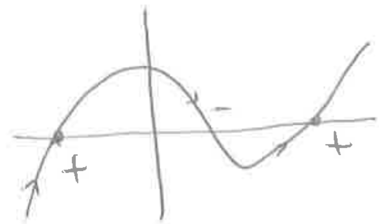
Speaker: Penka Georgieva
Talk Title: Real Gromov-Witten Theory
Date and Time: January 19, 2018 @ 9:am.

- How many roots does a poly. of degree d have?

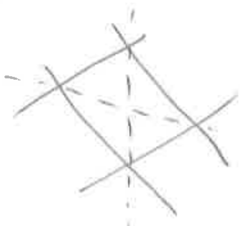
  Over $\mathbb{C}$ = d
  Over $\mathbb{R}$ = ?



signed counts
<u>lower bound</u>

- How many lines pass through four lines in $\mathbb{P}^3$?

  $\mathbb{C}$: 2



$\mathbb{R}$: 2

0 → Kollàr: we can take the 4-lines to be non-intersecting $\subset Q$-quadric with empty real locus

any line $\cap$ {4 lines} $\subset Q$
→ has a real locus $\mathbb{P}^1 \subset \mathbb{P}^3$ invariant under conj.

$\Rightarrow$ A line $\cap$ {4 lines} cannot be real.

Kharlamov

- How many deg. 3 rational curves pass through 8 points in $\mathbb{P}^3$?

  - Over $\mathbb{C}$ = 12
  - over $\mathbb{R}$ = 8, 10, 12

<u>Fibration</u>: $\mathbb{P}^2 - 9\text{pts} \longrightarrow \mathbb{P}^1$

$pt \in \mathbb{P}^2 \to Q[a,b] \to [a,b] \in \mathbb{P}^1$

$aQ_1 - bQ_2$

$Bl_9 \mathbb{P}^2 \to \mathbb{P}^1$ singular elliptic fibration.

we take $\chi(Bl_9\mathbb{P}^2) = \#$ singular fibers $= \#$rational cubics
$\quad\overset{\shortparallel}{12} \qquad\qquad \chi(\text{sing. fiber}) = 1$

Over $\mathbb{R}$

real rat. cubics passing through 8 pts $\in \mathbb{R}\mathbb{P}^2$

$\chi(Bl_9\mathbb{R}\mathbb{P}^2) =$  

$\overset{\shortparallel}{-8} \qquad +1 \qquad -1$

We count real curves with sign $(-1)^{\#\text{isolated real nodes}}$

- Welschinger's invariants: If $\dim_{\mathbb{C}} X = 2$, we take collection of r real points and 2s conjugate points in X. Then the number of real rational curves through this collection counted with $(-1)^{\#\text{isol. real nodes}}$ is independent of the position of the points.

(stab.) (S)pin structure is a trivialization $TX^{\phi} \oplus \det TX^{\phi}$

ϕ is anti-hol. involution
X^{ϕ} - fixed locus of ϕ.

and on a curve $TX^{\phi}|_{C} = TC \oplus \nu$ → splitting provides local spin structure. The comparison with a global one gives the above sign.

$\downarrow$
$C^{\phi} \subset X^{\phi}$

- Spinor states signs coming from comparison of local/global spin structure on X with $\dim_{\mathbb{C}} X = 3$.

Open Gromov-Witten

X symplectic mfld $\supset L$ - Lagrangian submanifold.

$\overline{\mathcal{M}}_{g,h;k,l}(X,\beta) = \Big\{$  $\xrightarrow{u}$  $+ \Big\} / \sim$

$\bar{\partial} u = 0$
$[u] = \beta \in H_2(X, L)$

$\in \overline{\mathcal{M}}_{g,h;k,l}$

↳ boundary & corners

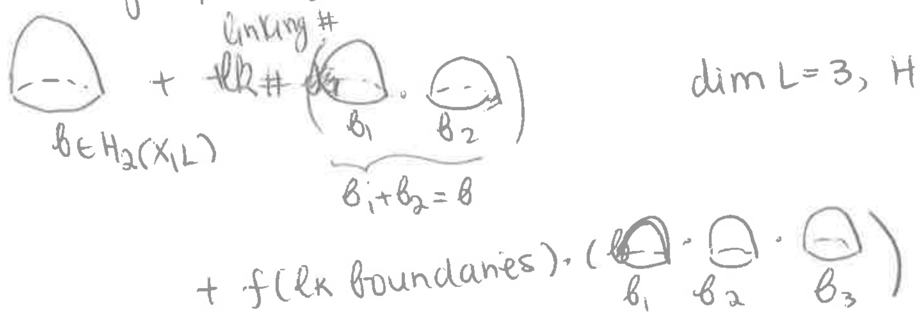
- not always orientable.

