Speaker: Konstanze Rietsch
Talk Title: Mirror symmetry for some homogeneous spaces
Date & Time: January 19, 2018 - 11 am

G/P $\quad$ G = reductive complex alg. group
P = parabolic.

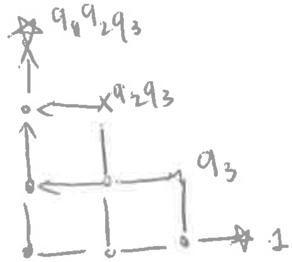
Q = What is "best" formulation of LG model for G/P

I. Laurent polynomial mirrors $\quad$ Givental SL_n/B
E H X $\quad$ $Gr_k(\mathbb{C})$
B, C-F, K, vS $\quad$ SL_n/P

II. Lie theoretic mirrors

III. "canonical" mirrors for G/P cominuscule

Eg SL_4/B



$\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_*$

$\mathcal{A}$ = arrows

$L_q : (\mathbb{C}^*)^{\mathcal{V}_0} \longrightarrow \sum_{a \in \mathcal{A}} t_{h(a)} t^{-1}_{t(a)}$ $\quad$ t_v = coordinate associated to v

$\left(\text{eg } = \dfrac{q_1q_2q_3}{t_{21}} + \cdots\right.$

Givental proved that $S(q) = \int_\Gamma e^{\frac{1}{\hbar}L_q} \bigwedge_{v \in \mathcal{V}_0} \dfrac{dt_v}{t_v}$

solves the q Toda lattice i.e. if $t_{ii} = e^{T_i}$, $S(T_i)$ satisfies

$$\det\left[\lambda \mathrm{Id} + \begin{pmatrix} t_1 \frac{\partial}{\partial T_1} & e^{T_1T_2} & & \\ -1 & t_{2} \frac{\partial}{\partial T_2} & e^{T_2T_3} & \\ & -1 & t_{3}\frac{\partial}{\partial T_3} & e^{T_3T_4} \\ & & -1 & t\frac{\partial}{\partial T_4} \end{pmatrix}\right] S = \lambda S$$

"q-diff eqs" of SL_n/B

Moreover critical points of L_q obey relations of $qH^*(SL_n/B)$

$$qH(SL_n/B) = \mathbb{C}[x_1,\dots,x_n,q_{n-1},\dots,q_1]/(E_1,\dots,E_n)$$

$\uparrow$ q elem. symm. pol's.

Rave: $qH^*(SL_n/B)_{[q^{-1}]} \to \mathbb{C}[(\mathbb{C}^*)^{\mathcal{V}_0}][q_i^{\pm}]/\left(\frac{\partial}{\partial t_v} L_q\right)_{v\in\mathcal{V}_0}$

but not isomorphic

II General G/P, Lie theoretic mirrors

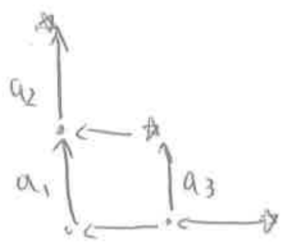
Replace $(\mathbb{C}^*)^{\mathcal{V}_0}$ by $R^\vee_{w_P w_0} = B^\vee w_P B^\vee_- \cap B^\vee_- w_0 B^\vee_- / B^\vee_-$

$\uparrow$ intersection of opposite Bruhat cells in $G^\vee/B^\vee$ $\leftarrow$ Langland's dual flag variety

Embed $(\mathbb{C}^*)^{\mathcal{V}_0} \times (\mathbb{C}^*)^{n_P} \hookrightarrow R^\vee_{w_P, w_0} \times (T^\vee)^{W_P}$

$$((t_{ij}), q_i) \longmapsto \left(\begin{pmatrix} 1 & & \\ & 1 & a_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & a_3 \\ & & 1 \end{pmatrix} B^\vee_-, A \right)$$

$\uparrow$ SL_3 example $\quad x_1(a_1) \quad x_2(a_2) \quad x_2(a_3)$



$a_i := t_{h(a_i)} t^{-1}_{t(a_i)}$

L_q extends to $W_{Lie}: R^\vee_{w_P, w_0} \times (T^\vee)^{W_P} \longrightarrow \mathbb{C}$ and generalizes to all G/P.

$\uparrow$ for SL_n/P

+ this function arose in work of Berenstein and Kazhdan on Geometric crystals.

Thm (D. Peterson) $qH^*(G/P)[q^{-1}] \cong \mathbb{C}[Y_P^*]$ for $Y_P^* \subset R^\vee_{w_P w_0}$ "stratum" of the Peterson variety $Y \subset G^\vee/B^\vee$.

$$Y = \{gB^\vee_- \mid g^{-1}.F \in [n_-, n_-]^\perp\} \subset G^\vee/B^\vee_-$$

$$Y \cap B^\vee_{w_P, w_0} = Y^*_P$$

relates to tridiag. matrices from description of $qH^*(SL_n/B)$ if g = upper triangular.

