January 18, 2018
(3:30 pm)

Speaker: Emily Clader

Talk Title: Double ramification cycles and tautological relations

(joint with F. Janda, S. Grushevsky, D. Zakharov)

Background:

- The moduli space of curves.

$$\overline{M}_{g,n} = \{$$  $$\mid \begin{matrix} \text{genus} = g \\ \text{stable} \end{matrix} \}/\sim$$

$$\cup\!|$$

$$M_{g,n} = \{\text{smooth curves}\}$$

- Attaching maps:

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \longrightarrow \overline{M}_{g_1+g_2,n_1+n_2}$$

$$\overline{M}_{g-1,n+2} \rightarrow \overline{M}_{g,n}$$

- Forgetful maps:

$$\overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$$

- Definition: The tautological ring(s) is the smallest family of subrings

$$R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$$

closed under pushforward by attaching forgetful maps.

- For example, the tautological ring contains "boundary classes"

e.g. $\Big[\{$  $\} \subseteq \overline{M}_{3,1}\Big]$

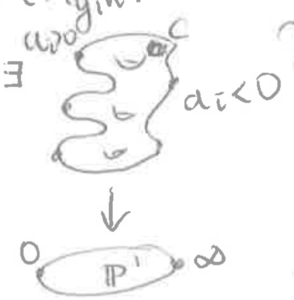
Also contains ψ-classes, κ-classes.

- The tautological ring is good to study because:
  1) a set of additive generators is known
  2) contains nearly every "geometrically interesting" class

Double Ramification Relations

- Fix $g \geq 0$ and a tuple of integers $A = (a_1, \dots, a_n)$ whose sum is zero.
- Define an element of $A^*(M_{g,n})$:

$$DR_{g,A} := \left[\left\{ (C; x_1, \dots, x_n) \,\middle|\, \exists \right. \right.$$



$$\left. \left. \right\} \right] \in A^g(M_{g,n})$$

- Another perspective:

Let $p_A : M_{g,n} \longrightarrow \mathrm{Jac}_{g,n} \{(C; x_1, \dots, x_n; L) \mid \deg(L) = 0\}$

$$p_A(C; x_1, \dots, x_n) = (C; x_1, \dots, x_n; \mathcal{O}_C(a_1[x_1] + \dots a_n[x_n]))$$

Then
$$DR_{g,A} = p_A^*[Z_g]$$
where
$$Z_g = \{(C; x_1, \dots, x_n; \mathcal{O}_C)\}$$

This perspective gives:

(1) Hain: A way to compute an explicit tautological formula for $DR_{g,A}$. Idea: Use that

$$[Z_g] = \frac{1}{g!} \Theta^g$$

$$\Rightarrow DR_{g,A} = p_A^* \left(\frac{1}{g!} \Theta^g \right) = \frac{1}{g!} (p_A^* \Theta)^g$$

$\therefore$ compute $p_A^* \Theta$ to compute $DR_{g,A}$

(2) Grushevsky - Zakharov: A collection of tautological relations in $A^*(M_{g,n})$ by using that

$$\Theta^d = 0 \quad \forall d > g$$

$$\Rightarrow (\rho_A^* \Theta)^d = 0 \quad \forall d > g$$

- This all works on $M_{g,n}^{ct}$

- Going from $M_{g,n}^{ct}$ to $\overline{M}_{g,n}$ is harder. The definition of

$$\overline{DR}_{g,A} \in A^g(\overline{M}_{g,n})$$

involves the virtual cycle of the moduli space of relative stable maps to $\mathbb{P}^1$.

- Given this definition, can we generalize the work of Hain (formula for $\overline{DR}_{g,A}$) and Grushevsky - Zakharov (relations)?
- Pixton: Gave an explicit tautological formula for a class

$$\Omega_{g,A} \in R^*(\overline{M}_{g,n})$$

(extension of $\exp(\varphi_A^* \Theta)$) and conjectured:

(a) $[\Omega_{g,A}]_{\text{codim } g} = \overline{DR}_{g,A}$

(b) $[\Omega_{g,A}]_{\text{codim} > g} = 0$

- Theorem [C-Janda]. Part (b) is true.
("double ramification relations")
- Janda - Pandharipande - Pixton - Zvonkine:
Part (a) is true

Pixton's Formula:

- Let $\overline{M}^r_{g,A} := \{(C, x_1, \ldots, x_n, L) \mid L^{\otimes r} \cong \mathcal{O}_C(-a_1[x_1] - \cdots - a_n[x_n])\}$

- There is a map

$$\phi : \overline{M}^r_{g,A} \longrightarrow \overline{M}_{g,n}$$

forgetting L.

- Universal family

$$\begin{tikzcd} \mathcal{L}_A \arrow[r, hook] & \mathcal{C}_A \arrow[d, "\pi"] \\ & \overline{M}^r_{g,A} \end{tikzcd}$$

- Set

$$\Omega^r_{g,A} = \frac{1}{r^{2g-1}} \phi_*\left(e^{r^2 c_1(-R\pi_* \mathcal{L}_A)}\right)$$

- <u>Fact</u>: $\Omega^r_{g,A}$ is polynomial in r for $r \gg 0$.

- <u>Def</u>: $\Omega_{g,A}$ is the constant term in this polynomial.

- <u>Idea of proof of DR relations</u>: Modify A to make $-R\pi_*\mathcal{L}_A$ a vector bundle of rank g. Thus higher Chern classes make sense, & we compare $\Omega^r_{g,A}$ to the total Chern class of $-R\pi_*\mathcal{L}_A$, which manifestly vanishes in codim $> g$.

<u>Consequences of the DR-relations</u>

- <u>Theorem</u> [conj. of Faber, proved by Looijenga and Ionel]. Any tautological class on $\overline{M}_{g,n}$ of codimension $\geq g$ is equivalent to a class supported on $\overline{M}_{g,n} \setminus M_{g,n}$.

- <u>New proof</u> [C-Grushevsky-Janda-Zakharov] Use the DR-relations. They are polynomial in $a_1, \ldots, a_n$, so the coefficient of each monomial gives a relation. There are enough such relations to get vanishing of all generators of $R^k(M_{g,n})$ with $k \geq g$.

- <u>Upshot</u>: Proof implies that every element of the tautological ring $R^k(\overline{M}_{g,n})$, $k \geq g$ has a <u>tautological</u> boundary expression.