- under top cond. on $L \rightsquigarrow$ orientable $(w_1(L), w_2(L), \dim(L))$

- boundary → we don't get an invariant ($\exists$ representatives of the constraints)

- Liu = invariants when there is S^1-action on (X,L)

- Welschinger, Fukaya, Solomon-Tukachinsky

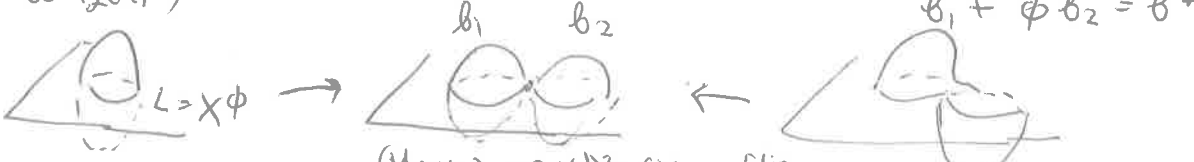


$\dim L=3$, $H_*(L,\mathbb{Q}) = H_*(S^3,\mathbb{Q})$

X sympl. $\phi: X \circlearrowleft$ $\phi^2 = id$, $\phi^* \omega = -\omega$

X^ϕ is Lagrangian

$b \in H_2(X,L)$



$(u_1,u_2): D^2 \times D^2 \to (X,L) \xrightarrow{\text{flip}} (u_1, \phi u_2 c_{D^2})$

Cho, Solomon dim 2,3:

- under top conditions the disk moduli spaces are orientable and in 1 parameter family the cut down moduli pass only through "good boundary"!

Pandharipande-Solomon-Walcher: Real MS in $g=0$.

$\overline{\mathcal{M}}^{\phi,\sigma}_{g,\mathbb{R},\ell}(X,A) = \{ u : $  $\updownarrow \sigma \longrightarrow X \updownarrow \phi \} / \sim_r$

$\phi \circ u \circ \sigma = u \quad \bar\partial u = 0$

k = real points
ℓ = # of pairs of conjugate pts.

Real Deligne-Mumford : $g = 0$, # of real pts $= 0 \to$ orientable
$g = 0$ # real points $\neq 0 \to$ not orientable

Q. Under topological conditions on X^ϕ, if we consider no real points the real moduli space $\overline{\mathcal{M}}^{\phi,\tau}_{0,\ell}(X,A)$ is orientable

 $\updownarrow \; \tau(z) = \frac{1}{\bar z} \to$ has fixed locus.

 in $\overline{\mathcal{M}}^{\phi,\tau}_{0,\ell}(X,A)$ this is a hypersurface (not bdry)

ε is the gluing param. $\in \mathbb{R}$

$\varepsilon > 0$  $\updownarrow \tau$

$\varepsilon < 0$  $\updownarrow \tau$

If $[u(\mathbb{R}\mathbb{P}^1)] \neq 0 \in H_1(L, \mathbb{Z}_2)$ there is no other boundary.

This is boundary for $\overline{\mathcal{M}}^{\phi,\tau}_{0,\ell}(X,A)$

 $\tau(z) = \frac{1}{\bar z}$

$\varepsilon \in \mathbb{R}$

$\varepsilon > 0$ $\quad$ $\varepsilon < 0$

 $\updownarrow \; \eta(z) = -\frac{1}{\bar z}$

$\overline{\mathcal{M}}^{\phi,\eta}_{0,\ell}(X,A) = \{ u :$  $\updownarrow \eta \to X \updownarrow \phi \} / \sim$