Thm Critical points of $W_{Lie,g} : R^\vee_{w_P, w_0} \longrightarrow \mathbb{C}$ recover Y_P^*

ie $\mathbb{C}[R^\vee_{w_P, w_0} \times (J^\vee)^{\ln q}]_{\partial W_g} \cong \mathbb{C}[Y_P^*] \cong qH^*(G/P)$

Thm: $R^\vee_{w_P, w_0}$ has a holomorphic volume form ω which is torus-invariant for every "Lusztig torus" in $R^\vee_{w_P, w_{P_0}}$

Conj: q-diff' equations of G/P have solutions of the form
$$S(q) = \int_\Gamma e^{\frac{1}{\hbar} W_{Lie}}\, \omega.$$

Status:
- ✓ for SL_n/B by Givental
- ✓ for G/B Gerasimov, Lebedev, Oblezin, R.
- $Gr_k(\mathbb{C}^n)$ Marsh-R
- all quadrics Pech, R, Williams
- miniscule G/P Lam-Templier

$\underline{\text{III. Canonical mirrors for cominiscule } G/P}$

replace $R^\vee_{w_P, w_0}$ by $\check{X} \subset P^\vee \backslash G^\vee$

G/P cominiscule means $P = P_{\omega_i}$ maximal parabolic where $\omega_i^\vee$ is miniscule as fundamental weight of $G^\vee$.

fundam rep. $V_{\omega_i^\vee} = IH^*(X_{\omega_i^\vee}) \cong H^*(G/P_{\omega_i})$

$$X_{\omega_i^\vee} \subset gr_G, \quad X_{\omega_i^\vee} \cong G/P_{\omega_i}$$

$$H^*(Gr_k(\mathbb{C}^n)) \cong \Lambda^k \mathbb{C}^n$$
$$\Downarrow$$
$$e_{i_1} \wedge \dots \wedge e_{i_k}$$

$\uparrow$

fundamental repr. of $SL_n = G^\vee$

$Gr_k(\mathbb{C}^n)$

$\circlearrowleft$

$PSL_n = G$

$P^\vee_{\omega_i} \backslash G^\vee \hookrightarrow \mathbb{P}(V^*_{\omega_i^\vee})$ therefore $P^\vee \backslash G^\vee$ has homogeneous coordinates in $V_{\omega_i^\vee} = H^*(G/P_{\omega_i})$

<u>Definition</u>:

$\check{X} \subset P^\vee \backslash G^\vee$

image of this projection

$$\begin{tikzcd}
& Z_p = B^\vee_- \cap B^\vee_+ \dot{w}_p \dot{w}_0^{-1} B^\vee_+ \subset G^\vee \arrow[dl, "\cong"'] \arrow[dr, "\wr"] & \\
X^\vee \times (T^\vee)^{W_p} \arrow[rr, "(p_i^\vee{,}t)", "\cong"'] \arrow[drr, "W_{can}"'] & & R^\vee_{w_p, w_0} \times (T^\vee)^{W_p} \arrow[d, "W_{Lie}"] \\
& & \mathbb{C}
\end{tikzcd}$$

$b \mapsto (p_i^\vee, t)$

$\mathbb{X} = Gr_{n-k}(\mathbb{C}^n) \rightsquigarrow \check{X} \subset Gr_k((\mathbb{C}^n)^*)$

$n\Big\{\Big(\;\Big)$, $\underbrace{\qquad}_{n-k}$ $\qquad$ $k\Big(\underbrace{\qquad\qquad}_{n}\Big)$

plucker coordinates labeled by $\lambda \subseteq n-k \,\overset{k}{\Box}$ Young diag.

Schubert classes σ^λ labelled by same λ's

$$\mathbb{X} = Gr_2(\mathbb{C}^5)\; W_q = \frac{p_{\ydiagram{1}}}{p_\emptyset} + \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}} + \frac{p_{\ydiagram{3,2}}}{p_{\ydiagram{2,2}}} + q\frac{p_{\ydiagram{3}}}{p_{\ydiagram{3,3}}} + \frac{p_{\ydiagram{3,1}}}{p_{\ydiagram{3}}}$$

Theorem (Marsh-R)

(1) $qH^*(Gr_{n-k}(\mathbb{C}^n)) \xrightarrow{\cong} \mathbb{C}[\check{X} \times \mathbb{C}_q^*] / (\partial_{X^\vee} W_q)$

$$\sigma^\lambda \longmapsto [p_\lambda]$$

(2) $S(q) = \sum_\lambda \left(\int e^{\frac{1}{\hbar} W_q} p_\lambda \omega \right) \sigma^{PD(\lambda)}$ is a flat section of Dubrovin Givental connection

$$\boxed{q \frac{d}{dq} S = \frac{1}{\hbar} \sigma^{\square} \star S}$$