the number of singular subvarieties with tangency conditions in higher dimension are still given by universal polynomial, because they are invariants of this new algebraic cobordism group.

There is a basis containing only products of $\mathcal{O}(1)$ and hyperplanes on projective spaces.

Computation

Now we will focus on the computation of universal polynomials of nodal curves in a linear system $|L|$ on smooth surfaces S with arbitrary tangency conditions (α, β) with a smooth divisor D.

By our theorem, they are polynomials of seven Chern numbers: L^2, LK_S, K^2, $c_2(S)$, D^2, $deg(L|_D)$, DK_S and a basis of the seven-dimensional space can be given from $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and the number of nodal curves with tangency conditions are completely computable by Caporaso-Harris' and Vakil's formulas!

Caporaso-Harris' formula for $\mathbb{P}^2$:

$$N^{d,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} k \cdot N^{d,\delta}(\alpha+e_k, \beta-e_k) + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}N^{d-1,\delta'}(\alpha',\beta')$$

Vakil's formula is similar except the line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ are (a,b). I wrote Java programs to implement these formulas and use interpolations to solve the coefficients of universal polynomials.

- First attempt: recursion. My laptop won't stop running when d is about 15 and a, b are about 8. Too small, not ample enough.
- Second attempt: dynamic programming. Need to design algorithms to build tables for these numbers.
- When solving universal polynomials, interpolation in Mathematica will give me inaccurate answer with lower degree rather than correct answer. So the structure of generating series is used to make sure we only need to solve a linear polynomial every time.

Then we found:

- All universal polynomials up to total degree 3 (around 30 such universal polynomials).
- The number of nodal curves on $\mathbb{P}^2$ tangent to a elliptic curve for small degrees and tangency multiplicities. We hope it is a nice series (modular?). But they are indexed by degree d and two integer sequences (α, β). How to organize them?
- The universal polynomial Block computed for nodal curves with tangency conditions on $\mathbb{P}^2$ does not match with our computation.

Thank you!