$\phi \circ u \circ \eta = u$

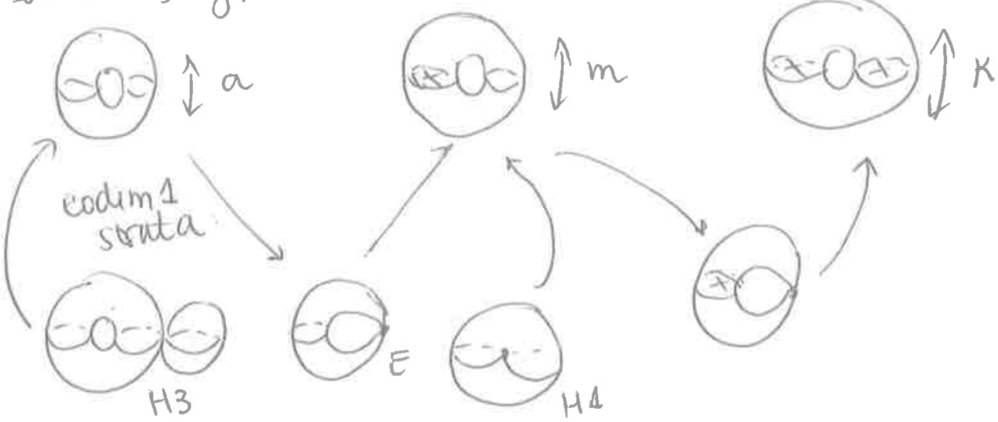
- Tehrani : if K_X has a real sq. root $\Rightarrow$ orientability of $\overline{\mathcal{M}}^{\phi\eta}_{0,\ell}(X,A)$
- On a genus g surface there are $\left|\frac{3g+4}{2}\right|$ top. types of order reversing involutions.

$\overline{\mathcal{M}}^{\phi}_{g,\ell}(X,A) = \bigcup \overline{\mathcal{M}}^{\phi,\sigma}_{g,\ell}(X,A) \longrightarrow$ has no boundary.

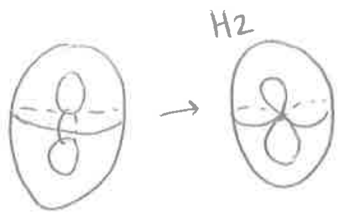
σ is an involution on Σ_g

↳ pairs of ex. conjugate points

$g=1$ 3 types



$g=2$



Real DM of only pairs of conj. points is not orientable if $g>0$

$\Lambda^{top} T\overline{\mathcal{M}}^{\phi}_{g,\ell}(X,A) \simeq \det \bar{\partial}^{\nabla}_{(TX,d\phi)} \otimes st^{*} \Lambda^{top}$ Real DM

Def: A real orientation (X,ω,ϕ) consists of • a complex line bundle $(L,\tilde{\phi}) \longrightarrow (X,\phi)$ such that

$\Lambda^{top}(TX,d\phi) \underset{Real}{\simeq} (L,\tilde{\phi})^{\otimes 2}$

and $w_2(TX^{\phi} \oplus 2L^{\phi}) = 0$

• a choice of a homotopic class of isom $(*)$

• a choice of a spin structure on $\widetilde{TX^{\phi} \oplus 2L^{\phi}}$ compatible with the orientation on X^{ϕ} induced by $(*)$.

Theorem: If $\dim_{\mathbb{C}} X$ is odd, and (X, ω, ϕ) is real orientable a choice of real orientation induces orientation on

$$\overline{\mathcal{M}}_{g,\ell}^{\phi}(X, A) \text{ for any } g, A.$$

(G. Zinger)

Examples
- $\mathbb{P}^{2n+1}$
- certain complete int.
- simply-connected CY's w/ $w_2(X^\phi) = 0$
- CY for which ϕ is anti-hol

Fano 3-folds:

$g=1$ real GW- invariants are enumerative.

Niu-Zinger: $\mu \in H^*(X)$

$$\underbrace{RGW_{g,B}^{X,\phi}(\mu)}_{\in \mathbb{Q}} = \sum_{\substack{0 \le h \le g \\ g-h \in 2\mathbb{Z}}} \tilde{C}_{h,B}^{X}\left(\tfrac{g-h}{2}\right) \underbrace{E_{h,B}^{X,\phi}(\mu)}_{\in \mathbb{Z}}$$

$$\sum_{g'=0}^{\infty} \tilde{C}_{h,B}^{X}(g') t^{2g'} = \left(\frac{\sinh(t/2)}{t/2}\right)^{h-1+c_1(B)/2}$$